FORM-FINDING OF COMPLEX TENGRITY STRUCTURES BY DYNAMIC RELAXATION METHOD

1. Introduction

Compared to conventional structural forms such as trusses, there are two distinct characteristics lying in tensegrity structures: (a) they are free-standing without any support; and (b) their cables in tension are continuous, while the struts in compression are discontinuous.

The existence of prestresses in tensegrity structures leads to difficulties in the determination of their (self-equilibrated) configurations associated with prestresses, since every node has to be balanced by the prestresses in the members. Being free-standing is another source of difficulties, because the existing methods for some other tension structures suspended to supports, such as cable-nets, cannot be directly applied to tensegrity structures.

The process of finding such self-equilibrated configurations for (tensegrity) structures is called form-finding or shape-finding. There have been a number of methods proposed for the form-finding problem of tensegrity structures. These methods can be categorized as intuition methods, analytical methods, and numerical methods.

Due to the high non-linearity of form-finding problems, only numerical methods are generally applicable to complex structures. In most of the existing numerical methods, it is not easy to find the final configuration with expected appearance. In some other exiting studies, the rigid-body motions have to be artificially constrained, see for instance Ref. 3.

To solve such problems for form-finding of complex tensegrity structures, we make use of Dynamic Relaxation Method (DRM) in this study. DRM starts from the initial configuration and prestresses given by the designers. Because the self-equilibrium conditions are usually not satisfied with the initial settings, the configuration changes due to the unbalanced forces at the nodes. Movements (motions) of the nodes are traced according to Newton’s second law, and the artificial damping is incorporated to make the system settle down at the static equilibrium state.

DRM has been successfully applied to form-finding of tension structures, including cable-nets, membrane structures, and non-regular tensegrity structures. However, the numerical

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examples studied in Ref. 6 are very simple, while the results are random which might not correctly reflect designers’ requirements on the final configurations. More importantly, convergence performance of DRM was not discussed.

In this study, we apply DRM to the design of complex tensegrity structures, for which tensegrity towers and tensegrity arches are used as examples. It will be demonstrated that the final self-equilibrated configurations of these structures are very close to their initial configurations as expected by the designers.

Moreover, it is well known that DRM suffers from poor convergence because it deals with the motion of each individual node separately. There have been some attempts, for example the drift damping in Ref. 4 and the fictitious mass in Ref. 8, to improve the convergence performance of DRM. In this study, to overcome the shortcoming of DRM in convergence for form-finding of complex tensegrity structures, we use a simple tensegrity structure to tune the analysis parameters including the viscous damping coefficient and time interval, which are necessary in the time history analysis.

Following this introductory section, the paper is organized as follows: Section 2 presents equilibrium equations that are necessary for self-equilibrium analysis of tensegrity structures. In Section 3, the basic idea of DRM is briefly introduced, and the algorithm for form-finding of tensegrity structures is given. A simple tensegrity structure is first studied in Section 4 to tune the analysis parameters; and the form-finding of complex tensegrity structures, including a ten-story tensegrity tower and a tensegrity arch, is further studied. In Section 5, we discuss the possibility of applying the proposed method to form-finding of more complex tensegrity structures, which concludes the study.

2. Equilibrium Equations

One of possible forms of the self-equilibrium equations, in terms of force density matrix associated with nodal coordinates, is presented in this section for the completeness of the paper. Details on derivation and some other equivalent forms of the equations can be found in Ref. 2.

Consider a three-dimensional tensegrity structure consisting of n nodes and m members. The x-, y-, and z-coordinates of the nodes are denoted by \( x, y, \) and \( z \) (\( \in \mathbb{R}^3 \)), respectively; and the generalized coordinate vector is defined as \( \mathbf{X} = (x^T, y^T, z^T)^T \in \mathbb{R}^{3n} \), where \( x^T \) is the transpose of \( x \). Connectivity of the structure is defined by using the connectivity matrix \( \mathbf{C} \in \mathbb{R}^{mn} \), in each row of which there are only two non-zero entries, 1 and -1, corresponding to the two nodes connected by a member.

The coordinate difference vectors \( \mathbf{d}^x, \mathbf{d}^y, \mathbf{d}^z (\in \mathbb{R}^n) \) are

\[
\begin{align*}
\mathbf{d}^x &= \mathbf{C} \mathbf{x}, \\
\mathbf{d}^y &= \mathbf{C} \mathbf{y}, \\
\mathbf{d}^z &= \mathbf{C} \mathbf{z}.
\end{align*}
\]

The length \( l_k \) of member \( k \) can then be computed as follows

\[
l_k = \sqrt{(d_x^k)^2 + (d_y^k)^2 + (d_z^k)^2}.
\]

Let \( s_k \) denote the prestress (axial force) in member \( k \), and let \( q_k \) denote its force density defined as \( q_k = s_k / l_k \). The prestress vector and the force density vector are denoted by \( \mathbf{s} \) and \( \mathbf{q} (\in \mathbb{R}^m) \), respectively.

Equilibrium equations in each direction are

\[
\begin{align*}
\mathbf{E}x &= f^x, \\
\mathbf{E}y &= f^y, \\
\mathbf{E}z &= f^z,
\end{align*}
\]

where \( f^x, f^y, f^z \in \mathbb{R}^n \) are the out-of-balance forces, and \( \mathbf{E} \in \mathbb{R}^{mn} \) is the force density matrix defined as

\[
\mathbf{E} = \mathbf{C}^\top \text{diag} (\mathbf{q}) \mathbf{C}.
\]

Note that diag(\( \mathbf{q} \)) \( \in \mathbb{R}^{mn} \) denotes the diagonal version of \( \mathbf{q} \).

When the structure is at the self-equilibrium state without external load, the (self-)equilibrium equations are reduced to

\[
\begin{align*}
\mathbf{E}x &= \mathbf{E}y = \mathbf{E}z = \mathbf{0}.
\end{align*}
\]

3. Dynamic Relaxation Method (DRM)

DRM is an explicit solution technique, which was originally developed for tidal flow computations. DRM deals with static problems in a dynamic manner. DRM has been applied in many different scientific fields, mainly owing to its simplicity in implementation.

Since DRM traces motion of every single node according to Newton’s second law against the external loads, there is no need to compute the (tangent) stiffness matrices as in conventional (non-)linear analysis. This is especially a distinct advantage for application to some special structures, for which the stiffness matrices might be (nearly) singular or difficult to obtain.

Since tensegrity structures are free-standing, their tangent stiffness matrices are singular, and therefore, they are generally non-invertible. Non-linear analysis of tensegrity structures needs artificial constraints\(^{10}\) or special techniques, such as generalized inverse matrix\(^{10}\). DRM can simply avoid this difficulty in the analysis.

In DRM, the motion of the system is traced at every step of small time increments. When the total kinetic energy of the system reaches its local peak, all velocity components are reset to zero, and the process is restarted from the current configuration and axial forces. This kinetic damping associated with the tracing and restarting procedures is continuously applied until the resultant out-of-balance forces are sufficiently small.

Let \( \mathbf{M} \in \mathbb{R}^{3n \times 3n} \) define the nodal mass matrix, and let \( \mathbf{D} \in \mathbb{R}^{3n \times 3n} \) define the damping matrix due to viscous damping at nodes. It is notable that we are only interested in the final static self-equilibrium state of the structure, hence, the nodal mass and viscous damping can be fictitious.

The equation of motion of the structure subjected to out-of-balance forces \( \mathbf{f} \in \mathbb{R}^n \) is

\[
\mathbf{M} \ddot{\mathbf{x}} + \mathbf{D} \dot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{f}.
\]
\[-Dv - f = Ma, \] (6)

where \( a \) and \( v ( \in \mathbb{R}^{3n} ) \) are the generalized acceleration and velocity vectors, respectively. \( f \) is defined as follows using \( f', f', f' \) defined in Eq. (3):

\[
f = \begin{pmatrix} f' \\ f' \\ f' \end{pmatrix}.
\] (7)

Let \( s, m ( \in \mathbb{R}^n ) \) denote the prestresses and lengths of the members in the initially stressed state. Suppose that the structure deforms to a new configuration due to nodal motions, and the current member lengths are \( l \in \mathbb{R}^n \). The current prestresses can then be updated as

\[
s = s + K(t - l),
\] (8)

where the non-zero entries in the diagonal matrix \( K \in \mathbb{R}^{n \times n} \) are stiffnesses of the members.

In the process of form-finding of a tensegrity structure, we are to search for the configuration associated with the prestresses satisfying the self-equilibrium equations in Eq. (5), or equivalently \( f = 0 \) in Eq. (7). Hence, DRM is terminated when the mean residual force \( \bar{R} \) defined as follows is small enough:

\[
\bar{R} = \sqrt{f^T f / (3n)}.
\] (9)

Moreover, to rule out the influence of masses in kinetic energy of the structure, we use the mean square of velocities defined by

\[
v = \sqrt{v^T v / (3n)}
\] (10)
as a criterion for restarting the nodal motions.

The algorithm making use of DRM for form-finding of tensegrity structures is summarized as follows, where the subscript \( t \) denotes the time.

Algorithm of DRM:

**Step 0.** Set the starting time as \( t = 0 \). Specify the initial configuration with the generalized coordinates \( X \) and prestresses \( s, \) and calculate the initial member lengths \( l \).

Specify time interval \( \Delta t \) for pseudo time history analysis with zero initial velocities \( v = 0 \).

**Step 1.** Solve the equation of motion in Eq. (6) to compute the nodal accelerations \( a_{v, x} \) at time \( t + \Delta t \) as

\[
a_{v, x} = -M' (Dv + f). \] (11)

Update the nodal velocities \( v_{v, x} \) as well as generalized nodal coordinates \( X_{v, x} \) at time \( t + \Delta t \) as

\[
v_{v, x} = v_i + a_{v, x} \Delta t, \]
\[
X_{v, x} = X_i + v_{v, x} \Delta t. \] (12)

**Step 2.** If the mean square of velocities reaches its local peak: i.e.,

\[
v_{v, x} \leq v_i \leq v_{v, x}, \] (13)

then reset the velocities to zero: i.e., \( v_{v, x} = 0 \).

**Step 3.** If the mean residual force \( \bar{R} \) is smaller than the specified small value \( \bar{R} \): i.e.,

\[
\bar{R} < \bar{R},
\] (14)

then terminate the algorithm: otherwise, return to Step 1 with the time updated as \( t \leftarrow t + \Delta t \).

4. Numerical Examples

In this section, we are to investigate the efficiency of the proposed approach for form-finding of complex tensegrity structures, and moreover, to demonstrate its ability in finding expected configurations. In particular, we consider a tensegrity tower and a tensegrity arch as numerical examples.

Without loss of any generality, the units in the numerical examples in this study will be omitted. This is because the configuration as well as prestresses of a self-equilibrated tensegrity structure can be scaled independently without affecting its self-equilibrium.

4.1. Tensegrity Towers and Tensegrity Arches

A tensegrity tower or arch is constructed by assembling the unit cells along a straight line or an arch-shaped curve.

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**Fig. 1** A prismatic tensegrity structure used as unit cell.

**Fig. 2** A two-story tensegrity tower.

**Fig. 3** Initial configuration of the one-story tensegrity.
In this study, the prismatic tensegrity structures are used as unit cells. Figure 1 shows an example of the symmetric prismatic tensegrity structure. It consists of eight nodes, four struts, eight horizontal cables, and four vertical cables.

In a tensegrity tower or a tensegrity arch, the struts directly come from the unit cells, and there are four different types of cables:

(a) Horizontal cables, which connect the nodes on the same horizontal plane at the top and bottom of the structure;
(b) Saddle cables, which connect the nodes on the lower horizontal plane of the upper unit cell and those on the upper horizontal plane of the lower unit cell;
(c) Vertical cables, which connect the nodes on the different horizontal planes of the same unit cell; and
(d) Diagonal cables, which connect the nodes on the upper (or lower) horizontal planes of different unit cells.

More detailed descriptions on connectivity of the tensegrity towers can be found in Appendix C of Ref. 2. As an example, a two-story tensegrity tower is shown in Figure 2, which is constructed by two prismatic structures as in Figure 1. The structure is composed of four struts, four vertical cables, and four diagonal cables in each story (unit cell); and there are four horizontal cables at each end of the structure, and eight saddle cables connecting different unit cells.

4.2. Parameter Tuning

In the form-finding of tensegrity structures, we are only interested in their final self-equilibrated configurations. However, the dynamic structural parameters are important to the convergence performance of DRM. Since these parameters do not necessarily correspond to real structures, they can be assigned arbitrarily so as to have better performance. In this subsection, we investigate the convergence performance of DRM with respect to damping coefficient as well as time interval. Because different member stiffnesses and initial prestresses result in different final configuration, their influence on converge performance will not be taken into consideration.

For the sake of simplicity, we consider the one-story tensegrity tower with four struts as shown in Figure 1 as an example. DRM starts from the initial configuration as shown in Figure 3, where the nodes connected by the same vertical cable have the same $x$-coordinates. Height of the structure is 15.0, and its radius is 10.0.

The following standard settings are used for all numerical examples in this study, if they are not separately defined. The mass of each node is set as 1.0; stiffnesses of the struts and cables are respectively set as 100.0 and 10.0, such that cables are more flexible than struts; the initial prestresses in struts and cables are respectively $-1.0$, and $3.0$, such that the corresponding initial strains are respectively 1.0% and 30.0%; and the time interval $\Delta t$ is 0.4. Moreover, the damping matrix is a diagonal matrix with the damping coefficients 1.0 for all diagonal components. The final configuration of the structure is shown in Figure 1. The algorithm is terminated when the mean residual force is smaller than 0.005.

First, we investigate the effect of time interval, which is varied from 0.02 to 1.0. The numbers of steps needed for Algorithm of DRM are plotted in Figure 4. Note that the cases that converge beyond 2,000 steps are omitted in the figure.

As in Figure 4, the convergence performance is significantly improved when $\Delta t$ is increased: this is because larger $\Delta t$ leads to larger deformation in each step, and therefore, the algorithm will not be trapped into small fluctuations. However, the algorithm performs poorly when it is too large, i.e., the algorithm does not converge within 2,000 steps if $\Delta t > 0.5$: this is because too large deformations in each step result in difficulty in settling down to the self-equilibrated configurations, especially at the end of the algorithm where small incremental deformations are required.

In the following examples, we use $\Delta t = 0.4$ as the standard setting for time interval. The smallest natural period of the structure with the initially specified configuration is 1.91. Note here that the geometrical stiffness is not taken into account in the computation of natural periods, since it has little influence on the smallest period in most cases. Although the smallest period varies slightly during the form-finding process due to configuration changes, the numerical stability of DRM might not be greatly influenced, since it is much larger than the time interval.
Next, we investigate the effect of viscous damping coefficient, which is varied from 0.0 to 4.0. The numbers of steps needed for Algorithm of DRM are plotted in Figure 5, which shows that the algorithm performs poorly when the damping coefficient is too small or too large. Hence, in the following examples, the standard values for viscous damping coefficient is set as 1.0.

The optimal parameter values are slightly different for each individual case, however, we do not tune the parameters for each example. Instead, the standard settings derived from this simple structure are used. This will be justified to be good enough for the complex structures considered later, because they have similar smallest natural periods to this simple structure.

4.3. Ten-story Tensegrity Tower

In this subsection, we consider a ten-story tensegrity tower, which consists of four struts in each unit cell. The initial configuration of the structure is shown in Figures 6(a), 7(a), and 7(b). For simplicity, each unit cell has the same height of 16.5, and overlaps between adjacent stories (unit cells) are 1.5. The nodes on the upper as well as lower horizontal planes of each unit cell lie on the circle with radius of 10.0.

Using the standard settings, the mean residual force $R_m$ of the initial configuration is 0.65. Obviously, the structure is not at the self-equilibrium state. The smallest natural period of the structure with the standard settings is 1.85. After 280 steps of tracing motions of the system, Algorithm of DRM settles down to the final configuration as shown in Figures 6(b) and 7(c). The mean square of velocities at each step of the form-finding process is plotted in logarithmic scale in Figure 8. By contrast, 16,397 iterative steps were needed in our previous study, in which the damping coefficient was 5.0 and the time interval was 0.1.

From the top view of the final configuration of the structure in Figure 6(b), we may observe that the nodes on each horizontal plane move around the longitudinal axis from the initial configuration to settle down to the positions with sufficiently small residual forces. The center line, on which the centers of each four nodes of the same level are lying, of the final configuration coincides with the center line of the initial configuration. This demonstrates that DRM can find the self-equilibrated configuration close to the initially specified one, which might be important in the preliminary design process.

We confirmed that the structure is stable, since its tangent stiffness matrix is positive semi-definite with six zero eigenvalues corresponding to the rigid-body motions.

4.4. Tensegrity Arch

In this subsection, we go further to consider the tensegrity arches to demonstrate the applicability of the proposed method to complex structures. A tensegrity arch has the same connectivity as the tensegrity tower in the previous subsection; however, it has a curved appearance as shown in Figure 9, in contrast to the straight appearance of the tensegrity tower.

The initial configuration of the tensegrity arch is shown in Figure 9. It is constructed from ten unit cells, with four struts in each. The half-opening angle of the structure is 90°, and the radius of the center line of the arch is 75.0. Other settings are the same as the ten-story tensegrity tower studied in the previous subsection. The smallest natural period of the structure with standard settings is 2.29.

After 236 steps, Algorithm of DRM converges at the final configuration as shown in Figure 10. By contrast, 14,496 iterative steps were needed in our previous study in Ref. 11 for the same structure, and the damping coefficient was 5.0 and the
time interval was 0.1. The mean square of velocities at each step is plotted in logarithmic scale in Figure 11.

The center line of the final configuration has a similar appearance to the specified arch, while the distances of the center line from the approximated arch are mostly under 2.5% of radius of the unit cells. Moreover, as can also be observed from Figure 10, the center nodes of the two ends have relatively large distances, which is about 7.5% of the radius. We confirmed that the structure with the self-equilibrated configuration is stable.

5. Conclusions

In this study, we adopted the basic idea of DRM to find the self-equilibrated configurations of complex tensegrity structures. The algorithm for such purpose is simple, because motion of every single node is traced separately according to Newton’s second law: and hence, there is no background knowledge on (non-linear) analysis necessary in DRM.

The tensegrity tower and tensegrity arch consisting of ten simple prismatic tensegrity structures as unit cells have been used as examples. In the numerical investigations, it has been demonstrated that DRM is good at finding the self-equilibrated configurations of complex tensegrity structures. Moreover, DRM can find the final configurations close to the initially specified ones, which might be important to designers.

Although DRM was well known to exhibit poor convergence, it is highly possible to improve its convergence performance by tuning the analysis parameters by using simpler structures.

In this study, the appropriate values of viscous damping coefficient and time interval were obtained by investigating convergence performance of a simple prismatic structure. As a result, DRM finds the self-equilibrated configurations for both of the tensegrity tower and tensegrity arch within acceptable computational costs. If the smallest natural periods are quite different, the masses can be tuned to make them have similar values, although this not the case in the examples studied here.

References

テンセグリティ構造はトラス構造と同じく線材をピン接合した構造である。しかし、トラス構造と比べて、テンセグリティ構造には以下の特徴がある：(a) 支点で支持しなくても自立する。 (b) 弾力をもつテション材は連続であり、圧縮力をもつ棒材が不連続である。

テンセグリティ構造に与える初期軸力を(張力)は、各節点で自己釣合い状態にある必要がある。したがって、その形状と軸力分布との関係には高い関連性があり、それそれぞれを独立に決定することができない。さらに、構造構造やケーブルネットなどのレジリオニクスに固定されている他の種類の張力構造と異なり、テンセグリティ構造は自立するため、その剛体変位が拘束されていない。そのため、他の種類の張力構造の形状決定のために用いる必要がある手法を、テンセグリティ構造にそのまま適用することができない。

テンセグリティ構造の形状決定については、今まで数多くの手法が開発されている。それらの手法は、直交的手法と、解析的手法および数値的手法に分類できる。テンセグリティ構造の形状決定は高次な非線形問題であるため、直交的手法と解析的手法の対向は単純な構造に限定され、複雑な構造には数値の手法のみが適用できる。しかし、既存の多くの数値的手法では、設計者が望ましい形状を求めることは困難である。また、一部の既存手法では、テンセグリティ構造の剛体変化を人工的に拘束する必要があるため、汎用性が低く、解析のプロセスも複雑となる。

上記の問題点を解決するため、本研究では高い汎用性のある動的緩和法を採用した。動的緩和法とは、静的釣合い問題を仮想的釣合い問題に変換し、動的システムを人工的に減衰により静止させることによって、元素的釣合い問題を解く方法である。動的緩和法においては、各節点の不平衡力による運動を個別に順に追跡するため、連続剛性行列をその逆行列も必要とせず、そのアルゴリズムの実現が容易である。

テンセグリティ構造の形状を決定するための動的緩和法は、与えられた初期形状及び初期張力から開始する。初期設定は自己釣合い条件を満たしたため、各点に不平衡力が生じる。そして、各節点がニュートンの運動法則に従って変位し、構造が形状を変化する。この動的システムの運動を早期に静止させるため、運動減衰および粘性減衰という人工的減衰を取り入れる。運動減衰は、システムの運動エネルギーが局所最大となったとき、すべての節点の速度をゼロにリセットすることで实现する。また、粘性減衰によって、節点にその速度に比例する減衰力を作用させる。動的緩和法は、既にケーブルネット、膜構造や単独なテンセグリティ構造などの張力構造の形状決定に応用されている。しかし、動的緩和法を複雑なテンセグリティ構造へ適用した例は見られない。

本研究においては、動的緩和法を複雑なテンセグリティ構造に適用し、設計者が指定した自己釣合い形状に近い形状に収束できる方法を開発する。特に、節点数と部材数が多いテンセグリティ・タワーおよびアーチを計算例題として採用する。これらの構造は、単純な角柱状テンセグリティ構造をユニットセルとして、それらを一つずつ直線状または曲線状に積み重ねることにより構成される。また、隣接するユニットセルの間にケーブルを追加することにより、異なるユニットセルが一体化される。

動的緩和法のアルゴリズムはシンプルメントが容易である反面、その収束性が悪いことがよく知られている。既存の研究では、その収束性を改善するため、仮想方向量や適応減衰など幾つかの試みがなされている。しかし、対象とする構造物ごとに異なる設定が必要であり、全てのケースに適応できる一般的な設定が存在しない。本研究では、動的緩和法の収束性を向上させるため、より単純な角柱状テンセグリティ構造（ユニットセル）を対象として、擬似的動的解析に伴う粘性減衰と時刻増分の値をチューニングする。

計算例題では、10 個のユニットセルで構成されたテンセグリティ・タワーおよびアーチを対象とした。提案手法により、いずれの例でも初期形状に近い自己釣合い形状が得られている。また、既往研究の計算コストと比較して、一つのユニットセルでチューニングされた解析プログラムを用いた本提案手法の収束性は、大幅に改善されること確認された。一つ重要な理由としては、テンセグリティ・タワーおよびアーチの最小の固有周期がユニットセルの数に関係せず、近い値となるため、同じ時刻増分を採用しても過大または過小となることが少ない。

本研究で提案した動的緩和法は、計算コストが低く、決定された自己釣合い形状が設計者の指定した形状に収束できるため、複雑なテンセグリティ構造の自由設計に有望な手法だと考えられる。

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