THE MINIMUM WEIGHT DESIGN OF A CLASS OF TALL MULTI-STORY FRAMES SUBJECTED TO LARGE LATERAL FORCES I*

By Ryo TANABASHI**
Tsuneyoshi NAKAMURA***

1. Introduction

This paper presents the minimum weight design of a broad class of tall multi-story frames subjected to a class of lateral loads which are large compared to the vertical loads on the beams. A frame considered here consists of horizontal beams and vertical columns with uniform cross-sections between joints. Each joint is rigid enough to transmit the fully plastic moment of any member framing into it. The problem of minimum weight design may be stated as follows: Given the center line dimensions of a frame and a set of static loads acting on it as shown in Fig. 1, what fully plastic moments should be assigned to the members in order that the loads may be just sustained and the material consumption is a minimum?

This problem has been investigated in a more general form by Heyman [1]****, Foulkes [2], [3], Livesley [4], Heyman and Prager [5], Stone [6], Kalker [7], Tanaka [8], and Prager [9]. The principal difficulty appears to consist in the discrepancy between the structural complexity and the size of the corresponding linear programming problems stemming from the very large number of inactive constraints. Even with Heyman's and Prager's method [5], Kalker's program [7] for an IBM 704 Computer can solve, in effect, a 6-story 8-bay frame. It has not been shown how powerful and convenient Tanaka's method is for multi-story frames, compared to the former. Although these computational methods based upon the linear programming technique are quite general and automatic, they furnish a solution, problem, by problem, numerically. The general qualitative features of the minimum weight designs can then hardly be studied for practical multi-story frames. Furthermore, in view of the approximate nature of the linear minimum weight design theory and the flatness of the permissible region, the exact solution of a problem may not have to be obtained for practical purposes with so much computational labor. Calladine [10] has studied the minimum weight design of an isolated story of a frame with equal spans. This special case appears to have revealed some interesting features of the minimum weight designs to a certain degree.

The purpose of the present paper is to establish in an extremely simple and explicit form the minimum weight design of a broad class of tall multi-story frames subjected to sufficiently large shear resultants compared to vertical loads. In view of the fact that the magnitude of vertical loads on beams hardly change from the topmost story through the first story, whereas lateral loads being accumulated downward, the proposed design may readily be modified for the case of comparatively large vertical loads in the upper stories, resulting in a good upper bound on the minimum weight. This will be illustrated by a numerical example.

The present minimum weight design should not, however, be adopted as a practically desirable design without the following modification for the effect of axial forces in twofold. First, the effect of axial forces in columns must be taken into account in the corresponding yield conditions. A prac-

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** Professor of Architecture, Kyoto University, Kyoto. Member of A.I.J.

*** Assistant Professor of Architecture, Kyoto University, Member of A.I.J.

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tical way of augmenting the column sections will be illustrated elsewhere. Secondly, the design must be modified for the additional moments induced by the presence of large axial forces in columns and large elastic-plastic deformation at the ultimate state of the frame. In view of the so-called Rankine formula [11], the present rigid-plastic collapse load design plays an important role as a fundamental design in establishing the real ultimate strength design [15].

Besides the customary assumptions made in the plastic analysis of structures, the following additional assumptions are made in this paper. (1) A designer has at his disposal the continuous spectrum of structural cross-sections available commercially. (2) The weight of a frame is estimated by a linearized weight function [9].

2. Frame moments and equilibrium equations

Since we deal with a frame acted upon by a set of sufficiently large lateral forces, it may be assumed that the equal extreme values of the bending moment in a member occur at its ends and any other section of the member may attain this extreme value. For the sake of simplicity, we replace here all the vertical loads on a beam by a single concentrated load acting upon its midspan [12]. The potentially critical sections of the beam are then its two ends and midspan. It should be observed that the collapse mechanisms in Foulkes' exhaustive study [4] on a portal frame must be a prototype of a more general mechanism for a multi-story frame. The minimum weight design of internal bays of an isolated story studied by Calladine [10] must also be a special prototype which serves to suggest us a more general approach. It seems quite natural then to ask if the extreme state can exist where plastic hinges have formed at all the end sections of the members of a frame at collapse as shown in Fig. 2. Some more hinges may form at some of the midspans of beams. This is the starting point of the present study. This approach is supported by the fact that, since a minimum weight design is in general represented by a vertex of a convex permissible region, it is a reasonable short cut toward the minimum weight design of the present class of frames, to minimize the weight with respect to a limited number of appropriate vertices and then to check if the corresponding mechanism satisfies the Foulkes' mechanism condition [3].

Lets us consider an f -story s -bay frame. The symbols are defined in Fig. 1. At such an extremely deteriorated state as mentioned above, all the end sections attain their fully plastic moments and hence the number of unknown end moments is equal to that of members if appropriate plus and minus signs are used. It is assumed here that the direction of the shear resultant in a story is always the same as that in the topmost story. Since under the above assumption the concentrated load on a beam merely determines the bending moment at its midspan, we shall first concentrate our attention to the lateral forces only. In view of the moment diagram in the elastic range, an appropriate choice of plus or minus signs may readily be made as shown in Fig. 3 and we may assume the corresponding plastic hinge rotations as shown in Fig. 2. The $f(s+2)$ equations of equilibrium, however, do not in general suffice to determine $f(2s+1)$ plastic moments uniquely except when $s=1$. The degree of redundancy is therefore $f(s-1)$. Instead of dealing with $f(2s+1)$ plastic moments, it seems convenient to define a "frame moment". In view of the linearity of the equations of equilibrium in terms of the plastic moments without any slope-continuity requirements, we may split or decompose the moment diagram at collapse into fs unit diagrams for the fs rectangles as shown in Fig. 4., and regard the original moment diagram as the composite consisting of these unit diagrams. It is noted that if the lower ends of the first story columns are all fixed, imaginary ground floorbeams may be supposed to exist for the frame moments of the first story rectangles. Each unit moment distribution is such that moment equilibrium is maintained at the four corners with the same absolute value in the manner shown in Fig. 4. Such a unit moment diagram is therefore completely defined by this equal corner moment, which will be called a "frame moment". The positive frame moment is defined as the one corresponding to the shear which has the same direction as that of the shear resultant in the story. The frame moments illustrated in Fig. 4 are positive when the shear resultant in a story is directed

Fig. 1

Fig. 2

Fig. 3

Fig. 4

from left to right. Let $M_{j,k}$ denote the frame moment of the k -th rectangle from the left in the j -th story. The k -th beam and column from the left in the j -th story will be called (j,k) -beam and (j,k) -column, respectively. Let $B_{j,k}$ and $C_{j,k}$ denote the plastic moments of the (j,k) -beam and (j,k) -column, respectively. It is noted that the subscript k in $C_{j,k}$ should run from 1 through $s+1$ since there are $s+1$ columns in a story. The plastic moments of any beams and columns may be expressed in terms of these frame moments as follows.

$$\left. \begin{aligned} B_{j,k} &= M_{j,k} + M_{j+1,k}, & B_{f,k} &= M_{f,k} \\ C_{j,1} &= M_{j,1}, & C_{j,k} &= M_{j,k-1} + M_{j,k}, & C_{j,s+1} &= M_{j,s} \\ (j &= 1, 2, \dots, f; & k &= 1, 2, \dots, s) \end{aligned} \right\} \dots\dots\dots (1)$$

If there are ground floor beams, $B_{0,k} = M_{1,k}$. The equations of equilibrium of moments at joints for the assumed moment diagram are given by

$$\left. \begin{aligned} B_{f,1} &= C_{f,1}, & B_{f,k-1} + B_{f,k} &= C_{f,k}, & B_{f,s} &= C_{f,s+1} \\ B_{j,1} &= C_{j,1} + C_{j+1,1}, & B_{j,k-1} + B_{j,k} &= C_{j,k} + C_{j+1,k}, & B_{j,s} &= C_{j,s+1} + C_{j+1,s+1} \\ (j &= 1, 2, \dots, f-1; & k &= 1, 2, \dots, s) \end{aligned} \right\} \dots\dots\dots (2)$$

for the $f(s+1)$ joints and are satisfied identically by these frame moments. Therefore we may only have to write down just one equation of sway equilibrium for each story. In other words, f conditions of sway equilibrium for the fs unknown frame moments leave $f(s-1)$ degree of redundancy as a whole. This reduction of the number of unknowns from $f(2s+1)$ to fs is one of the advantages of introducing the frame moments. The f equations of sway equilibrium may be written as

$$4 \sum_{k=1}^s M_{j,k} = h_j \sum_{i=j}^f P_i \quad (j=1, 2, \dots, f) \dots\dots\dots (3)$$

where P_i is the lateral load applied at the i -th story beam level.

Some inequality constraints must be imposed upon the frame moments. The principal assumption in defining the frame moments has been the one that plastic hinges have formed at all the end sections

of the members, no other sections having violated the yield conditions. This assumption remains satisfied even if comparatively small vertical loads are applied together with the lateral loads. If the simple-beam bending moment diagrams are superposed on the assumed diagram in Fig. 2, the resulting moment at any section of a member should not exceed the plastic moment of the member. Conversely therefore, for a specific set of vertical loads, there must exist the limiting smallest value to which the plastic moment of a beam may be decreased in order to achieve the minimum weight. Let $V_{j,k}$ denote the concentrated load acting upon the midspan of the (j,k) -beam. Then the minimum values of the plastic moment of the (j,k) -beam may be obtained as follows

$$\left. \begin{aligned} B_{f,k} &= M_{f,k} \geq (1/4)V_{f,k}l_k \\ B_{j,k} &= M_{j,k} + M_{j+1,k} \geq (1/4)V_{j,k}l_k \quad (j=1,2,\dots,f-1) \end{aligned} \right\} \dots\dots\dots(4a)$$

As far as $B_{j,k}$ and $C_{j,k}$ are all non-negative, $M_{j,k}$ does not necessarily have to be non-negative. But we shall assume that the minimum weight design satisfies the condition

$$M_{j,k} \geq 0 \dots\dots\dots(4b)$$

Then the problem of minimization may readily be dealt with by means of the standard techniques of linear programming parametrically [16].

3. The minimum weight design

Any moment distribution represented by a set of $M_{j,k}$ satisfying (3) and (4) provides us with a complete solution since it satisfies all the equilibrium equations, does not violate any yield conditions and corresponds to an overcomplete collapse mechanism. Let D denote the set of all the designs satisfying (3) and (4). Let $\bar{D}$ denote the set of all the possible designs including D and those designs which do not correspond to the present particular overcomplete collapse mechanisms. The minimum weight design may be established in the following two steps of a semi-inverse method.

- (I) Assuming that the minimum weight design $D_{\min}$ in D also gives the minimum weight among all the designs in $\bar{D}$, the weight is minimized subjected to the constraints (3) and (4) by means of linear programming.
- (II) It is then proved that the overcomplete collapse mechanism corresponding to $D_{\min}$ indeed satisfies Foulkes' mechanism condition for the class of frames of practical interest. This proof justifies the above assumption (I) since $D_{\min}$ then satisfies all the necessary and sufficient conditions for the minimum weight design [3].

The linearized weight function [3],[9] may be written as

$$\begin{aligned} G &= g \left[\sum_{j=1}^f h_j \sum_{k=1}^{s+1} C_{j,k} + \sum_{k=1}^s l_k \sum_{j=1}^f B_{j,k} \right] \\ &= g \left[\sum_{j=1}^f 2h_j \sum_{k=1}^s M_{j,k} + \sum_{k=1}^s l_k \left\{ M_{1,k} + 2 \sum_{j=2}^f M_{j,k} \right\} \right] \dots\dots\dots(5) \end{aligned}$$

where g is a proportional constant. The problem is to minimize G given by (5) subject to the constraints (3) and (4). The first step of linear programming is to convert the inequalities (4a) into equivalent equalities by introducing fs slack variables $x_{j,k}$ as follows.

$$\left. \begin{aligned} M_{f,k} - x_{f,k} &= (1/4)V_{f,k}l_k \\ M_{j,k} + M_{j+1,k} - x_{j,k} &= (1/4)V_{j,k}l_k, \quad x_{j,k} \geq 0 \end{aligned} \right\} \dots\dots\dots(6)$$

There are now $fs+f$ constraints in $2fs$ non-negative variables. In order to choose $fs+f$ basic variables appropriately, it is helpful to consider the geometrical significance of the constraints (3) and (4) in a frame moment space of fs -dimensions whose cartesian coordinates are $M_{j,k}$. The constraints (3) define an $f(s-1)$ dimensional intersection of f hyperplanes in the fs -space. Only that part of the intersection which satisfies all the constraints in (4) is the permissible convex region. A vertex of this permissible region may be defined by a set of $f(s-1)$ equalities in (4a), provided that (4b) are also satisfied. It is noted that for a specific value of k , the f frame moments $M_{j,k}$ may be uniquely determined by the f simultaneous equalities for $M_{j,k}$ in (4a). In other words, the fs equalities in (4a) may be subdivided into s sets of those f equalities. We may then choose any $(s-1)$ out of the

s sets to define a vertex. The weight function (5) represents a family of hyperplanes in this f -space for various values of G . The minimum weight design is represented by the point or points of tangency when the weight hyperplane is tangent to the permissible region. With the help of this geometrical consideration, we may choose $x_{j,k} (k \neq m)$ as the $f(s-1)$ non-basic variables where the subscript m is defined by

$$l_m = \min_{k=1}^s \{l_k\}$$

The subscript m satisfying this condition may not in general be unique. Let N denote the set of all the subscripts corresponding to the minimum beam length. Then any subscript $r \in N$ may be chosen as the subscript m . The basic variables $M_{j,k}$ and $x_{j,m}$ may readily be expressed in terms of the $f(s-1)$ non-basic variables as follows.

$$\left. \begin{aligned} M_{f,k} &= (1/4)V_{f,k}l_k + x_{f,k} \\ M_{f-1,k} &= (1/4)(V_{f-1,k} - V_{f,k})l_k + x_{f-1,k} - x_{f,k} \\ &\dots\dots\dots \\ M_{j,k} &= (1/4)l_k \{V_{j,k} - V_{j+1,k} + \dots\dots + (-1)^{f-j}V_{f,k}\} + x_{j,k} - x_{j+1,k} + \dots\dots + (-1)^{f-j}x_{f,k} \\ M_{j,m} &= (1/4)h_j \sum_{i=j}^f P_i - \sum_{k \neq m} (1/4)l_k \{V_{j,k} - V_{j+1,k} + \dots\dots + (-1)^{f-j}V_{f,k}\} \\ &\quad - \sum_{k \neq m} \{x_{j,k} - x_{j+1,k} + \dots\dots + (-1)^{f-j}x_{f,k}\} \\ x_{f,m} &= (1/4)h_f P_f - (1/4) \sum_{k=1}^s l_k V_{f,k} - \sum_{k \neq m} x_{f,k} \\ x_{j,m} &= (1/4)h_j \sum_{i=j}^f P_i + (1/4)h_{j+1} \sum_{i=j+1}^f P_i - (1/4) \sum_{k=1}^s l_k V_{j,k} - \sum_{k \neq m} x_{j,k} \end{aligned} \right\} \dots\dots (7)$$

The corresponding basic solution may immediately be obtained if we set $x_{j,k} = 0$. The basic solution is feasible if the following conditions are all satisfied.

$$\left. \begin{aligned} V_{j,k} - V_{j+1,k} + \dots\dots + (-1)^{f-j}V_{f,k} &\geq 0 \\ h_j \sum_{i=j}^f P_i - \sum_{k \neq m} l_k \{V_{j,k} - V_{j+1,k} + \dots\dots + (-1)^{f-j}V_{f,k}\} &\geq 0 \\ h_f P_f - \sum_{k=1}^s l_k V_{f,k} &\geq 0 \\ h_j \sum_{i=j}^f P_i + h_{j+1} \sum_{i=j+1}^f P_i - \sum_{k=1}^s l_k V_{j,k} &\geq 0 \end{aligned} \right\} \dots\dots\dots (8)$$

In many practical problem where $V_{j,k}$ does not increase radically downward compared to the downward increase in the shear resultant, it may readily be found that if these conditions are satisfied at the topmost story, they will in general be satisfied at any other story. For instance, if $V_{j,k} = V_k$ for all j , (8 a) is always satisfied and the remaining three conditions may be satisfied if

$$h_j \sum_{i=j}^f P_i \geq \sum_{k=1}^s l_k V_k.$$

Our attention will be confined here to those cases where the conditions (8) are all satisfied.

If we substitute equations (7) into the expression (5), we obtain

$$G = g \left[\sum_{j=1}^f (1/2)h_j^2 \sum_{i=j}^f P_i + l_m \left\{ (1/4)h_1 \sum_{i=1}^f P_i + (1/2) \sum_{j=2}^f h_j \sum_{i=j}^f P_i \right\} \right. \\ \left. + \sum_{k \neq m} (l_k - l_m) \sum_{j=1}^f \left\{ (1/4)V_{j,k}l_k + x_{j,k} \right\} \right] \dots\dots\dots (9)$$

If m is the only subscript which indicates the minimum beam length, then $l_k - l_m > 0$ for all $k \neq m$. Then the basic solution obtained above by setting $x_{j,k} = 0 (k \neq m)$ in (7) is the unique minimal solution. If on the other hand, N contains more than one element, then $l_k - l_m = 0$ for $k \in N$ and $l_k - l_m > 0$ for $k \notin N$. In this case the coefficients of $x_{j,k} (k \in N)$ become zeros. Hence the minimal basic solution obtained by setting $x_{j,k} = 0 (k \neq m)$ in (7) is not unique and $x_{j,m}$ may be exchanged with any other $x_{j,r}, r \in N$, giving the same minimum weight. Any linear combination of these minimal solutions will

also provide a minimum weight design. In the case where N contains the sole element m , the corresponding plastic moments may readily be obtained from equations (1).

$$\left. \begin{aligned} B_{j,k} &= (1/4)l_k V_{j,k} \quad (k \neq m), & B_{f,m} &= (1/4)h_f P_f - \sum_{k \neq m}^s (1/4)l_k V_{f,k} \\ B_{j,m} &= (1/4)h_j \sum_{i=j}^f P_i + (1/4)h_{j+1} \sum_{i=j+1}^f P_i - \sum_{k \neq m}^s (1/4)l_k V_{j,k} \\ C_{j,k} &= (1/4)l_k \sum_{t=j}^f (-1)^{t-j} V_{t,k} + (1/4)l_{k+1} \sum_{t=j}^f (-1)^{t-j} V_{t,k+1} \quad (k \neq 1, m-1, m, s+1) \\ C_{j,n} &= (1/4)h_j \sum_{i=j}^f P_i - (1/4) \sum_{\substack{k \neq n, \\ k \neq n+1}}^s l_k \sum_{t=j}^f (-1)^{t-j} V_{t,k} \quad (n = m-1, m; m \neq 1, s) \\ C_{j,e} &= (1/4)l_e \sum_{t=j}^f (-1)^{t-j} V_{t,e} \quad (e = 1, s+1) \end{aligned} \right\} \dots\dots\dots(10)$$

Similar expressions may be obtained for the case $m=1$ or $m=s$. It is worth noticing that the $f(s-1)$ beams with length l_k 's ($k \neq m$) are to be designed to carry vertical loads in their respective spans and that the corresponding bays contribute little to supporting the lateral shear. The bay with the minimum beam length l_m , on the other hand, must then be designed to carry the remaining part of the lateral shear in each story. It should also be pointed out that in certain cases the present solution results in columns of zero plastic moments. This theoretical solution is a necessary consequence under the present assumption but appears unpractical. Although the columns may be augmented by taking into account the effect of axial forces as will be shown elsewhere, a more practical design may also be obtained by imposing further restrictions on the plastic moments. Such a special class of minimum weight designs will be discussed later.

Let us now consider Foulkes' mechanism condition and step (II). Since the present problem involves $f(2s+1)$ independent variables $B_{j,k}$ and $C_{j,k}$, a minimal design can always be found which has $f(2s+1)$ alternative collapse mechanisms. It is noted, in view of the solution obtained above, that

Fig. 5 a

Fig. 5 b

Fig. 5 c

Fig. 5 d

since at least $f(s-1)$ midspans attain their fully plastic moments, plastic hinges may form not only at their end sections but also their midspans. Hence there are $f(s-1)$ independent hinge rotations at the $f(s-1)$ midspans as shown in Fig. 5c. There are $f(s+1)$ independent joint rotations and f different sway mechanisms. The total number is equal to $f(2s+1)$. Foulkes' mechanism condition may be stated as follows.

A mechanism must be found by an appropriate combination of these alternative mechanisms such that, for every beam and column, $\sum [\text{Hinge rotations in } (j,k)\text{-beam (or } (j,k)\text{-column)}] = a \text{ const.} \times [\text{the length of } (j,k)\text{-beam (or } (j,k)\text{-column)}], \text{ respectively.} \dots\dots\dots [A]$

It should be observed that all the (j,k) -columns with a fixed number j in the j -th story have the same length h_j . Hence the condition [A] may be satisfied between all the (j,k) -columns in the j -th story if the same hinge rotations θ_j appear through all the upper and lower end sections of the j -th story columns as shown in Fig. 5a. Then f s ratios between $f(s+1)$ sums of hinge rotations in the columns satisfy the condition [A], leaving only f different sums for further study. In a similar manner, the condition [A] may be satisfied between all the (j,m) -beams ($m \in N$) if the same hinge rotations ψ_1 due to joint rotations and sways appear through all the (j,m) -beams as shown in Fig. 5b. Since $l_k > l_m$, it is apparent that some more hinge rotations will be required to occur in (j,k) -beams ($k \notin N$). As has been pointed out, a plastic hinge may appear at the midspan of any (j,k) -beam ($k \notin N$). Hence we may consider the collapse mechanism shown in Fig. 5c where the same hinge rotations $2\phi_k$ occur through all the hinges at the midspans of (j,k) -beams with a fixed subscript k ($k \notin N$). A similar mechanism may be considered for every different $k \notin N$. Then the $(f-1)s$ ratios between f s sums of hinge rotations in beams satisfy the condition [A], leaving s different sums for further study. It is now shown that the condition [A] may be satisfied by a mechanism formed by an appropriate linear combination of these $f+s$ different mechanisms. Fig. 5d shows the resulting overcomplete mechanism. The sums of hinge rotations for the members may be tabulated as shown in Table 1.

Table 1

Member	Plastic moment	Length	Subscripts		Sum of hinge rotations
			j	k	
(j,m) -beam	$B_{j,m}$	l_m	$1, 2, \dots, f; m$		$2\psi_1$
(j,k) -beam	$B_{j,k}$	l_k	$1, 2, \dots, f; 1, 2, \dots, m-1, m+1, \dots, s$		$2\psi_1 + 2\phi_k$
(j,k) -column	$C_{j,k}$	h_j	$2, 3, \dots, f; 1, 2, \dots, s+1$		$2\theta_j$
(l,k) -column	$C_{1,k}$	h_1	$1; 1, 2, \dots, s+1$		$2\theta_1 + \psi_1$

The condition [A] may be satisfied if the angles of rotations are determined as follows.

$$\psi_1 = (1/2)l_m\theta, \quad \phi_k = (1/2)(l_k - l_m)\theta, \quad \theta_j = (1/2)h_j\theta, \quad \theta_1 = (1/2)(h_1 - l_m/2)\theta \dots\dots\dots (11)$$

where θ is a positive constant.

Since $\psi_1 \geq \phi_k \geq 0$ and $\theta_1 \geq 0$ by assumption, the condition [A] can be satisfied by those frames which satisfy

$$2l_m \geq l_k \geq l_m \quad \text{and} \quad 2h_1 \geq l_m \dots\dots\dots (12)$$

Thus it has been shown that the design defined by eqs. (10) for the frames satisfying the restrictions (12), indeed satisfy the three necessary and sufficient conditions for the minimum weight design. The minimum weight design for those frames which violate the latter of the restrictions (12) may be obtained by modifying the first story members of the present design.

(Continued to Part II)

大きな水平力をうける 1 クラスの高層架構の最小重量設計 I*

正 会 員 棚 橋 諒**

正 会 員 中 村 恒 善***

1. 序論 この論文の目的は、一つのクラスの外荷重の作用をうける高層架構の最小重量設計を非常に簡潔な解形式で一般的に確立することである。架構のすべての部材はそれぞれ一様な断面を有するものとし、そのすべての節点は接続する部材の全塑性モーメントを伝達するに十分な強度を有するとする。今、一つの高層架構の構造設計に際し、その各階高、及び張間長が与えられ、図 1 のように梁の鉛直荷重が単一集中荷重によって置換えられて梁スパン中央に、かつそれに比べて十分大なる水平力が各層床面に、準静的に比例載荷されるとして、それらの最大値が決定されているとする。この最大値の下で丁度崩壊点に達し、かつ架構構成材料の総重量が最小であるようにするには、どのように各部材の全塑性モーメントを選定すべきであるかという問題を考える。

この問題については、文献 [1]～[9] において種々の観点から論ぜられた。その LP による方法は十分一般で全く自動的ではあるが、計算が煩雑かつ大規模であるのみならず、単に問題ごとに数値解を与えるに過ぎず、高層架構の最小重量設計の一般的性質を明らかにし得ていない。さらに、線型最小重量設計理論の近似的性格や、重量超平面が接する部分の近傍における許容領域の偏平性から見て、大きな計算労力を要しないですむような一般解を確立する必要があると思われる。このような観点から本論は、この一般解を確立し、その性格、特徴を明らかにし、さらに若干の特別な種類の最小重量設計についても簡潔かつ explicit な解を見出そうとするものである。

2. 構モーメントと釣合条件式 横力の十分大なる場合を取扱うので、1 部材の曲げモーメントの最大値は必らずその両部材端に生じ、他のどの断面でもこれと同じ値に到達してよいと仮定する。梁に作用する鉛直荷重をそのスパン中央に作用する単一集中荷重によって置換えて考えるので、関節可能断面としては両梁端とスパン中央とだけ考えればよい。崩壊時に架構の全部材端に塑性関節が形成されてしまっているような極端な過完全崩壊機構に、図 2 のようにさらに若干のスパン中央に塑性関節が同時に形成されていてもよい、より極端な崩壊状態を想定する。このような状態では、未知の材端モー

メントの数は、その正負の符号さえ適正に選択すれば、部材数に等しい。まず水平力にのみ注目すれば、その選択は図 3 の M 図から容易になされ得るが、今この $f(2s-1)$ 個の部材の塑性モーメントを取扱う代りに、図 3 の M 図を図 4 のように分解した形で、「構モーメント」という概念を導入すると好都合である。変形の連続条件が不要で、塑性モーメントで表現され得る釣合条件式が線型であることから、崩壊時の M 図を図 4 のように fs 個の単位矩形に対する fs 個の単位 M 図に分解する。各単位 M 図として、図 4 に示す形式の、4 隅の M の絶対値が等しくて不連続がないもののみを考える。この時各単位 M 図はこの等しい隅モーメントの値で完全に定義されるので、これを「構モーメント」と称する。正の構モーメントとしてその矩形のある階に作用している層せん断力と同方向のせん断力に対応するものを選ぶ。原 M 図をこれら単位 M 図の合成と考えて構モーメントによって釣合条件式を表現すれば層方程式のみでよい。図 1 のように、 $M_{j,k}$ を j 階の左から k 番目の構モーメントとし、 j 階の左から k 番目の梁と柱をそれぞれ (j,k) -梁、 (j,k) -柱と呼び、それらの塑性モーメントをそれぞれ $B_{j,k}$, $C_{j,k}$ で表わす。 $B_{j,k}$ および $C_{j,k}$ と、 $M_{j,k}$ との関係は (1) 式で与えられ、節点方程式 (2) が $M_{j,k}$ によって満足されていることは直接の代入で確かめられる。かくて $M_{j,k}$ の導入により、 $f(2s+1)$ 個の未知量を fs 個に減少せしめ得るという利点もたらされた。各層について唯 1 個の層方程式を $M_{j,k}$ について書けば、(4) 式となる。所で材端塑性関節以外の如何なる断面においても降伏条件が満足されているためには、置換集中荷重による単純梁としての M 図が、図 3 の仮定 M 図に重ね合せられた時に得られる M 図で、如何なる断面の M もその部材の塑性モーメントを超えてはならない。逆に、与えられた鉛直荷重に対して、 $B_{j,k}$ がとり得る限界最小値が存在するはずである。この条件から $M_{j,k}$ が満足すべき不等式制約条件 (4a) が得られる。

3. 最小重量設計 (3) と (4) とを満足する如何なる M 分布も 1 個の完全解を与える。何となれば然る時には、すべての釣合条件は満足され、降伏条件も乱されていないし、かつ過完全崩壊機構に対応するからである。今 D を以て (3) と (4) とを満足する設計のすべての集合を表わす。また $\bar{D}$ を以て、 D 及びここに想定したごと

* 日本建築学会近畿支部昭和 38 及 39 年研究発表会にて発表

** 京都大学教授 工博 *** 京都大学助教授

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き過完全崩壊機構に対応しない設計の両者すべてを含む集合を表わす。然る時は最小重量設計は、次の一種の半逆解法の二段階を経て確立され得る。

(I) まず D の中の最小重量設計 $D_{\min}$ が同時に $\bar{D}$ の中でも最小重量設計を与えると仮定し、LPの手法で、(3) 及び (4) の制約条件の下に重量を最小ならしめる。

(II) 次にこの $D_{\min}$ に対応する過完全崩壊機構が実用上興味のあるクラスの架構について Foulkes の機構条件を満足することを証明する。然る時は $D_{\min}$ は最小重量設計の必要十分条件のすべてを満足し、(i) の仮定も正当化される。

かくて問題は、まず (5) 式で表現される線型重量関数 G を (3) と (4) の下に最小化することにある。スラック変数 $x_{j,k}$ を導入して (4a) を等式になおした形では、 $2fs$ 個の変数について (3) と (6) との $f(s+1)$ 個の制約条件がある。 $M_{j,k}$ を直交座標とする fs 次元の構モーメント空間においては、(3) は f 個の超平面の共通部分として $f(s-1)$ 次元空間を定義する。この $f(s-1)$ 次元空間のうち、(4) をすべて満足する部分が許容領域である。この許容領域の頂点は、(4a) の $f(s-1)$ 個の等式と残りの不等式を満足する点として定義される。このように定義される頂点のうちの特定の 1 個または数個が実際に、 fs -空間内で G を最小ならしめるような、この許容領域と重量超平面との接点または接点の集合であることが次のように証明される。最小梁長に対応する指標のすべての集合 N の中から 1 個の指標 m をとり、 $f(s-1)$ 個の非基本変数として $x_{j,k} (k \neq m)$ を選ぶ。基本変数 $M_{j,k}$ と $x_{j,m}$ とを $x_{j,k} (k \neq m)$ で表わせば式 (7) となり、対応する基本解は $X_{j,k}=0 (k \neq m)$ において得られる。この基本解は (8) の条件がすべて満足されている場合に基本可能解である。実際上興味あるクラスの P_i と $V_{j,k}$ の組合せについて、(8) が最上階で満足されている時、一般階でも満足される。最上階から数層だけ満足されない時は、その部分のみ塑性モーメント分配法で修正して最小重量設計の良好な上界が得られる。ここでは (8) の満足される場合を考え、(5) の表現を (7) に代入すると (9) 式が得られる。(9) の表現から、 N が m を唯 1 個含む時、最小解の構モーメントに対応する塑性モーメントの割当は (10) 式で表現されることがいえる。 m が 1 や s の時にも類似の表現が得られる。 N が 2 個以上の指標を含む時は最小基本解は唯一ではなく、指標数だけの頂点で重量超平面と接し、各頂点に対応する塑性モーメントの一次結合も最小解である。注目すべきは (10) において、 $l_k (k \neq m)$ なる長さの $f(s-1)$ 個の梁が各スパンに作用している集中荷重のみ支持するように設計され、 l_m なる最小スパン長の梁及びこれに連結する柱が横力の大部分を支持するように設計される時に D の中の $D_{\min}$ が達成されることである。

次に Foulkes の機構条件を考える。この問題には $f(2s+1)$ 個の独立変数 $B_{j,k}$ と $C_{j,k}$ が含まれているから、 $f(2s+1)$ 個の異なる崩壊機構を有するような最小重量設計が見出され得る。上に得られた解から見て、 $f(s-1)$ 個の梁中央断面で全塑性モーメントに達しているから、Fig. 5c に示すように $f(s-1)$ 個の梁中央断面で $f(s-1)$ 個の独立な関節回転角が存在し得る。これ以外に $f(s+1)$ 個の独立な節点回転角と、 f 個の異なる sway 機構が存在し、合計 $f(2s+1)$ 個となる。Foulkes の機構条件は次のように表現できる。

すべての梁及び柱について、 (j,k) -梁 (または (j,k) -柱) に形成される塑性関節回転角の総和
 = 定数 $\times$ (その (j,k) -梁または (j,k) -柱の長さ)
 という条件の満足されているような崩壊機構が上記の $f(2s+1)$ 個の関節回転角の適当な組合せによって見出されねばならない。…… [A]

まず j 階の柱がすべて同じ長さ h_j を有することに注目すれば、Fig. 5a のように j 階すべての柱の上下両端で θ_j なる等しい回転角の関係が形成されるような機構によって、 j 階のすべての柱の間で、条件 [A] は満足される。従って $f(s+1)$ 個の柱に形成される関節回転角の各柱についての $f(s+1)$ 個の総和の間の fs 個の比が条件 [A] を満足し、それらの間になお f 個の異なる総和のみが検討さるべく残っている。同様に Fig. 5d のようにすべての (j,k) -梁の両端に sway による節点回転角に等しいだけの関節回転角が形成されるならば、同じ指標 k を有する梁の組ごとに、条件 [A] が満足される。しかるに $l_k > l_m$ であるから、 (j,m) -梁以外のすべての (j,k) -梁でさらにスパン中央に塑性関節が形成されて、それらの部材の回転角の総和を増加せしめる必要がある。そこで同じ k -指標をもつすべての (j,k) -梁のスパン中央に等しい関節回転角 $2\varphi_k$ が形成されているような Fig. 5c に示す崩壊機構を考える。Fig. 5c はすべての異なる $k (k \notin N)$ について $2\varphi_k$ を考慮した崩壊機構である。 fs 個の梁の各々についての関節回転角の総和の中の $(f-1)s$ 個の比はかくて条件 [A] を満足し、残り s 個の総和の間の比について検討の余地を残すのみとなる。従って合計 $f+s$ 個の総和の間の比が [A] を満足するように、 θ_j と φ_k と ψ_i の計 $f+s$ 個の独立な回転角の組合を決定せねばならない。Fig. 5d はこの $f+s$ 個の回転角を含む過完全崩壊機構であって、この場合に形成される塑性関節回転角の各部材についての総和は、表 1 に示す通りである。回転角が (11) 式のように決定されれば、条件 [A] は満足される。(11) 式から見て、(12) を満足するサイズの架構において条件 [A] は満足され得る。かくて (12) を満足する架構に対する (10) で定義される設計が最小重量設計の 3 個の必要十分条件のすべてを満足することが証明され得た。