DYNAMIC INTERACTIVE BEHAVIOR BETWEEN A POTENTIAL FLUID AND ELASTIC CONTAINER IN LARGE DEFORMATIONS

— Two-dimensional interaction problems of potential fluid and elastic bar (Part 1) —

ポテンシャル流体と弾性容器の大変形動的連成場における相互作用問題
二次元連成場における定式化

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Liquid-filled tanks on a shaking table showed vibratory responses that contradicted the prediction of elementary vibration theory. The author has been studying the responses that might be caused by the dynamic geometric nonlinear behavior of the system and showed the Lagrangian function that governs the interactive behavior between a potential fluid and elastic container in large deformations. Arbitrary Lagrangian-Eulerian Elements (ALE) were applied to the functional of the two-dimensional model, proposed a new procedure to analyze the problem, and obtained various nonlinear vibration responses corresponding to the vibratory responses in experiments using a shaking table.

Keywords : *nonlinear response of tank, interactive behavior of fluid and container, 1/2 subharmonic oscillation, summed combination resonance*

タンクの振動, 円筒タンクのおバリング, タンクの分数調波振動応答, 定常応答の不安定領域

1. Introduction

Until the 1950s, for the problem related to interaction of a fluid and its elastic container, the formulation that appends the mass effect of the fluid to the container was used. Subsequently, a functional was developed for the interaction problem of a potential fluid and its elastic container on the basis of the infinitesimal deformation field. Based on this, a linear theory was formulated that yielded useful knowledge^{2,3,4)} for engineering. J. C. Luke¹⁾ proved that the function that integrates the pressure formula for a potential fluid over the fluid field is a strict Lagrangian function governing the fluid movement. Using this, the nonlinear vibratory response of a fluid in a rigid tank was analyzed and comparison of responses obtained in the experiment was reported⁸⁾. In addition, using the finite element and boundary element methods, the processing and analysis of the conditions in the interaction of a fluid and its elastic container were reported¹⁸⁾. It is difficult to process a fluid at a moving boundary using the finite element method in large deformation fields; therefore, an ingenious solution is required to ensure the stability of the numerical analysis using the boundary element method.

Considering the fluid field deformation accompanying the elastic displacement of the container, the author has shown how to obtain the strict functional for a dynamic interaction problem of a potential fluid

and its elastic container^{6,7)} for large deformation. The nonlinear response obtained from experiments using cylindrical tanks could be theoretically analyzed on the basis of the derived functional. In this study, an analysis by discretizing the system on the basis of this functional¹⁹⁾ was attempted. Because the pressure integral for the potential fluid is given by Euler's method and that for the elastic container is normally given by the Lagrange method, precisely processing compatibility conditions for deformation and movement of the interaction surface between the fluid and elastic container was difficult.

In the early 1990s, arbitrary Lagrangian-Eulerian (ALE) elements¹⁸⁾ were developed. These elements introduced moving reference coordinates and enabled the measurement of fluid motion using a coordinate system that is independent of the motion of the fluid particles.

Limiting the investigation to incompressible, irrotational and inviscid fluids by using the functional that governs the interaction behavior of a fluid and an elastic body, the equation for the discretized field can be derived. First, it is shown that the appropriate fundamental equations and boundary conditions of the system are acquired from the first variation of the functional. Subsequently, a direct method is formulated for discretizing the strict functional in the dynamic interaction problem of the potential fluid and the elastic container in large deformations.

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When a cylindrical tank containing water is subjected to periodic ground motions at a certain frequency caused by an external force, subharmonic vibrations and nonlinear vibration responses occur. The author has experimentally studied the intense subharmonic vibration that occurs in cylindrical tanks^{9,10}.

In this paper, the numerical analysis of the subharmonic vibrations generated by the container's response in the two-dimensional interaction problem of the fluid and elastic container is discussed, and the relationship between the frequency of the external force and the subharmonic vibration as well as the modes of oscillation are analyzed, and finally, the usefulness of the proposed analysis method is demonstrated.

The simple two-dimensional model gives not only a new formulation of the coupling problem to solve the nonlinear vibration response, but also significant nonlinear response phenomena of the system.

2. Functional and a discretization method for the potential fluid and the elastic container in large deformations

2.1 Functional in the interaction field of a potential fluid and elastic container

An elastic container that exists on a moving base at velocity $\mathbf{v}_0$, which holds a potential fluid and its free surface, is considered. By introducing the velocity potential φ and the velocity $\hat{\mathbf{v}}$ of moving coordinates to measure fluid motion by an observer positioned on this base, the functional $I_m(\varphi, \boldsymbol{\eta}, \mathbf{u})$ ^{6,7,17} for the interaction field of the potential fluid and the elastic container in large deformations, which is seen by the observer on the base, is expressed by the following formula.

$$I_m = \int_{t_0}^{t_1} \left[\iint_{A_L(\boldsymbol{\eta}, \mathbf{u})} \rho_L \{ \dot{\varphi} + \nabla \varphi \cdot (\nabla \varphi / 2 - \hat{\mathbf{v}}) + \dot{\mathbf{v}}_0 \cdot \mathbf{r} + g(y - y_f) \} dx dy - \int_0^\ell \{ (\rho_E A_E \dot{\mathbf{u}} / 2 + \mathbf{v}_0) - \Pi(\mathbf{u}) \} ds \right] \quad (1)$$

Here, $\dot{\mathbf{v}}_0$ is the acceleration of the base, $\boldsymbol{\eta}$ is the wave height vector on the free surface, $\mathbf{u}$ is the displacement vector of the elastic container, $\mathbf{r}$ is the position vector after deformation, Π is the potential energy of the elastic container, A_L is the fluid domain, ρ_L is the fluid mass density, A_E is the cross-sectional area of the container, y_f is the coordinate value of free surface, ρ_E is the container mass density, and g is the acceleration due to gravity.

The first variation of the equation^{6,7,17} is given by the following formula:

$$\begin{aligned} \delta I_m = & \int_{t_0}^{t_1} \left[\int_{S_f} \delta \varphi \rho_L (\nabla \varphi - \hat{\mathbf{v}} - \dot{\eta} \mathbf{i}_y) \cdot \mathbf{n}_L ds - \iint_{A_L(\boldsymbol{\eta}, \mathbf{u})} \delta \varphi \rho_L \nabla^2 \varphi dx dy \right. \\ & + \int_{S_f} \delta \eta \rho_L \{ \dot{\varphi} + \nabla \varphi \cdot (\nabla \varphi / 2 - \hat{\mathbf{v}}) + \dot{\mathbf{v}}_0 \cdot \mathbf{r} + g(y - y_f) \} \cdot \mathbf{n}_L ds \\ & + \int_{S_i} \rho_L \delta \varphi (\nabla \varphi - \hat{\mathbf{v}} - \dot{\mathbf{u}}) \cdot \mathbf{n}_L ds + \int_{S_0} \rho_L \delta \varphi (\nabla \varphi - \hat{\mathbf{v}}) \cdot \mathbf{n}_L ds \\ & + \int_{S_i} \rho_L \delta \mathbf{u} \{ \dot{\varphi} + \nabla \varphi \cdot (\nabla \varphi / 2 - \hat{\mathbf{v}}) + \dot{\mathbf{v}}_0 \cdot \mathbf{r} + g(y - y_f) \} \cdot \mathbf{n}_L ds \\ & \left. + \int_0^\ell \delta \mathbf{u} \{ (\rho_E A_E \dot{\mathbf{u}} + \mathbf{v}_0) - \partial \Pi / \partial \mathbf{u} \} ds \right] \quad (2) \end{aligned}$$

where $\mathbf{n}_L$ is the outward normal vector to the fluid at each boundary, S_f is the fluid free surface, S_i is the interaction surface, and S_0 is the fixed boundary surface, $\mathbf{i}_y$ is a unit vector of coordinate y .

The formula (2) leads to the basic equation and the expressions for

the boundary conditions at each boundary as below.

$$1) \text{ Basic equation for the fluid interior } A_L: \nabla^2 \varphi = 0 \quad (3)$$

2) Fluid free surface S_f :

$$(\nabla \varphi - \hat{\mathbf{v}} - \dot{\eta} \mathbf{i}_y) \cdot \mathbf{n}_L |_{y=y_f+\eta} = 0 \quad (4)$$

$$\{ \dot{\varphi} + \nabla \varphi \cdot (\nabla \varphi / 2 - \hat{\mathbf{v}}) + \dot{\mathbf{v}}_0 \cdot \mathbf{r} + g \eta \mathbf{i}_y \} \cdot \mathbf{n}_L |_{y=y_f+\eta} = 0 \quad (5)$$

$$3) \text{ Fluid fixed boundary } S_0: (\nabla \varphi - \hat{\mathbf{v}}) \cdot \mathbf{n}_L = 0 \quad (6)$$

4) Interaction surface between fluid and container S_i :

$$(\nabla \varphi - \hat{\mathbf{v}} - \dot{\mathbf{u}}) \cdot \mathbf{n}_L = 0 \quad (7)$$

$$\rho_L \{ \dot{\varphi} + \nabla \varphi \cdot (\nabla \varphi / 2 - \hat{\mathbf{v}}) + \dot{\mathbf{v}}_0 \cdot \mathbf{r} + g(y - y_f) \} \cdot \mathbf{n}_L |_{\mathbf{u}} + \{ \rho_E A_E (\dot{\mathbf{u}} + \mathbf{v}_0) + \partial \Pi / \partial \mathbf{u} \} = 0 \quad (8)$$

Equation (3) is the fundamental equation for the fluid. Because the fluid velocity is being measured in a coordinate system with a moving velocity, the velocity measured in the fixed coordinate system is expressed as $\nabla \varphi - \hat{\mathbf{v}}$. Equation (4) states that the fluid velocity matches the normal direction component of the wave height velocity on the free surface, and equation (5) expresses the mechanical condition of the component of the fluid pressure and the wave height in the direction normal on the fluid free surface. Equation (6) states that the velocity in the direction normal to the fixed boundary is zero.

Equation (7) states that the fluid velocity and container velocity are equal in the direction normal to the fixed boundary. Equation (8) expresses the mechanical condition of the fluid pressure acting in the direction normal to the container. The functional gives the physically appropriate fundamental equations and boundary conditions.

The formulation and analysis technique for the interaction system model in which the potential fluid filling a two-dimensional container undergoes a sine wave horizontal ground motion are shown below. In this system, the container, the initial free surface, and the base are parallel and both left and right sides of the fluid have a rectilinear shape. A direct method that adopts the reference coordinates is applied and equation (1) is discretized.

2.2 Moving boundary

The fluid that generates waves on the free surface S_f moves on the elastic container and its boundary surface depending on the wave height. If the velocity of the fluid motion is defined in conjunction with the movement of the fluid element coordinates, the fluid velocity includes the movement velocity of the coordinates. To convert the fluid velocity for Euler's method, the reference coordinates¹⁸ are introduced, which are independent of the moving velocity of the fluid motion.

The equation for the discretized field in the dynamic interaction problem of the fluid and container can be derived by first, creating the displacement field of wave height, fluid field, and elastic container that satisfy the kinematic condition, then discretizing the functional in equation (1); followed by computing the first variation.

The two-dimensional model with a moving boundary is divided into a skewed grid region, which is macroscopically fitted to the initial shape of the fluid domain (shown by a dashed line in Fig. 1). These fluid nodes are not points that move in tandem with the fluid motion, but rather form reference coordinates. These grid intersections will be described

hereafter as fluid nodes.

A model with both sides of the fluid field as interaction surfaces is adopted, in which the elastic container consists of elastic linear members. Because two interaction surfaces S_i exist, they are distinguished as $S_{i\alpha}$, where $\alpha=1$ for the interaction surface on the right side and $\alpha=2$ for that on the left side. For simplicity, the shape of the interaction surface S_{i1} following deformation is illustrated in Fig. 1. A formulation that considers both sides can be obtained using this diagram.

x is the horizontal direction, y is the reverse vertical direction, and y_f is the free surface coordinate. The movement of the fluid nodes is set using the two direction vectors of the skewed grid (the direction vectors $\mathbf{i}_\mu, \mathbf{i}_\nu$ for point $k_{\mu\nu}$ are shown in Fig.1). The fluid nodes fall into five types.

1. Nodes k_{fa} exist on the fluid free surface and the interaction surfaces (hereafter, nodes on the two moving boundaries).
2. A node on the fluid free surface is expressed as point $k_{f\mu}$.
3. A node on the interaction surfaces S_{ia} is expressed as point $k_{ia\nu}$.
4. The nodes on the fluid fixed boundary surface S_0 do not deviate from the initial coordinates. A base node corresponds to this case.
5. Interior nodes $k_{\mu\nu}$ in the interior of the fluid do not make contact with the boundary.

Displacements occur in the fluid domain as independent variables. They apply to the wave height of the free surface and the displacement of the elastic container on the interaction surfaces on both sides. The position of the fluid nodes on these boundaries is first determined for consistency, and then, a suitable coordinate movement of the interior nodes is determined, and finally, the reference coordinates are configured. The position of each node is defined as follows.

1) Movement of fluid nodes on the interaction surfaces

The position coordinates of the elastic linear members are defined using the coordinates at the centroid of the cross-section of the linear member s_α , which means distance from the base of the container. If the nodes k_{fa} on the two moving boundaries are measured in the coordinate system of the elastic member, they are defined as having the initial coordinates s_{af} that move after deformation to the elastic member coordinate $s_{af} + \eta_\alpha$. That is, the “wave height” η_α of these points is defined by the amount of movement in the direction of the tangent vector $\mathbf{i}_{\alpha s}$ at the centroid position of the elastic member. Considering the displacement vector $\mathbf{u}_\alpha(s_{af} + \eta_\alpha)$ of the node, the movement of the points on these two moving boundaries from before and after deformation is defined by the following formula.

$$\Delta \mathbf{r}_{fa} = \mathbf{r}(s_{af} + \eta_\alpha) - \mathbf{r}(s_{af}) + \mathbf{u}_\alpha(s_{af} + \eta_\alpha) \quad (9)$$

Fig.1 Reference Coordinates of a Two Dimensional Tank

The position after deformation of the nodes k_{fa} on the two moving boundaries is then the moving position of these points.

Equation (9) guarantees that the nodes on the two moving boundaries exist on the elastic member even after deformation. There is a distance of around half the member thickness between the centroid of the member and the interaction surfaces, which is possible to consider. However, here, this distance is assumed to be small and is ignored. The symbol η_α is used to display the vector for the wave height η_α ($\alpha = 1, 2$). This is a subset of all wave height vectors η_e .

Similarly, the position of the other fluid node $k_{ia\nu}$ on the interaction surfaces following deformation is defined with the following formula using the interior division ratio of the height of these points from the tank base to the free surface height, where y_ν and s_{av} are the initial y and s_α coordinates of the node, respectively.

$$\Delta \mathbf{r}_{av} = (\eta_\alpha y_\nu / y_f) \mathbf{i}_{as} + \mathbf{u}_\alpha(s_{av} + \eta_\alpha y_\nu / y_f) \quad (10)$$

Using this procedure, the kinematic conditions satisfy that the fluid nodes and elastic member after deformation closely fit at the fluid node positions. Equation (10) uses the displacement representation for the elastic member at the fluid node positions after deformation. The position of the fluid nodes and the node positions of the elastic member are not identical, but the node positions of the elastic member present no problem if the displacements are determined using the node vectors for the elastic member.

2) Movement of nodes on the free surface

The fluid node $k_{f\mu}$ on the free surface except for those on the two moving boundaries are assumed to move in the direction of the wave height $\eta_\mu \mathbf{j}_\mu$ (Fig. 1.2). The movement in the horizontal direction is defined by the following formula by internally dividing the horizontal displacement of nodes k_{f1} on the two moving boundaries with the initial coordinate from the left-side interaction boundary,

$$\Delta \mathbf{r}_{f\mu} = \eta_\mu \mathbf{j}_\mu + \{ \mathbf{u}_1(s_{1f} + \eta_1) \cdot \mathbf{i}_\mu (x_\mu / X_\mu) + \mathbf{u}_2(s_{2f} + \eta_2) \cdot \mathbf{i}_\mu (1 - x_\mu / X_\mu) \} \mathbf{i}_\mu \quad (11)$$

Here, where x_μ is the initial X coordinate of the node, X_μ is the free surface width and the last term corresponds to the interaction surface $\alpha = 2$.

3) Movement of interior nodes

The movement of the interior node $k_{\mu\nu}$ (initial coordinates vector $\mathbf{r}_{\mu\nu} = (x_\mu, y_\nu)$) is defined by the following formula. The movement in the horizontal direction $\mathbf{i}_\mu$ of the skewed grid, which gives an internally divided quantity depending on the x coordinate of the horizontal displacement of the two nodes at each edge of the interaction surfaces with the same water level, and the corresponding movement in the vertical direction $\mathbf{j}_\nu$ of the skewed grid, which gives an internally

divided quantity between the free surface wave height and the container base, are simply used together as the movement of the vector sum (Fig.1.3).

$$\begin{aligned} \Delta \mathbf{r}_{\mu\nu} = & \eta_{\mu}(y_{\mu}/y_f) \mathbf{j}_{\nu} \\ & + \{\mathbf{u}_1(s_{1\nu} + \eta_1/y_{\mu}/y_f) \cdot \mathbf{i}_{\mu}(x_{\mu}/X_{\mu}) \\ & + \mathbf{u}_2(s_{2\nu} + \eta_2/y_{\mu}/y_f) \cdot \mathbf{i}_{\mu}(1 - x_{\mu}/X_{\mu})\} \mathbf{i}_{\mu} \end{aligned} \quad (12)$$

2.3 Discretization method

An arbitrary position in the fluid interior is defined using the finite element method. Four nodes are used as a quadrilateral element for the fluid velocity potential. The shape function for each fluid node i using the $\xi_1 - \xi_2$ coordinate system is expressed by the following formula.

$$\begin{aligned} N_i(\xi_1, \xi_2) = & (1 + \xi_1 \xi_{2i})(1 + \xi_2 \xi_{1i})/4, \\ i = & 1, \dots, 4, -1 \leq \xi_1, \xi_2 \leq 1 \end{aligned} \quad (13)$$

Expressing the initial x and y coordinate vectors of the four nodes as $\mathbf{x}_e$ and $\mathbf{y}_e$, respectively, the following equation is derived.

$$\mathbf{r}_0 = \{x_0, y_0\}, \quad x_0 = N(\xi_1, \xi_2) \mathbf{x}_e, \quad y_0 = N(\xi_1, \xi_2) \mathbf{y}_e \quad (14)$$

Using the linear rod element, the total strain energy is defined for the elastic container. The displacement vector $\mathbf{u}(s_{\alpha})$ is defined by the following formula.

$$\mathbf{u} = \{u, v\} \quad (15)$$

Here, u is the axial displacement and v is the displacement in the normal direction.

For the shape functions of these elements, first- and third-order algebraic functions are used. The axial strain $\mathcal{E}_s$ for the linear member and the curvature $\mathcal{K}_s$ use the following formulas. A nonlinear term for the axial strain is adopted.

$$\mathcal{E}_s = u_{,s} + v_{,s}^2/2, \quad \mathcal{K}_s = -v_{,ss} \quad (16)$$

The movement that accompanies the movement of the fluid nodes is considered, namely one fluid node on an element moves on the adjacent element. At each interaction surface, each fluid element makes contact with the linear element at two points. The displacement of the container at each fluid motion node position is defined using the generalized coordinates (node displacement vectors) of the two edge nodes of the linear element. Because the fluid element contacts the linear member at two points, a node displacement vector having minimum two points and maximum four points is required to determine the fluid node motion. For the entire system, as there are two interaction surfaces on both the left and right sides, a node displacement vector with a minimum of four points and a maximum of eight points is required (each node has three degrees of freedom) in order to determine the amount of displacement of the four nodes for each fluid element. With these node displacement vectors $\mathbf{d}_e$ for the elastic linear element and the wave height η_{α} of the nodes on the two moving boundaries as parameters, the displacement vector of the four nodes for each fluid element on the linear element is determined with the following formula.

$$\begin{aligned} \mathbf{u}_{\alpha j} = & (u_{\alpha j}, u_{\alpha yj}), \\ u_{\alpha j} = & \mathbf{C}_x(s_{\alpha}, \eta_{\alpha}) \mathbf{d}_e, u_{\alpha yj} = \mathbf{C}_y(s_{\alpha}, \eta_{\alpha}) \mathbf{d}_e \end{aligned} \quad (17)$$

$\mathbf{C}_x$ and $\mathbf{C}_y$ are the shape functions for the linear element. The node coordinates after deformation of the fluid element are a function of the

wave height η_{α} . In formula (17), the direction cosines for the fluid nodes on the two moving boundaries are used. When θ_{α} is the angle formed by the tangent vector of the elastic member and the x direction of the global coordinates (positive counterclockwise direction), the change in the direction cosines is strictly defined in the following equations using the rotation $v_{,s}$ of the elements.

$$\cos \Delta \theta_{\alpha} = 1/\sqrt{1+v_{,s}^2}, \quad \sin \Delta \theta_{\alpha} = v_{,s}/\sqrt{1+v_{,s}^2} \quad (18)$$

The movement of the interior nodes is expressed with the following formula using equations (12) and (17).

$$\begin{aligned} \mathbf{u}_{mj} = & (u_{xm}, u_{ym}), \\ u_{xm} = & \mathbf{E}_{xm} \eta_m + \mathbf{C}_{xm} \mathbf{d}_e, \quad u_{ym} = \mathbf{E}_{ym} \eta_m + \mathbf{C}_{ym} \mathbf{d}_e \end{aligned} \quad (19)$$

In the above formulas, $\mathbf{E}_{xm}$ and $\mathbf{E}_{ym}$ are given using the initial coordinates for each point. If the amount of movement and displacement for these nodes is substituted into the coordinate system of equation (14), the position vector $\mathbf{r} = \{x, y\}$ of an arbitrary point in the fluid interior after deformation is defined by the following formulas using the wave height vector $\boldsymbol{\eta}_e$ and the node displacement vector $\mathbf{d}_e$ for the linear elements.

$$x = N\{\mathbf{x}_e + \mathbf{E}_x \boldsymbol{\eta}_e + \mathbf{C}_x \mathbf{d}_e\}, \quad y = N\{\mathbf{y}_e + \mathbf{E}_y \boldsymbol{\eta}_e + \mathbf{C}_y \mathbf{d}_e\} \quad (20)$$

These can be expressed with the following equation.

$$\mathbf{r} = \begin{Bmatrix} x \\ y \end{Bmatrix} = \mathbf{r}_0 + \mathbf{B}_{\eta} \boldsymbol{\eta}_e + \mathbf{B}_d \mathbf{d}_e \quad (21)$$

$$\text{Here, } \mathbf{r}_0 = \begin{Bmatrix} N\mathbf{x}_e \\ N\mathbf{y}_e \end{Bmatrix}, \mathbf{B}_{\eta} = \begin{bmatrix} N\mathbf{E}_x(\mathbf{d}_e) \\ N\mathbf{E}_y(\mathbf{d}_e) \end{bmatrix}, \mathbf{B}_d = \begin{bmatrix} N\mathbf{C}_x(\boldsymbol{\eta}_{\alpha}) \\ N\mathbf{C}_y(\boldsymbol{\eta}_{\alpha}) \end{bmatrix}.$$

Then, a definite integration is performed from the $x - y$ coordinate system to the fluid element $\xi_1 - \xi_2$ coordinates system. The Jacobian matrix $\mathbf{J}$ is defined by the following equation.

$$\begin{Bmatrix} \partial/\partial \xi_1 \\ \partial/\partial \xi_2 \end{Bmatrix} = \begin{bmatrix} \partial x/\partial \xi_1 & \partial y/\partial \xi_1 \\ \partial x/\partial \xi_2 & \partial y/\partial \xi_2 \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = \mathbf{J}(\boldsymbol{\eta}_e, \mathbf{d}_e) \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} \quad (22)$$

Using this, the operator ∇ is expressed by the following equation.

$$\nabla = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = \mathbf{J}^{-1}(\boldsymbol{\eta}_e, \mathbf{d}_e) \begin{Bmatrix} \partial/\partial \xi_1 \\ \partial/\partial \xi_2 \end{Bmatrix} \quad (23)$$

Using the shape function, the velocity potential ϕ is expressed as

$$\phi = N \boldsymbol{\phi}_e \quad (24)$$

Here, $\boldsymbol{\phi}_e$ is the fluid velocity potential vector. Substituting this into equation (23), the following expression is derived.

$$\nabla \phi = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \partial y/\partial \xi_2 & -\partial y/\partial \xi_1 \\ -\partial x/\partial \xi_2 & \partial x/\partial \xi_1 \end{bmatrix} \begin{Bmatrix} \partial \phi/\partial \xi_1 \\ \partial \phi/\partial \xi_2 \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \mathbf{A}_e \boldsymbol{\phi}_e \quad (25)$$

$$\text{Here, } \mathbf{A}_e = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \partial y/\partial \xi_2 & -\partial y/\partial \xi_1 \\ -\partial x/\partial \xi_2 & \partial x/\partial \xi_1 \end{bmatrix} \begin{Bmatrix} \partial N/\partial \xi_1 \\ \partial N/\partial \xi_2 \end{Bmatrix}.$$

The determinant of the Jacobian matrix $|\mathbf{J}| = |\mathbf{J}(\boldsymbol{\eta}_e, \mathbf{d}_e)|$ is a function of the wave height vector $\boldsymbol{\eta}_e$ and the node displacement vector $\mathbf{d}_e$ for the elastic element.

Moreover, the velocity of the fluid element coordinates that reflect the fluid node motion is the movement velocity. Therefore, the movement velocity $\hat{\mathbf{v}}$ of the reference coordinates for each ALE element is given by the following equation using equation (21).

$$\hat{\mathbf{v}} = \partial \mathbf{r} / \partial t = \dot{\mathbf{r}} \quad (26)$$

Substituting the expression formulas of these unknown quantities into equation (1) and converting the integral into the fluid element coordinates, an expression for the discretized functional is obtained.

$$I_m^{ALE}(\boldsymbol{\varphi}_e, \boldsymbol{\eta}_e, \mathbf{d}_e) = \int_{t_0}^{t_1} \left[\sum_{m_e=1}^{\ell} \left\{ \int_{-1}^1 \int_{-1}^1 \rho_L (N \boldsymbol{\varphi}_e | \mathbf{J} | - \boldsymbol{\varphi}_e' A_e' \dot{\mathbf{r}}) \right. \right. \\ \left. \left. + \frac{1}{2|J|} \boldsymbol{\varphi}_e' N' N \boldsymbol{\varphi}_e + (\mathbf{a} \cdot \mathbf{r} | \mathbf{J} | + \mathcal{G}(y - y_f) | \mathbf{J} |) \right\} d\xi_1 d\xi_2 \right. \\ \left. - \sum_m \sum_a \int_0^{\ell} \left\{ \rho_E h \left(\frac{1}{2} (\dot{\mathbf{u}}^2 + \dot{\mathbf{v}}^2) - \mathbf{u} \cdot \mathbf{a} \right) - \Pi(\mathbf{u}) \right\} ds \right] dt \quad (27)$$

Here, $\Pi(\mathbf{u}) = Eh(u_{,3} + v_{,3}^2/2)^2/2 + Eh^3 v_{,3}^2/24$ is the strain energy for the elastic member with a unit width, E is the Young's modulus, h is a thickness of the member and the acceleration $\mathbf{a} = \dot{\mathbf{v}}_0$ of the base.

In equation (27), the functional is assumed to be a function of the fluid velocity potential vector $\boldsymbol{\varphi}_e$, the wave height vector $\boldsymbol{\eta}_e$ and the elastic element node displacement vector $\mathbf{d}_e$. Bearing this in mind, if the first variation is evaluated with respect to the $\boldsymbol{\varphi}_e, \boldsymbol{\eta}_e, \mathbf{d}_e$ -argument function for the functional, the equation for the discretized field can be obtained.

The determinant $|J|$ is a scalar quantity. $A_e \boldsymbol{\varphi}_e$ in equation (25) is the column vector and the incremental expression for the $\boldsymbol{\varphi}_e, \boldsymbol{\eta}_e, \mathbf{d}_e$ -argument function can be written with the following increment representation.

$$\Delta |J| = D_{\eta} \Delta \boldsymbol{\eta}_e + D_{\alpha} \Delta \boldsymbol{\eta}_{\alpha} + D_d \Delta \mathbf{d}_e \\ \Delta(A_e \boldsymbol{\varphi}_e) = A_e \Delta \boldsymbol{\varphi}_e + S_{\eta} \Delta \boldsymbol{\eta}_e + S_{\alpha} \Delta \boldsymbol{\eta}_{\alpha} + S_d \Delta \mathbf{d}_e \\ \Delta(B_{\eta} \boldsymbol{\eta}_e) = B_{\eta} \Delta \boldsymbol{\eta}_e + T_d \Delta \mathbf{d}_e, \quad \Delta(B_d \mathbf{d}_e) = B_d \Delta \mathbf{d}_e + B_{\alpha} \Delta \boldsymbol{\eta}_{\alpha} \\ B_r = B_d + T_d \quad (28)$$

If these are used, the variation formula can be expressed for equation (27) from terms 1 to 4. For the sake of simplicity, the element summation symbols are omitted. The nonlinear terms for the elastic container are included in term 4.

$$\rho_L \delta \iint_A (\dot{\varphi} - \nabla \varphi \cdot \dot{\mathbf{r}}) dx dy = \rho_L \delta \int_{-1}^1 \int_{-1}^1 (N \boldsymbol{\varphi}_e | \mathbf{J} | - \boldsymbol{\varphi}_e' A_e' \dot{\mathbf{r}}) d\xi_1 d\xi_2 \\ = -\delta \boldsymbol{\eta}_e' \rho_L \int_{-1}^1 \int_{-1}^1 [(N' D_{\eta} + A_e' B_{\eta}) \dot{\boldsymbol{\eta}}_e + (N' D_{\alpha} + A_e' B_{\alpha}) \dot{\boldsymbol{\eta}}_{\alpha} \\ + (N' D_d + A_e' B_d) \dot{\mathbf{d}}_e] d\xi_1 d\xi_2 + \delta \boldsymbol{\eta}_e' \rho_L \int_{-1}^1 \int_{-1}^1 [(D_{\eta}' N + B_{\eta}' A_e) \dot{\boldsymbol{\varphi}}_e \\ + (B_{\eta}' S_{\eta} - S_{\eta}' B_{\eta}) \dot{\boldsymbol{\eta}}_e + (B_{\eta}' S_{\alpha} - S_{\eta}' B_{\alpha}) \dot{\boldsymbol{\eta}}_{\alpha} \\ + (B_{\eta}' S_d - S_{\eta}' B_r) \dot{\mathbf{d}}_e] d\xi_1 d\xi_2 \\ + \delta \boldsymbol{\eta}_{\alpha}' \rho_L \int_{-1}^1 \int_{-1}^1 [(D_{\alpha}' N + B_{\alpha}' A_e) \dot{\boldsymbol{\varphi}}_e + (B_{\alpha}' S_{\eta} - S_{\alpha}' B_{\eta}) \dot{\boldsymbol{\eta}}_e \\ + (B_{\alpha}' S_{\alpha} - S_{\alpha}' B_{\alpha}) \dot{\boldsymbol{\eta}}_{\alpha} + (B_{\alpha}' S_d - S_{\alpha}' B_r) \dot{\mathbf{d}}_e] d\xi_1 d\xi_2 \\ + \delta \mathbf{d}_e' \rho_L \int_{-1}^1 \int_{-1}^1 [(D_d' N + B_r' A_e) \dot{\boldsymbol{\varphi}}_e + (B_r' S_{\eta} - S_d' B_{\eta}) \dot{\boldsymbol{\eta}}_e \\ + (B_r' S_{\alpha} - S_d' B_{\alpha}) \dot{\boldsymbol{\eta}}_{\alpha} + (B_r' S_d - S_d' B_r) \dot{\mathbf{d}}_e] d\xi_1 d\xi_2 \quad (29.1)$$

$$\rho_L \delta \frac{1}{2} \iint_A \text{grad} \varphi \cdot \text{grad} \varphi dx dy = \delta \boldsymbol{\varphi}_e' \rho_L \int_{-1}^1 \int_{-1}^1 (A_e' A_e | \mathbf{J} | d\xi_1 d\xi_2 \boldsymbol{\varphi}_e \\ + \delta \boldsymbol{\eta}_e' \rho_L \int_{-1}^1 \int_{-1}^1 [S_{\eta}' A_e \boldsymbol{\varphi}_e | \mathbf{J} | - \frac{1}{2} D_{\eta}' (\boldsymbol{\varphi}_e' A_e' A_e \boldsymbol{\varphi}_e) / | \mathbf{J} |^2] d\xi_1 d\xi_2 \\ + \delta \boldsymbol{\eta}_{\alpha}' \rho_L \int_{-1}^1 \int_{-1}^1 [S_{\alpha}' A_e \boldsymbol{\varphi}_e | \mathbf{J} | - \frac{1}{2} D_{\alpha}' (\boldsymbol{\varphi}_e' A_e' A_e \boldsymbol{\varphi}_e) / | \mathbf{J} |^2] d\xi_1 d\xi_2 \\ + \delta \mathbf{d}_e' \rho_L \int_{-1}^1 \int_{-1}^1 [S_d' A_e \boldsymbol{\varphi}_e | \mathbf{J} | - \frac{1}{2} D_d' (\boldsymbol{\varphi}_e' A_e' A_e \boldsymbol{\varphi}_e) / | \mathbf{J} |^2] d\xi_1 d\xi_2 \quad (29.2)$$

$$\rho_L \delta \iint_A \mathbf{a}^t \cdot \mathbf{r} dx dy = \delta \int_{-1}^1 \int_{-1}^1 \rho_L (\mathbf{a}^t \cdot \mathbf{r} | \mathbf{J} | + \mathcal{G}(y - y_f) | \mathbf{J} |) d\xi_1 d\xi_2 \\ = \delta \boldsymbol{\eta}_e' \int_{-1}^1 \int_{-1}^1 \rho_L [B_{\eta}' | \mathbf{J} | + D_{\eta}' (r_0 + B_{\eta} \boldsymbol{\eta}_e + B_d \mathbf{d}_e)] \cdot \mathbf{a} d\xi_1 d\xi_2 \\ + \delta \boldsymbol{\eta}_{\alpha}' \int_{-1}^1 \int_{-1}^1 \rho_L [B_{\alpha}' | \mathbf{J} | + D_{\alpha}' (r_0 + B_{\eta} \boldsymbol{\eta}_e + B_d \mathbf{d}_e)] \cdot \mathbf{a} d\xi_1 d\xi_2 \\ + \delta \mathbf{d}_e' \int_{-1}^1 \int_{-1}^1 \rho_L [B_d' | \mathbf{J} | + D_d' (r_0 + B_{\eta} \boldsymbol{\eta}_e + B_d \mathbf{d}_e)] \cdot \mathbf{a} d\xi_1 d\xi_2 \quad (29.3)$$

The fourth term is the well-known discretized equation of motion for both sides of an elastic container.

$$-\delta \int_{t_0}^{t_1} \int_0^{\ell} \left\{ \rho_E A_e (\dot{\mathbf{u}}^t \cdot \dot{\mathbf{u}}/2 - \mathbf{u} \cdot \mathbf{a}) - \Pi(\mathbf{u}) \right\} ds dt \\ = \delta \mathbf{d}_e' [\mathbf{M}_d \ddot{\mathbf{d}}_e + \mathbf{K}_d \mathbf{d}_e + \mathbf{f}_e(\mathbf{a}, \mathbf{d}_e)] \quad (29.4)$$

Here, $\mathbf{M}_d$ is the container mass matrix, $\mathbf{K}_d$ is the container stiffness matrix, and $\mathbf{f}_e$ is the external force term. The stiffness matrix and external force term include nonlinear terms with respect to the generalized coordinates.

These terms correspond to discretizing the following into various equations: based on the coordinate system before deformation in equation (1) and using the finite element method, the wave height and the displacement of the container are measured after deformation and the functional is also evaluated after deformation. That is, under the system of fluid and elastic elements that are used in equation (29), the equation for the field is given by approximately discretizing equations (3) to (9). The equation obtained for the field for this system is generally expressed as in the following equation.

$$\begin{bmatrix} 0 & \mathbf{R}_1 & \mathbf{R}_2 \\ -\mathbf{R}_1' & \mathbf{R}_{\eta} & \mathbf{R}_{12} \\ -\mathbf{R}_2' & -\mathbf{R}_{12}' & \mathbf{R}_d \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{\varphi}}_e \\ \dot{\boldsymbol{\eta}}_e \\ \dot{\mathbf{d}}_e \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{\varphi} & \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_1' & \mathbf{K}_{\eta} & \mathbf{K}_{12} \\ \mathbf{K}_2' & \mathbf{K}_{12}' & \mathbf{K}_d \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varphi}_e \\ \boldsymbol{\eta}_e \\ \mathbf{d}_e \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}_{\varphi} \\ \mathbf{f}_{\eta} \\ \mathbf{M}_d \ddot{\mathbf{d}}_e + \mathbf{f}_d \end{Bmatrix} = \mathbf{0} \quad (30)$$

Here, $\mathbf{R}_{\eta} = -\mathbf{R}_{\eta}'$, $\mathbf{R}_d = -\mathbf{R}_d'$ and the vectors $\mathbf{f}_{\varphi}$ and $\mathbf{f}_{\eta}$ are external force terms including nonlinear terms. Cholesky decomposition of the matrix $\mathbf{M}_d$ is performed, followed by substitution in the equation.

$$\mathbf{M}_d = \mathbf{W}' \mathbf{W}, \quad \mathbf{d}_2 = \mathbf{W} \dot{\mathbf{d}}_e \quad (31)$$

Matrix $\mathbf{W}$ is an upper triangular matrix. Using equation (31), the matrix $\mathbf{M}_d$ in equation (29.4) is converted to get the following general formula.

$$\begin{bmatrix} 0 & \mathbf{R}_1 & \mathbf{R}_2 & 0 \\ -\mathbf{R}_1' & \mathbf{R}_{\eta} & \mathbf{R}_{12} & 0 \\ -\mathbf{R}_2' & -\mathbf{R}_{12}' & \mathbf{R}_d & \mathbf{W}' \\ 0 & 0 & -\mathbf{W} & 0 \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{\varphi}}_e \\ \dot{\boldsymbol{\eta}}_e \\ \dot{\mathbf{d}}_e \\ \dot{\mathbf{d}}_2 \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{\varphi} & \mathbf{K}_1 & \mathbf{K}_2 & 0 \\ \mathbf{K}_1' & \mathbf{K}_{\eta} & \mathbf{K}_{12} & 0 \\ \mathbf{K}_2' & \mathbf{K}_{12}' & \mathbf{K}_d & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varphi}_e \\ \boldsymbol{\eta}_e \\ \mathbf{d}_e \\ \mathbf{d}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{f}_{\varphi} \\ \mathbf{f}_{\eta} \\ \mathbf{f}_d \\ 0 \end{Bmatrix} = \mathbf{0} \quad (32)$$

Linear terms have been used in the matrix and nonlinear terms have been expressed within the external force term. Equation (32) can be abbreviated as follows.

$$\mathbf{R} \dot{\mathbf{q}} + \mathbf{K} \mathbf{q} + \mathbf{f} = \mathbf{0} \quad (33)$$

In equation (33), matrix $\mathbf{R}$ is a real skew-symmetric matrix and matrix $\mathbf{K}$ is a real symmetric matrix. These matrices include the generalized coordinate vectors $\boldsymbol{\eta}_e$ and $\mathbf{d}_e$.

A frequency equation of equation (33) has real eigenvalues, which is shown in Appendix.

3. Solving procedure for the discretized equations

3.1 Analzing method for the eigenfrequencies

A method for calculating the natural frequencies of the system from the initial shape is illustrated below. In the initial shape, when all generalized coordinates are zero, the nonlinear terms are zero and using only the nonzero components of equation (30), they are expressed with the following equation. A realistic approach for solving the equation for the discretized field (30) is illustrated.

$$\begin{bmatrix} 0 & R_1 & R_2 \\ -R_1^t & 0 & 0 \\ -R_2^t & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi}_e \\ \dot{\eta}_e \\ \dot{d}_e \end{Bmatrix} + \begin{bmatrix} K_\phi & 0 & 0 \\ 0 & K_\eta & K_{12} \\ 0 & K_{12}^t & K_d \end{bmatrix} \begin{Bmatrix} \phi_e \\ \eta_e \\ d_e \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ M_d \ddot{d}_e \end{Bmatrix} = 0 \quad (34)$$

In the equation, $\det K_\phi = 0$, which makes the following equation true.

$$K_\phi \phi_0 = 0, \quad \phi_0 = \{1, \dots, 1\} \quad (35)$$

Vector ϕ_0 signifies the uniform movement⁽¹⁰⁾ of the fluid field. The incidental conditional expression in the following equation is incorporated for the orthogonality of this vector and the unknown vector ϕ_e .

$$\lambda \phi_0' \bullet \phi_0' = 0 \quad \lambda : \text{Lagrange's multiplier} \quad (36)$$

Substituting equation (36) into equation (34), the following equation is derived.

$$\begin{bmatrix} 0 & 0 & R_1 & R_2 \\ 0 & 0 & 0 & 0 \\ -R_1^t & 0 & 0 & 0 \\ -R_2^t & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi}_e \\ \dot{\lambda} \\ \dot{\eta}_e \\ \dot{d}_e \end{Bmatrix} + \begin{bmatrix} K_\phi & \phi_0 & 0 & 0 \\ \phi_0' & 0 & 0 & 0 \\ 0 & 0 & K_\eta & K_{12} \\ 0 & 0 & K_{12}^t & K_d \end{bmatrix} \begin{Bmatrix} \phi_e \\ \lambda \\ \eta_e \\ d_e \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_d \ddot{d}_e \end{Bmatrix} = 0 \quad (37)$$

From equation (37), vector $\{\phi_e, \lambda\}$ is obtained.

$$\begin{Bmatrix} \phi_e \\ \lambda \end{Bmatrix} = - \begin{bmatrix} K_\phi & \phi_0 \\ \phi_0' & 0 \end{bmatrix}^{-1} \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\eta}_e \\ \dot{d}_e \end{Bmatrix} \quad (38)$$

Differentiating both sides of equation (38) with respect to time and substituting into equation (37), the following equation is derived.

$$[\bar{M}] \begin{Bmatrix} \ddot{\eta}_e \\ \ddot{d}_e \end{Bmatrix} + [\bar{K}] \begin{Bmatrix} \eta_e \\ d_e \end{Bmatrix} = 0 \quad (39)$$

$$\text{Here, } [\bar{K}] = \begin{bmatrix} K_\eta & K_{12} \\ K_{12}^t & K_d \end{bmatrix}, \quad [\bar{M}] = \begin{bmatrix} 0 & 0 \\ 0 & M_d \end{bmatrix} + \begin{bmatrix} R_1^t & 0 \\ R_2^t & 0 \end{bmatrix} \begin{bmatrix} K_\phi & \phi_0 \\ \phi_0' & 0 \end{bmatrix}^{-1} \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}$$

From equation (39), the generalized eigenvalue problem for the real symmetric matrix can be derived with the following equation.

$$\bar{K} \phi = \omega^2 \bar{M} \phi \quad (40)$$

Solving this eigenvalue problem and numbering the natural frequencies ω^2 starting from the smallest, then the corresponding eigenvector ϕ_k is normalized as per the following equation.

$$\bar{K} \phi_k = \omega_k^2 \bar{M} \phi_k, \quad \phi_k \bullet \phi_k = 1 \quad (41)$$

where an eigenvector ϕ_k is a natural vibration mode.

3.2 Damping terms

Assuming Rayleigh damping for the damping of the elastic container, a damping term that is the minimum critical damping ratio h_c for a vibration frequency ω_c is added in the third row of equation (32) as the term in the following equation.

$$R_d = h_c (M \omega_c + K_d / \omega_c) \quad (42.1)$$

Assuming that damping matrix of the wave of the fluid is proportional to the matrix K_η and using the critical damping ratio h_η for the vibration frequency ω_η , the following term can be added to the second row of equation (32).

$$R_\eta = 2(h_\eta / \omega_\eta) K_\eta \quad (42.2)$$

Description of this damping will be appended as postscript in the section about the response analysis.

3.3 Time integration

The choice of time integration method is an important decision to make. The author adopts the Crank–Nicolson method,^(10, 17) which is widely used in fluid analysis as it is easily applied to this problem and has good stability. Various quantities for the n -step are obtained and by applying this method to equation (33), the equation for the $(n+1)$ -step is expressed with the following equation.

$$(R_{n+1} + R_n) \frac{q_{n+1} - q_n}{\Delta t} + (K_{n+1} q_{n+1} + K_n q_n + f_{n+1} + f_n) = 0 \quad (43)$$

The above equation establishes the velocity between steps using various quantities at the step center. This is the dynamic interaction problem for large deformation of the fluid and the elastic container and it is predicted to be highly nonlinear. A convergence operation is performed in each step using the Newton–Raphson method. If q_{n+1}^γ is made the approximation for the γ th time of the $(n+1)$ -step and Δq_{n+1} the incremental for the next step, the incremental equation is formally expressed as follows.

$$\begin{aligned} & \left[\frac{(R_{n+1} + R_n)}{\Delta t} + \frac{\partial R_{n+1}}{\partial q'} \frac{q_{n+1}^\gamma - q_n}{\Delta t} + K_{n+1} + \frac{\partial K_{n+1}}{\partial q'} q_{n+1}^\gamma + \frac{\partial f_{n+1}}{\partial q'} \right] \Delta q_{n+1} \\ & = - \left\{ (R_{n+1} + R_n) \frac{q_{n+1}^\gamma - q_n}{\Delta t} + (K_{n+1} q_{n+1}^\gamma + K_n q_n + f_{n+1} + f_n) \right\} \end{aligned} \quad (44)$$

Using equation (44), the following response is gained.

$$q_{n+1}^{v+1} = q_{n+1}^v + \Delta q_{n+1}, \quad \dot{q}_{n+1} = (q_{n+1}^{v+1} - q_n) / \Delta t \quad (44.a)$$

The $(n+1)$ -step solution is found by repeating the convergence operation several times.

4. Numerical analysis

4.1.1 Numerical analysis model

The two-dimensional fluid domain has a rectangular shape⁽¹⁰⁾ of height 0.6 m and width 0.9 m, and is contained by the bottom and side walls of the container. The side walls on the left and right are composed of steel element of identical thickness (height 0.99 m and Young's modulus $E=205.8$ GPa). The mass densities of the steel and fluid are $\rho_E = 7850 \text{ kg/m}^3$ and $\rho_L = 1000 \text{ kg/m}^3$, respectively. In the finite element method, eight elements were used in the height direction, 16 elements were used in the horizontal direction, and 12 elements were used for the container as a two-dimensional linear element (Fig. 2). The linear element is fixed at the rigid base.

4.1.2 Organizing the response data

The behavior can be represented using the response vector q_n from the time history of the response analysis. This response also uses the Fourier spectrum and the vibration frequency components of the response, which are obtained with a finite Fourier transform. Moreover, using each natural vibration mode, this response was resolved into natural vibration modes as in the following equation and then was shown in diagrams.

$$q_n = \zeta_1 \phi_1 + \dots + \zeta_m \phi_m \quad (45)$$

The generalized coordinates in the above equation $\zeta_k (k=1, \dots, m)$ are the components of the response spanned in a space formed by natural vibration modes.

Because the analysis model possesses a bilateral symmetrical shape, the natural vibration modes are separated into an antisymmetric mode,

Fig.2 Model Mesh for Analyses

Table1 Natural Frequencies(8x16M)

記号	5mm	10mm	30mm	Rigid
ω_{a1}	5.40	5.74	5.78	5.78
ω_{s1}	7.05	7.18	7.19	7.20
ω_{s2}	7.96	8.32	8.35	8.36
ω_{a2}	9.92	10.30	10.34	10.35
ω_{s3}	11.66	12.09	12.13	12.14
ω_{a3}	13.28	13.79	13.83	13.85
ω_{s4}	14.86	15.48	15.53	15.55
ω_{a4}	16.39	17.20	17.25	17.28
ω_{s5}	17.97	18.99	19.05	19.07
ω_{a5}	19.42	20.85	20.93	20.95
ω_{s6}	21.07	22.79	22.88	22.92
ω_{a6}	22.55	24.78	24.90	24.93
ω_{s7}	24.39	26.75	26.89	26.93
ω_{a7}	26.08	28.57	28.75	28.79
ω_{s8}	28.04	30.10	30.31	30.35
ω_{a8}	29.69	31.14	31.35	31.39
ω_{s9}	31.14	31.63	31.72	31.77
ω_{a9}	33.19	49.70	154.3	—
ω_{s10}	33.52	50.92	155.8	—
only Container				
ω_{oc1}	26.48	52.96	158.9	—

or alternatively, a symmetric mode with respect to the axis of the bisector of the container. Each mode is represented by the suffix “a” or “s.” In other words, the minimum natural frequencies of the antisymmetric vibration mode or symmetric vibration mode are represented as ω_{a1} or ω_{s1} , respectively. The natural vibration modes corresponding to these are represented by ϕ_{a1} or ϕ_{s1} , which are described as the A1 mode and the S1 mode. Depending on the size of the natural frequencies, they are numbered to satisfy $\omega_{a1} \leq \omega_{a2} \leq \dots$ and $\omega_{s1} \leq \omega_{s2} \leq \dots$.

4.2 Natural frequencies

The natural frequencies obtained when adopting three different kinds of the container thicknesses (5 mm, 10 mm, 30 mm) and the natural frequencies for a rigid container are shown in Table 1. The minimum natural frequencies just for containers with no water are shown in the same table, where the left and right sides of the containers have identical values. The natural frequencies for the antisymmetric vibration modes and the symmetric vibration modes have been annotated with symbols

from the smallest up to the 9th or 10th modes. The natural frequencies for the 30 mm thick container and the rigid container were basically identical and the difference was around 0.15% at the most.

In the model of the 30 mm and 10 mm thick containers, the vibration modes that induced major container displacement corresponded to the natural frequencies ω_{a9} and ω_{s10} . In response to the mass effect of the fluid, the natural frequencies were smaller by approximately 3% and 6% than the values for only the container. On the other hand, when the container thickness was 5 mm, the interaction between the fluid and container had considerable effects on all the natural frequencies shown in the table. The influence is small for the natural frequencies of sloshing modes. But it is hard to detect the natural frequency of “bulging mode”. This has an important effect on the lowest natural frequency corresponding to symmetric and antisymmetric vibration modes of the 5mm thick container.

4.3 Basic response of the container

4.3.1 Response of containers with high rigidity

For the external force frequency region that caused the main resonance region of the water, the response of the 30 mm thick, and a rigid container when subjected to a horizontal external vibration force was analyzed and investigated. The sine wave external force frequency was set to 5.5 rad/s and the force acceleration amplitude was set to $a_x = 0.12 \text{ m/s}^2$. In addition, the response of the 30 mm thick, and a rigid container (Fig. 2) was analyzed using a time interval of $\Delta t = 0.01 \text{ s}$. The response time history of up to 90 s for the wave height at the end of the free surface is shown in Fig. 3. Because the sine wave frequency is near the minimum natural frequency of the system, the initial response increases with time. Its transient response can be observed as a process in which the response converges.

The finite Fourier coefficient (hereafter expressed as the Fourier spectrum) obtained using 1024 response datapoints for a time interval of $\Delta t = 0.01 \text{ sec}$ over a time history of $t = 80\text{--}90 \text{ s}$ is shown in Fig. 4. The dominant Fourier spectra for elastic and rigid containers at a external frequency of 5.59 rad/s, which is near the frequency of main resonance, were calculated as 52 mm and 48 mm, respectively. The response spectrum for the rigid container is simply the fundamental vibration component and that for the 30 mm thick container is roughly similar; however, a second harmonic vibration spectrum of 2.8 mm can be observed at a frequency of 11.04 rad/s. The difference between the two containers appears to be small for the steady-state response.

Next, the response time history of the wave height at the free surface edge of the 5 mm thick container and that for the 30 mm thick container, which are subjected to an identical external force, is shown in Fig. 5. The Fourier spectrum of both, derived from the response over identical periods, is shown in Fig. 6. The wave height response at the free surface edge for the 5 mm thick container was observed as the fundamental vibration component of 109.5 mm, a second harmonic vibration component of 19.4 mm, a third harmonic vibration

component of 1.7 mm, a constant term of 16.7 mm, and so on.

A disparity in the responses of the antisymmetric and symmetric modes can be seen for the 5 mm thick container. Resolving the response to these modes of oscillation, the Fourier spectra corresponding to the antisymmetric vibration modes A3, A4, and A6 and the symmetric vibration modes S3, S6, and S8 are shown in Figs. 7 and 8, respectively. Here, the symbols A3 and S3 in the Figures correspond to the modes for the natural frequencies ω_{a3} and ω_{s3} , respectively.

Apart from the fundamental vibration component, the response spectrum for the antisymmetric vibration mode has a third harmonic vibration component and fifth and seventh odd-order harmonic vibration components. On the other hand, the symmetric vibration mode has second, fourth, and sixth even-ordered harmonic vibration components and a constant term component. The constant term in the symmetric vibration mode response indicates that a drift occurs with vibration, or in other words, the average value of the vibration is not the horizontal plane. These response spectra do not seem to change as long as they do not branch out into subharmonic vibrations that would

be described in a following paper. The displacement and bending moment for the container were examined when the interaction between the fluid and elastic container was considered.

For the 5 mm and 30 mm thick containers, the Fourier spectra of the horizontal displacement response at the free surface level before deformation and those of the difference bending moment response from initial static state at the fixed edge of the container base are shown in Figs. 9 and 10, respectively. The horizontal displacement spectrum for the 30 mm thick container is shown in Figures 9. The horizontal displacement spectrum for the 5 mm thick container is 4.1 mm in the vicinity of the fundamental vibration and 0.3 mm in the vicinity of the second harmonic vibration. The spectrum at this vibration frequency for the 30 mm thick container is 0.022 mm.

In the difference bending moment response spectrum, although 97.3 kNm/m for the 30 mm thick container stands out corresponding to the fundamental vibration component, a few amount of second harmonic vibration frequency components are also observed. This is the same for the 5 mm container and apart from the fundamental vibration

Fig.3 Time History Responses of Wave at End sec

Fig.4 Response Spectra of EndWave rad/s

Fig.7 Response Spectra of AntiSym. Modes rad/s

Fig.5 Time History Responses of Wave at End s

Fig.6 Response Spectra of EndWave rad/s

Fig.8 Response Spectra of Sym. Modes rad/s

component of 89.3 kNm/m, second, third, fourth, fifth, and sixth harmonic vibration components are seen. The fundamental vibration spectrum for the 5 mm thick container is roughly 10% smaller than that for the container with the 30 mm thick container.

5. Conclusion

1. Using the functional for the dynamic interaction problem for the large-deformation field for the potential fluid and elastic container, a method was demonstrated to formulate the two-dimensional problem.
2. The response of the rigid container that was subject to sine wave horizontal ground motion and the response of the fluid contained in the elastic container was analyzed. The response of the wave height was shown in the first-order resonance region of the sloshing motion in these systems as well as the Fourier spectrum for the stress of the elastic container.
3. The solving method formulated here enables numerical analysis, and then, computation of the nonlinear vibrational response in the dynamic interaction problem of a potential fluid and its elastic container. The method is applicable to not only periodic force but also an arbitrary seismic force.

Then, nonlinear vibration responses and a dynamic instability of the container would be examined in a following paper, which is one of the main purpose to develop the solving procedure.

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Appendix A Frequency Equation of Expression (33)

A homogeneous equation of expression (33) is given by

$$R\dot{q} + Kq = 0 \quad (\text{a.1})$$

Ignoring the nonlinear terms and damping terms, and assuming a periodic solution $q = q_0 \text{Exp}(-i\omega t)$, where ω is natural frequency and i is imaginary unit, the following an eigenvalue problem is derived.

$$\omega[iR]q_0 = [K]q_0 \quad (\text{a.2})$$

Matrices $[iR]$ and $[K]$ are Hermitian matrices. Then, eigenvalues of the expression (a.2) are all real, which means the coupled systems also have real number natural frequencies.

和文要約

1. はじめに

Euler の圧力式を変形後の流体領域で積分した式が、剛な容器に入ったポテンシャル流体の厳密な Lagrangian となることを J.C.Luke¹⁾ は示した。この式を容器の弾性変形をも含む変形後の流体領域で積分して Hamilton 関数に取り込むと、この問題の大変形場における厳密な汎関数となること^{6, 7, 9)}を示した。この汎関数へ遷移座標を導入し、二次元問題の定式化、および解析を行う。

2. ポテンシャル流体と容器の動的相互作用の汎関数と離散化

2.1 汎関数

ポテンシャル流体を入れた容器が載っている基盤が水平方向にある速度で移動する系をモデルとする。基盤に立つ観測者が観察する流体に速度ポテンシャルを導入する。流体運動を測定する遷移座標系の移動速度 $\hat{y}$ を導入した二次元の汎関数を (1) 式に示す。この式の第一変分を (2) 式に示す。これから得られる基礎式、および境界条件を (3) 式-(8) 式に示す。これらの式が表す物理的な意味を記述する。

この問題の定式化を行うために、流体と容器の大変形時の運動学的な条件を満足する場を構成する必要がある。

2.2 流体領域に対する動く境界条件

流体の自由表面 S_f および流体と容器の相互作用面 S_i における運動を記述する表示方法を示す。流体節点は 5 種類に分類される。流体節点は流体の運動を計測する遷移座標系を構成し、流体の移動とは独立した扱いを導入する。流体自由表面と相互作用面の二つの境界に載る流体節点を 2 移動境界上の節点とし、変形後のこの節点位置は容器座標系を利用して特定される。斜交座標系を流体領域へ導入し、流体内部の節点移動は流体初期形状を用いて、各軸の両端の境界での移動量を内分して定める。

2.3 汎関数の離散化

有限要素法を利用し、変形後の離散化された汎関数を定義する。この汎関数は流体ポテンシャルベクトル、波高ベクトル、および容器の変位ベクトルを変関数とする。第一変分を誘導し、場の方程式 (33) を得る。

3. 離散化された場の方程式の解法

流体の速度ポテンシャルは一樣膨張の有意でない解を持つ。これを避けて、ポテンシャル流体と弾性容器が連成する系の固有振動数を解析する方法を示す。

3.1 固有振動数の解析

(33) 式の非線形項を無視し定常解を仮定すれば、Hermite 行列を持つ系の固有値問題が得られる。有意な固有振動数は一樣膨張の解と直交するモードを有するとして、(39) 式の固有値問題を誘導する。これを解析して固有振動数の数値解析を得る。

3.2 減衰項の導入

容器の振動に対してはレーリー減衰を導入し、流体は波高の剛性行列に減衰を入れる。流体、および容器それぞれ単独の最小固有振動数に対して臨界減衰比を導入する。

3.3 時刻積分

クランク・ニコルソン法を適用して、離散化された場の方程式を解析する方法を示す。各ステップにおいて、ニュートン・ラフソン法を

用いた収束演算法も行なう。

4. 数値解析

4.1.1 数値解析モデル

数値解析モデルを Fig.2 に示す。水高 0.6m、幅 0.9m、および容器高さ 0.99m を有する。この流体領域を 8x16 の流体要素メッシュにモデル化した。容器は板厚 5mm、10mm、および 30mm の直線材を 6 要素に分割し、両側を合わせて計 12 要素を利用した。応答計算において、容器の最小固有振動数に対して臨界減衰比 0.0075 の減衰を導入する。

4.1.2 応答データの処理

時刻応答、応答の有限フーリエ級数スペクトルを定義する。

4.2 固有振動数の数値解析

容器の剛性 (板厚) をパラメータとした系の固有振動数を Table.1 に示す。

板厚 30mm の容器と剛な容器のスロッシング振動の固有振動数はほぼ同一である。板厚 10mm のスロッシング振動の固有振動数は僅かに小さくなるものの、ほぼ板厚 30mm のものと一致する。板厚 30mm、および 10mm モデルの容器の最小固有振動数は容器のみの振動数と比較してそれぞれ 3% および 9% 低下する。

一方、板厚 5mm の容器の固有振動数はスロッシング振動と (容器の振動が主となる) パルジング振動の区別は困難となる。すべての固有振動数に容器剛性の影響が及ぶ。流体の剛性はこの容器の固有振動数にかなりの影響を与える。

4.3 流体を有する弾性容器の振動応答

容器の剛性 (板厚) をパラメータとした系の応答を解析し、剛な容器に入った流体の応答と比較する。また、弾性容器の時刻歴応答、および応答スペクトルを示す。

二つの容器 (板厚 30mm、および剛な容器) に入ったモデルを解析する。外力は振動数 5.5rad/s、振幅 0.12m/s² の正弦波である。時間刻み $\Delta t = 0.01$ sec を用いて得られた時刻歴応答 (0-90sec) を Fig.3 に示す。両者の 89-90sec の 1024 個のデータから算出される自由表面端部の波高の応答スペクトルを Fig.4 に示す。

次に、板厚 10mm のモデル容器の応答を上記と同一の条件で解析し、時刻歴応答を板厚 30mm モデルの応答と共に Fig.5 に示す。両者の自由表面端部の波高の応答スペクトルを Fig.6 に示す。板厚 10mm のモデル容器の応答に高調波振動成分が出現する。波高の応答を容器の初期形状の 2 等分線に関する逆対称振動モードと対称振動モードに分解して応答を表現した。逆対称振動モードは奇数次の高調波振動応答を含む (Fig.7) 対称振動モードは偶数次の高調波振動成分を含む (Fig.8)。

容器の自由表面位置の水平変位応答のスペクトル、および容器の基盤位置 (境界条件は固定) の差分モーメント応答を Fig.9、および Fig.10 に示す。この応答に高調波振動成分が観察される。

5. 結論

水平地動を受ける二次元のポテンシャル流体と容器の連成系の動的問題を解析する手法を示し、この連成系の数値解析を行い、系の時刻歴応答、応答の振動成分を示した。

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