Several formulations are known for robust compliance optimization of structures subjected to uncertain static external loads. Aiming at providing clear perspective of the problem, this paper establishes connection between three formulations: a semidefinite programming formulation, a formulation as minimization of the maximum eigenvalue of a symmetric matrix, and a formulation using a generalized eigenvalue problem. Equivalence of these formulations is shown by using a fundamental property of the Schur complement of a symmetric positive semidefinite matrix. A series of numerical examples is presented to show that an optimal solution of the robust compliance optimization problem can possibly have eigenvalues of large multiplicity.

**Keywords**: robustness, uncertainty, robust optimization, eigenvalue optimization, multiple eigenvalue, semidefinite programming

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**1 Introduction**

Real-world structures inevitably encounter uncertainties stemming from limitation of knowledge of input disturbance. Robust structural optimization against uncertainty has in turn drawn large attention. This paper discusses three different but closely-related formulations of robust compliance optimization under uncertain external loads and establishes mathematical connection between these formulations. A possibilistic (or bounded-but-unknown) model rather than a probabilistic model is employed to represent the uncertainty and the compliance optimization problem is then treated within the framework of robust optimization. Namely, given the set of static external loads, we attempt to minimize the maximal compliance that corresponds to the worst loading.

Attention of this paper is focused on the following three formulations of robust compliance optimization:

(i) A semidefinite programming (SDP) formulation due to Ben-Tal and Nemirovski.

(ii) An eigenvalue optimization formulation using a standard eigenvalue problem due to Takezawa et al.

(iii) An eigenvalue optimization formulation using a generalized eigenvalue problem due to Cherkaev and Cherkaev.

This paper attempts to give clearer perspective to connections between these three formulations. Specifically, provided that the stiffness matrix is nonsingular, it is shown how one of these formulation can be reduced to the others. Although equivalence of these formulations is mathematically quite trivial, it is not necessarily recognized well in the engineering community.

Formulations (i)–(iii) involve the minimum or maximum eigenvalue of a certain matrix in their constraints or objective function. It is often that an optimal solution of such an optimization problem has multiple eigenvalues. A set of instances can be constructed so that the multiplicity actually increases as the instance size increases. Since a multiple eigenvalue is not differentiable, conventional gradient-based optimization algorithms may possibly fail to solve formulations (ii) and (iii). An advantage of the SDP approach might be that it can be solved efficiently with a primal-dual interior-point method and its computational efficiency is usually independent of multiplicity of eigenvalues of a variable matrix. On the other hand, the robust optimization problem of interest can be recast as an SDP problem only in limited cases, including robust truss topology optimization in which the stiffness matrix depends linearly on the design variables. This is not the case with, e.g., continuum-based (robust) topology optimization. Then the problem is categorized...
as a nonlinear SDP problem; see, for nonlinear SDP, Jarre \cite{Jarre2014} and the references therein.

The paper is organized as follows. Section 2 states a well-known property of the Schur complement of a symmetric matrix. Section 3 defines a robust compliance optimization problem and recalls formulation (i). Section 4 establishes relation between formulations (i) and (ii), while section 5 establishes relation between formulations (i) and (iii). Section 6 presents problem instances that have optimal solutions with repeated eigenvalues. Conclusions are drawn in section 7.

A few words regarding our notation: For set $C \subseteq \mathbb{R}^n$, we use $int C$ and $bd C$ to denote the interior and boundary of $C$, respectively. We use $I_n$ and $O_{n,n}$ to denote $n \times n$ identity matrix and $m \times n$ zero matrix, respectively. We omit subscripts if the matrix size is clear from the context. We use $\mathbb{R}^n$ to denote the set of $n \times n$ real symmetric matrices. For $X, Y \in \mathbb{R}^n$, notation $X \succeq Y$ means that matrix $X - Y$ is positive semidefinite. Particularly, notation $X \succeq O_{n,n}$ denotes that $X$ is positive semidefinite. We write $X > O_{n,n}$ if $X$ is positive definite.

2 Lemma on Schur complement

The following fundamental property of the Schur complement, which can be found in, e.g., Boyd et al. \cite{Boyd2004} and Horn and Johnson \cite{Horn1985}, plays a key role in this paper.

**Lemma 2.1.** Suppose that symmetric matrix $Z \in \mathbb{R}^{n+m}$ is partitioned as

$$Z = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix},$$

where $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^m$ are square blocks and $C \in \mathbb{R}^{n \times m}$. Assume $A > O$. Then $Z \succeq O$ if and only if $B - C^T A^{-1} C \succeq O$.

**Proof.** Since $A$ is assumed to be positive definite, it is nonsingular. Consider the following congruence transformation of $Z$:

$$\begin{bmatrix} I_n & O_{n,m} \\ -C^T A^{-1} & I_m \end{bmatrix} \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \begin{bmatrix} I_n & -A^{-1} C \\ O_{n,m} & I_m \end{bmatrix} = \begin{bmatrix} A & O_{m,n} \\ O_{n,m} & B - C^T A^{-1} C \end{bmatrix}.$$

Since the obtained matrix is a block-diagonal matrix, it is positive semidefinite if and only if $B - C^T A^{-1} C \succeq O$. On the other hand, from the property of congruence transformation, the obtained matrix is positive semidefinite if and only if $Z \succeq O$. \hfill \square

Matrix $B - C^T A^{-1} C$ in Lemma 2.1 is called the Schur complement of $A$ in $Z$.

3 Robust compliance optimization and its semidefinite programming formulation

This section recalls a robust compliance optimization problem. Section 3.1 defines an uncertainty model of external loads. Section 3.2 defines the robust optimization problem and section 3.3 recalls its SDP formulation due to Ben-Tal and Nemirovski \cite{Ben-Tal1998}.

3.1 Uncertainty in external load

Consider a finitely discretized linear elastic structure. Let $f \in \mathbb{R}^n$ denote the external load vector, where $n$ is the number of degrees of freedom of displacements. We use $x \in \mathbb{R}^m$ to denote the vector of design variables. In the case of truss optimization, for instance, suppose that a truss with $m$ candidate members is given according to a conventional ground structure approach. Then $x_t$ denotes the cross-sectional area of member $i$.

Throughout the paper we assume that only $f$ cannot be known precisely. Suppose that $f$ can possibly take any value in the following compact convex set:

$$F = \{Qe \mid ||e|| \leq 1\}. \tag{1}$$

Here, $e \in \mathbb{R}^n$ is the vector of unknown parameters, $l$ is the number of the unknown parameters, $||e||$ is the Euclidean norm of $e$, and $Q \in \mathbb{R}^{n \times l}$ is a constant matrix satisfying $\text{rank} \, Q = l$.

3.2 Robust compliance optimization

Let $u \in \mathbb{R}^n$ and $K(x) \in \mathbb{R}^n$ denote the displacement vector and the stiffness matrix, respectively. For given external load $f$, the compliance, denoted $\pi(x; f)$, is defined by

$$\pi(x; f) = \sup \{2f^T u - u^T K(x) u \mid u \in \mathbb{R}^n\} \tag{2}$$

that is, $\pi(x; f)$ is the least upper bound of $2f^T u - u^T K(x) u$.

We use $c_i$ to denote the structural volume for unit value of $x_i$ and write $e = (c_1, \ldots, c_m)^T \in \mathbb{R}^m$. For instance, $c_i$ is the undeformed length of member $i$ of a truss. The conventional compliance optimization problem, against fixed load $f$, is formulated as

$$\begin{aligned}
\min_x & \pi(x; f) \tag{3a} \\
\text{s.t.} \quad c^T x &\leq v; \tag{3b} \\
x &\geq 0; \tag{3c}
\end{aligned}$$

where $v > 0$ is the specified upper bound for the structural volume.

As a robust counterpart of problem (3), we attempt to minimize the compliance in the worst case among all values of $f$ included in the uncertainty set, $F$. This robust optimization problem is formulated as

$$\begin{aligned}
\min_x & \sup \{\pi(x; f) \mid f \in F\} \tag{4a} \\
\text{s.t.} \quad c^T x &\leq v; \tag{4b} \\
x &\geq 0. \tag{4c}
\end{aligned}$$

Precisely speaking, the model of uncertainty adopted by Cherkaev and Cherkaev \cite{Cherkaev1997, Cherkaev1998} and Takezawa et al. \cite{Takezawa2012} is slightly different from $F$ in (1). There, external load $f$ is supposed to satisfy

$$f \in \text{bd} \, F = \{Qe \mid ||e|| = 1\}.$$

Nonetheless, the formulations are also true when $f \in F$ is supposed. This validity follows from the next proposition.

**Proposition 3.1.** The compliance satisfies

$$\sup \{\pi(x; f) \mid f \in \text{bd} \, F\} = \sup \{\pi(x; f) \mid f \in F\}.$$
Proof. The assertion can be obtained by showing the following inequality:
\[
\sup\{\pi(x; f) \mid f \in \text{bd} \mathcal{F}\} \geq \sup\{\pi(x; f) \mid f \in \text{int} \mathcal{F}\}.
\]
From definition (2), the compliance satisfies \(\pi(x; f) \geq 0\) for any \(f \in \mathbb{R}^n\), because with \(\tilde{u} = 0\) we have that \(2f^\top \tilde{u} - \tilde{u}^\top K(x) \tilde{u} = 0\).

Let \(f \in \text{int} \mathcal{F}\). Since \(F\) is a closed ball centered at the origin, there exists real number \(\gamma > 1\) satisfying \(\gamma f \in \text{bd} \mathcal{F}\). By putting \(f = \gamma f\) in (2) and using \(\pi(x; f) \geq 0\), we obtain
\[
\pi(x; \gamma f) = \gamma^2 \sup\{2\gamma f^\top u - u^\top K(x) u \mid u \in \mathbb{R}^n\} = \gamma^2 \gamma \sup\{2f^\top (u/\gamma) - (u/\gamma)^\top K(x)(u/\gamma) \mid u \in \mathbb{R}^n\} = \gamma^2 \pi(x; f) \geq \pi(x; f).
\]
This inequality implies (5).

3.3 Semidefinite programming formulation
This section recalls the SDP reformulation of problem (4).

Ben-Tal and Nemirovski\textsuperscript{2)} showed that \(w\) and \(x\) satisfy
\[
w \geq \sup\{\pi(x; f) \mid f \in \mathcal{F}\}
\]
if and only if they satisfy
\[
\begin{bmatrix}
K(x) & Q \\
Q^\top & wh_l
\end{bmatrix} \succeq O.
\]
(7)
The main points of the proof are repeated in Appendix for the reader’s convenience. From this fact it follows that problem (4) can be rewritten equivalently as
\[
\begin{aligned}
\min_{x, w} & \quad w \\
\text{s.t.} & \quad \begin{bmatrix}
K(x) & Q \\
Q^\top & wh_l
\end{bmatrix} \succeq O, \\
& \quad c^\top x \leq v, \\
& \quad x \geq 0.
\end{aligned}
\]
(8)
In particular, the stiffness matrix of a truss can be written as
\[
K(x) = \sum_{i=1}^m a_i K_i,
\]
(9)
where \(K_1, \ldots, K_m \in \mathbb{S}^n\) are constant matrices. In this case, constraint (8b) is a linear matrix inequality in terms of \(x\) and \(w\), and hence problem (8) is an SDP problem.

4 Formulation as eigenvalue optimization
Takezawa et al.\textsuperscript{18)} showed that, when the stiffness matrix, \(K(x)\), is nonsingular at given \(x\), the worst-case compliance is equal to the maximum eigenvalue of positive definite matrix \(Q^\top K(x)^{-1}Q \in \mathbb{S}^l\). Therefore, problem (4) is reduced to a minimization problem of the maximum eigenvalue of this matrix. To guarantee that \(K(x)\) is nonsingular during the optimization process, one should introduce a small positive lower bound, denoted \(\varepsilon\), for \(x_1, \ldots, x_m\). With this modification, problem (4) is rewritten as
\[
\begin{aligned}
\min_{x, w} & \quad \mu_{\max}(QQ^\top, K(x)) \\
\text{s.t.} & \quad c^\top x \leq v, \\
& \quad x_i \geq \varepsilon, \quad i = 1, \ldots, m.
\end{aligned}
\]
(13a)
Note that small constant \(\varepsilon > 0\) in (13c) is used to ensure that \(K(x)\) is positive definite.

Problem (13) can be converted to problem (8) as follows. Let \(w \in \mathbb{R}\) be an upper bound for the maximum eigenvalue, i.e.,
\[
w \geq \mu_{\max}(QQ^\top, K(x)) = \frac{\lambda_{\max}(QQ^\top, K(x))}{\lambda_{\max}(K(x))},
\]
(13b)
where the equality follows from the fundamental property of the Rayleigh–Ritz ratio\textsuperscript{13}). Condition (13c) is equivalent to
\[
w \geq \frac{\varepsilon}{\lambda_{\max}(K(x))} (\forall z \neq 0),
\]
(13c)
which can further be rewritten as
\[ \mathbf{z}^T(w\mathbf{K}(\mathbf{x}) - \mathbf{QQ}^T)\mathbf{z} \geq 0 \quad (\forall \mathbf{z} \in \mathbb{R}^n). \]

As a consequence, we see that (14) is equivalent to
\[ w\mathbf{K}(\mathbf{x}) - \mathbf{QQ}^T \succeq \mathbf{O}. \tag{15} \]

Since \( \mathbf{QQ}^T \neq \mathbf{O} \) is positive semidefinite and \( \mathbf{K}(\mathbf{x}) \) is assumed to be positive definite, (15) implies \( w > 0 \). Therefore, (15) can be rewritten as
\[ \mathbf{K}(\mathbf{x}) - (1/w)\mathbf{QQ}^T = \mathbf{K}(\mathbf{x}) - \mathbf{Q}^{-1}\mathbf{Q}^T \succeq \mathbf{O}. \tag{16} \]

Application of the property of the Schur complement in Lemma 2.1 shows that (16) is equivalent to
\[ \begin{bmatrix} \mathbf{K}(\mathbf{x}) & \mathbf{Q} \\ \mathbf{Q}^T & w\mathbf{I} \end{bmatrix} \succeq \mathbf{O}, \]

which is (8b). Thus problem (13) is essentially equivalent to problem (8), as far as \( \mathbf{K}(\mathbf{x}) \) is nonsingular.

6 On multiplicity of eigenvalues

The two formulations studied in section 4 and section 5 are stated in forms of minimization of the maximum eigenvalue. For such an optimization problem, it is often that the optimal solution has a multiple eigenvalue. Since a multiple eigenvalue is not differentiable, the directional derivatives or the generalized gradient are usually computed in sensitivity analysis. As mentioned in section 1, sensitivity analysis of eigenvalues is not required when we solve SDP problem (8) with a primal-dual interior-point method.

Takezawa et al.\(^{18}\) and Brittain et al.\(^{51}\) solved problem (10) and problem (13), respectively, numerically and reported that the obtained optimal solutions have simple maximum eigenvalues. In contrast, Herkovits et al.\(^{113}\) found an optimal solution with fivefold maximum eigenvalue by using their algorithm for nonsmooth convex optimization. This section shows that, for truss structures, a series of simple problem instances can be constructed so that multiplicity of the optimal solution increases as the problem size increases.

In the following numerical experiments, the SDP formulation in (8) is solved to find robust optimal solutions. Computation was carried out on a 1.9 GHz Intel Core i7 processor with 8 GB RAM. SDP problems were solved by SDPT3 ver. 4.0\(^{108}\) on MATLAB ver. 8.0.

Consider the ground structure shown in Figure 1, where only the nodes are depicted. The nodes are aligned on a \((p + 1) \times (2p + 1)\) square grid. Any two nodes are connected by a member, but overlapping of members is avoided by removing the longer member when two members overlap. The leftmost nodes are pin-supported. Figure 1 shows the case of \( p = 4 \). The elastic modulus is 200 GPa. As for uncertainty in the external load, we suppose that external forces, not greater than 10 kN and in any direction, can be applied at any of the nodes depicted as filled circles, while the rightmost center node is subjected to vertical nominal load 100 kN. More precisely, matrix \( \mathbf{Q} \) in (1) is supposed to be given by
\[ \mathbf{Q} = \begin{bmatrix} f + \alpha \mathbf{O}_{(p-1,p)} \\ \alpha \mathbf{O}_{(p-1,p)} \\ \mathbf{O}_{p,p} \end{bmatrix} \]

with \( f = 100 \) kN and \( \alpha = 10 \) kN, where, without loss of generality, the degrees of freedom of the nodes at which uncertain forces are applied are assumed to be 1, \ldots, \( p \) and that corresponding to the vertical nominal load is assumed to be 1. For instance, if \( p = 4 \), then \( l = 20 \), because there exist 10 nodes depicted as filled circles in Figure 1. The upper bound for the structural volume is given by \( v = 2p^2 \times 10^6 \) mm\(^3\). The constraint on the structural volume becomes active at the obtained optimal solutions.

Figure 2 collects the optimal solutions obtained by solving SDP on a 1.9 GHz Intel Core i7 processor with 8 GB RAM. SDP problems were solved by SDPT3 ver. 4.0\(^{108}\) on MATLAB ver. 8.0.

<table>
<thead>
<tr>
<th>( p )</th>
<th>Opt. val. (J)</th>
<th>Multiplicity</th>
<th>( n )</th>
<th>( m )</th>
<th>( l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>121.5000</td>
<td>3</td>
<td>20</td>
<td>74</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>122.2222</td>
<td>5</td>
<td>42</td>
<td>251</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>123.1250</td>
<td>7</td>
<td>72</td>
<td>632</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>124.2000</td>
<td>9</td>
<td>110</td>
<td>1361</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>125.4444</td>
<td>11</td>
<td>156</td>
<td>2542</td>
<td>42</td>
</tr>
<tr>
<td>7</td>
<td>126.8571</td>
<td>13</td>
<td>210</td>
<td>4441</td>
<td>56</td>
</tr>
<tr>
<td>8</td>
<td>128.4375</td>
<td>15</td>
<td>272</td>
<td>7180</td>
<td>72</td>
</tr>
</tbody>
</table>

Table 1: Computational results of robust optimization with \( \varepsilon = 0 \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \varepsilon ) (( \text{mm}^2 ))</th>
<th>( \lambda_1 ) (J)</th>
<th>( \lambda_{2p-1} ) (J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.1</td>
<td>123.7817</td>
<td>113.5207</td>
</tr>
<tr>
<td>4</td>
<td>0.01</td>
<td>123.1904</td>
<td>122.0912</td>
</tr>
<tr>
<td>4</td>
<td>0.001</td>
<td>123.1315</td>
<td>123.0208</td>
</tr>
<tr>
<td>5</td>
<td>0.1</td>
<td>125.3373</td>
<td>109.5745</td>
</tr>
<tr>
<td>5</td>
<td>0.01</td>
<td>124.3128</td>
<td>122.5604</td>
</tr>
<tr>
<td>5</td>
<td>0.001</td>
<td>124.2113</td>
<td>124.0340</td>
</tr>
<tr>
<td>6</td>
<td>0.1</td>
<td>127.2152</td>
<td>105.8352</td>
</tr>
<tr>
<td>6</td>
<td>0.01</td>
<td>125.6193</td>
<td>123.1554</td>
</tr>
<tr>
<td>6</td>
<td>0.001</td>
<td>125.4621</td>
<td>125.2016</td>
</tr>
</tbody>
</table>

Table 2: Eigenvalues of the optimal solutions with positive lower bounds for member cross-sectional areas.
positive definite, (15) implies $w$ as eigenvalue. Since a multiple eigenvalue is not differentiable, the di-

forms of minimization of the maximum eigenvalue. For such an op-

sis of eigenvalues is not required when we solve SDP problem (8) with

skovits et al. optimal solutions have simple maximum eigenvalues. In contrast, Her-

shows that (16) is equivalent to

$\mathbf{K}(\mathbf{x})^{-1}\mathbf{Q}$ at the optimal solution. More precisely, $K(\mathbf{x})$ is singular at the optimal solution, because some nodes in the ground structure vanish at the optimal solution. Hence, we constructed the stiffness matrix of the optimal solution only with respect to the remaining degrees of freedom of displacements and computed its inverse to evaluate multiplicity of the eigenvalue. Note that multiplicity of the maximum eigenvalue of formulation (13) is same as that of (10), be-

cause the eigenvalues of generalized eigenvalue problem (12) coincide with the eigenvalues of $\mathbf{Q}^T K(\mathbf{x})^{-1}\mathbf{Q}$. This multiplicity is related to non-uniqueness of the worst-case loads. Multiplicity of the minimum eigenvalue of the SDP formulation, i.e., multiplicity of zero eigenvalue of the matrix in (8b), is in general larger than the multiplicity listed in Table 1, because zero eigenvalue can stem not only from the non-

uniqueness of the worst-case loads but also from removal of nodes. It is observed in Table 1 that multiplicity of the maximum eigenvalue in-

tcreases two by two as $p$ increases.

In contrast, if the lower bound for the member cross-sectional areas, $\varepsilon$, is positive, then the obtained optimal solutions have simple max-

imum eigenvalues. The computational results are listed in Table 2. Here, $\lambda_k$ denotes the $k$th-largest eigenvalue, where $\lambda_1$ is equal to the optimal value. It is observed that the optimal value decreases as $\varepsilon$ de-

creases. Also, the optimal values in Table 2 are larger than those in Table 1. Furthermore, $\lambda_1 - \lambda_{2p-1}$ decreases as $\varepsilon$ decreases, although all of $\lambda_1, \ldots, \lambda_{2p-1}$ are distinct.

7 Summary and discussion

In this paper we have compared existing three formulations for robust compliance optimization. They can be readily reduced to each other by making use of a fundamental property of the Schur complement of a symmetric positive semidefinite matrix. This reduction, requiring knowledge of elementary linear algebra only, is almost trivial from a mathematical point of view but has not been recognized well in the engineering community.

When the stiffness matrix depends linearly on the design variables, the robust optimization problem is reduced to an SDP problem which can be solved efficiently with a primal-dual interior-point method even if an optimal solution has large multiplicity of minimum eigenval-

ues. With this approach this paper has presented a series of problem instances such that the multiplicity increases as the problem size in-

creases. If the small positive lower bound is given for the design vari-

ables, all the eigenvalues of these problem instances become distinct, although they distribute very closely.

Problem (10), due to Takezawa et al. 18), has an advantage that the size of the matrix, the maximum eigenvalue of which is to be mini-

mized, is small when uncertain external forces are supposed to be ap-

plied only at a small number of nodes. A potential disadvantage of this formulation is that the inverse of the stiffness matrix is required. Problem (13), due to Cherkaev and Cherkaev7,8), has subsequently been studied further in Brittain et al. 5).

In continuum-based topology optimization methods, e.g., the SIMP (simplified isotropic material with penalization) approach, the stiffness matrix depends nonlinearly on the design variables17). In this case the robust optimization problem is reduced to a nonlinear SDP problem. Numerical solution methods based upon this formulation remain to be explored.

As for the uncertainty model of the external load, in this paper we have restricted our attention to a homogeneous model. Some formul-
tions have been extended to a non-homogeneous model,
\[ f \in \{ \tilde{f} + Q x | \| e \| \leq 1 \}, \]
where \( \tilde{f} \in \mathbb{R}^n \) denote the nominal value, or the best estimate, of \( f \).
Extensions of SDP formulation (8) to this model are due to Calafiore and Dabbene \(^b\) (Proposition 2) and Ben-Tal \textit{et al.} \(^1\) (problem (8.2.15)). An alternative approach to the non-homogeneous model can be found in de Gournay \textit{et al.} \(^10\). There, a descent direction used in the algorithm is computed by solving a certain SDP problem.

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Appendix
Equivalence of (6) and (7) in section 3.3 was shown by Ben-Tal and Nemirovski \(^2\) (Lemma 2.2). Essentials of the proof are repeated here for the convenience of the reader.

It follows from (1) and (2) that the compliance in the worst case is written as
\[ \sup \{ \pi(x,f) | f \in F \} = \sup \{ 2(Qe)^T u - u^T K(x)u | u \in \mathbb{R}^n, \| e \| \leq 1 \}. \]
Hence, \( w \) and \( x \) satisfy (6) if and only if
\[ w - 2(Qe)^T u + u^T K(x)u \geq 0 \quad (\forall u \in \mathbb{R}^n, \forall e : \| e \| \leq 1) \]
holds. The left-hand side of (17) attains the minimum value when \( \| e \| = 1 \). Hence, (17) is equivalent to
\[ w(e^T e) - 2(Qe)^T u + u^T K(x)u \geq 0 \quad (\forall u \in \mathbb{R}^n, \forall e : \| e \| = 1), \]
which can be written in the quadratic form of a symmetric matrix as
\[
\begin{bmatrix}
    u^T & \bar{Q} \\
    \bar{Q}^T & wI
\end{bmatrix}
\begin{bmatrix}
    u \\
    -e
\end{bmatrix} \geq 0 \quad (\forall u \in \mathbb{R}^n, \forall e : \| e \| = 1).
\]
This is equivalent to (7).

References
和文要約

1 はじめに
静的な外力の不確実性を考慮した構造物のロバスト最適設計法には、多くの研究がある。この論文では、ロバスト性を考慮したコンプライアンス最適化問題について、既存の三つの定式化をとりあげ、その関係性を調べる。つまり、(i) 一般正定値計画としての定式化、(ii) 標準固有値問題の最大固有値の最小化問題としての定式化、(iii) 一般化固有値問題の最大固有値の最小化問題としての定式化の三つが、どのようにして等価な形に変形できるかを明らかにする。さらに、最適解で多くの固有値が重複するような例題が実際に存在することを明らかにする。

2 Schur の補元に関する補題
準備として、実対称半正定値行列の Schur の補元が一つ、よく知られる性質（Lemma 2.1）を紹介している。この性質は、4 節および 5 節の結果を導くために重要な役割を果たす。

3 ロバストコンプライアンス最適化とその半正定値計画による定式化
外力の不確実性の下でコンプライアンスを最小化するロバスト構造最適化問題の定義を述べる。この問題は、外力の存在可能範囲がコンパクトな凸集合として与えられたときに、コンプライアンスの最小値（つまり、最大値）を最小化する設計解を求めることである。次に、Ben-Tal and Nemirovski による、この問題の半正定値計画問題としての定式化（問題 (8)) を確認する。

4 固有値最適化としての定式化
Takezawa et al. は、ロバスト最適設計問題に、ある対称行列の最大固有値問題の最大固有値の最小化問題として定式化できることを示した。ここでは、Lemma 2.1 を用いることで、この問題が半正定値計画問題 (8) に帰着できることを明らかにする。つまり、最適化の過程で剛性行列が正定値であるという条件の下では、二つの定式化は本質的に等値である。

5 一般化固有値問題を用いた定式化
ここでは、Cherkaev and Cherkaev による一般化固有値問題の最大固有値の最小化問題としての定式化をとりあげる。そして、Lemma 2.1 を用いて、この問題がどのようにして半正定値計画問題 (8) に帰着できるかを説明する。

6 固有値の重複について
トラスのロバスト最適化問題を数値的に解くことにより、最適解で多くの固有値が重複する例が実際に存在することを明らかにする。重複固有値は一般に微分不可能であるため、例えば方向微分などを用いた最適化手法を選択する必要がある。一方で、半正定値計画問題としての定式化を主対角線法で解く場合には、固有値の配列を必要としないため、固有値の重複に関係なく容易に最適解を得ることができる。しかし、剛性行列が設計変数の非線形な関数である場合は一様には半正定値計画問題として定式化できないため、この手法はトラスなどの限られた場合のみ適用可能である。

7 まとめと議論
以上の議論をまとめ、外力の存在範囲が不確実性の大きさに対しオーダーをもたらない場合についての定式化の拡張性を論じている。

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