A PROBABILISTIC ANALYSIS OF LENGTH OF STAY IN HOSPITAL

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1. INTRODUCTION

1.0 Significance of Length of Stay in Hospital Planning

Throughout the world of today, an adequate provision of health services has been recognized as one of the essential community responsibilities. An emerging concept of a hospital as the center of the comprehensive health service system in the community has been displacing the old idea of a hospital as an institution specialized in care of the sick.¹ The change calls for a critical review of various indices used in the assessment of hospital performance. This is an essential step toward the rational planning of hospital buildings and management.

One of such indices is the length of stay in hospital. The index indicates the period in hospital per patient admitted, usually in terms of days. Both the day of admission and the day of discharge are counted according to the definition set by the Ministry of Welfare in Japan. The length of stay is related to the number of hospital beds in a manner expressed by the equation,

$$B = \frac{A \times L}{u} \dots\dots\dots (1)$$

where B = number of hospital beds

A = average number of admission per day

L = average length of stay in days and

u = bed-occupancy ratio.

Two points should be made on this equation. First, given an average number of admissions per day, a required number of hospital beds in the community changes in direct proportion to the average length of stay and in inverse proportion to the bed-occupancy ratio. This aspect alone would be enough to justify indepth studies of the length of stay in hospital. Second, the bed-occupancy ratio, which is often taken as an index of hospital efficiency, may be altered by the changing length of stay. If there is a big variation in the occurrence of cases requiring hospitalization, the bed-occupancy ratio must be kept low to make emergency admissions possible. Thus, a true assessment of hospital efficiency cannot be made without a full knowledge about the length of stay.

The significance of the length of stay as an index of hospital performance goes beyond these two points. A hospital in-patient service may be considered as a service process in the theory of queues in operations research. The length of stay corresponds to the duration of service or simply service time. A knowledge and understanding of the distribution of service time is a necessary prerequisite for the application of the theory to hospital management.

The index has many implications also for hospital architecture. For example, certain facilities in the hospital, such as diagnostic facilities, should be given a capacity in proportion to the number of new admissions rather than to the number of beds. A reduction in the length of stay is likely to increase the demand for such facilities, other things being equal.

1.1 Study objectives and method

The primary objective of the study is to develop a practical tool for the investigation into the

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1. R. Llewelyn-Davies, Baron and H.M.C. Macaulay. Hospital Planning and Administration, (W.H.O.: Geneva, 1966) Monograph series No. 54, pp. 23-24.

behaviour of the length of stay in hospital for the purposes mentioned earlier. The pioneering work was undertaken by professor Y. Yoshitake of the University of Tokyo and reported by him as early as in 1960.²⁾ He found that the distribution curve of the length of stay in hospital could be obtained by differentiating the distribution function of "time after admission" of patients currently hospitalized on a randomly selected survey day. He also found that the latter function could be approximated by a sum of two exponential functions each with a different mean. Two shortcomings of this method may be pointed out. One is the influence of fluctuations in admissions and the other is a need for a subjective judgment in determining parameters of the mixed exponential function. This present study is an attempt to improve upon the original method by formulating a model on the theory of probability and also by eliminating the element of arbitrariness from the process of curve fitting as much as feasible. The model will be tested against two sets of observed data for evaluation.

2. MODELS

2.0 Preliminaries

A hospital stay is a service process whereby a patient is admitted, receives services, and is eventually discharged. In this process, each patient has his characteristic random variable, l , called his length of stay. For a population of patients, we then define a probability density function of the length of stay by the equation,

$$f(x) = \lim_{\Delta x \rightarrow 0+} \frac{\text{prob}(x > l \leq x + \Delta x)}{\Delta x} \dots\dots\dots (2)$$

with

$$\int_0^{\infty} f(x) dx = 1. \dots\dots\dots (3)$$

The function is defined zero for negative x . The quantity $f(x)$ is a rate because its dimensions are probability divided by time. If we imagine a group of patients all admitted at the same time and discharged at different times, $f(x)$ would measure the mean rate at which their hospital stay is ending at time x .

As Yoshitake pointed out in his paper, however, a direct observation of the completed length of stay requires a collection of historical data over a long period of time. On the other hand, it is relatively easy to obtain the dates of admission of the patients staying in hospital on a particular survey day. It is also easy to obtain the past record of the number of admissions per day or some other time unit. A quantity obtained by dividing the number of patients admitted during a certain period in the past and still hospitalized on a survey day by the total number of new admissions during the corresponding period indicates the probability that the length of stay of this group of patients will be longer than the period they already spent in hospital. This quantity $F(x)$ is related to $f(x)$ by the equation,

$$F(x) = \int_x^{\infty} f(x) dx \dots\dots\dots (4)$$

It should be noted here that the shape of function $F(x)$ should be exactly the same as the frequency distribution of time after admission of the current patients in Yoshitake's paper, although dimensions are different, provided that there is no fluctuation in the number of admissions over time. However, there are occasions that this assumption may likely lead to an erroneous conclusion. One is the case of expanding hospitals with relatively long average length of stay, such as an expanding mental hospital. The simple frequency distribution of the time after admission is likely to overrepresent recent patients and distort the reality in this case. Another case is concerned with hospital services which experience a wide variation in admissions, such as obstetrics. The proposed probabilistic formulation permits us to eliminate these possible distortions by removing the influence of fluctuations in admissions.

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2. Yasumi Yoshitake. "On the length of Stay in Hospital Planning", Transactions of the Architectural Institute of Japan, No. 66, October 1960 and also "Planning of Long-term Care Medical Facilities", Transactions of the Architectural Institute of Japan, No. 71, April 1962.

In this paper, following the convention of the probability theory, we call $f(x)$ a probability density function and distinguish it from $F(x)$, a probability distribution function. The average length of stay, L , is calculated by the equation,

$$L = \int_0^{\infty} xf(x)dx = \int_0^{\infty} F(x)dx. \dots\dots\dots (5)$$

In other words, the area under curve $F(x)$ in Figure 1 turns out to equal the average length of stay in hospital.³ This relation holds regardless of the type of function. This definition of the average length of stay seems to be the most appropriate one particularly when the institution is undergoing a rapid expansion or contraction.

At this moment, it should be noted that $f(x)$ and $F(x)$ are mathematically equivalent. Therefore, any information concerning the length of stay may be obtained from either of the two functions. In fact, it is usually more convenient to work with $F(x)$ than $f(x)$ in the application of the theory of queues. This is another point in favour of the adoption of $F(x)$ as the basis for the investigation into the length of stay. Our next step is to analyse the actual form of $F(x)$ to gain insight into the operation of the service process.

Fig.-1 Density and Distribution Functions

2.1 Exponential Model

If the chance of occurrence of discharge of a patient is independent of the time he already spent in hospital, the theory of probability indicates that the probability density function of the length of stay is exponential, that is,

$$f(x) = \lambda e^{-\lambda x} \dots\dots\dots (6)$$

and therefore

$$F(x) = \int_x^{\infty} \lambda e^{-\lambda x} dx = e^{-\lambda x} \dots\dots\dots (7)$$

where λ denotes the probability of immediate discharge of the patient, called the rate parameter. The average length of stay in this case turns out to equal the reciprocal of the parameter λ .

Because of this normative characteristic of the exponential model, it is a good practice to plot the observed curve of $F(x)$ on a semi-logarithmic scale. A straight line corresponding to the average length of stay calculated by equation (5) may then be compared to the observed curve. If the two curves show a good fit, the service process can be assumed to be an exponential service channel. In the exponential distribution, the coefficient of variability, that is, a ratio of the standard deviation to the mean, should be unity.

2.2 Mixed-exponential Model

There are cases where the distribution shows a greater variability than the exponential channel, that is, its coefficient of variability is greater than unity. One interpretation of these cases is that the group of patients are composed of two or more homogenous sub-groups, each having an exponential distribution with a different mean. Its distribution curve shows a concave shape on the semi-logarithmic scale. This is exactly what Professor Yoshitake found applicable to many length of stay distributions.⁴ As he pointed out, it is usually sufficient for practical purposes to assume two such homogenous sub-groups, one with a short mean value and the other with a relatively long mean value.

We can visualize the mixed exponential model as an arrangement in parallel of channels with different service times. The theory of queues shows that a greater efficiency is obtained by creating two separate physical facilities for the two groups, if it is possible to classify the two groups in advance or shortly after admission.

The mixed exponential model involves three parameters; two mean values and a ratio between the

3. For proof, see Philip M. Morse, Queues, Inventories and Maintenance. (John Wiley and Sons: New York, 1958), p. 9

4. Yoshitake, op. cit.

number of patients in each group. Two obvious methods are, first, curve-fitting by hand and, second, least square approximation. The former was described in Professor Yoshitake's paper in detail and will not be discussed in this paper further. The latter is theoretically feasible by the application of the Gauss-Newton iteration method.⁵. However, the length of stay data is usually truncated or not very reliable due to the paucity of samples beyond a certain time span. The least square method does not necessarily yield a satisfactory solution in these cases. Thus in this paper, two other practical alternatives are proposed and discussed below.

a) Mean-variance Method

This method calculates the three parameters mentioned above from the mean of $f(x)$, and the mean and variance of $F(x)$. Suppose that the probability density function $f(x)$ is expressed by the equation,

$$f(x) = \theta \lambda_1 e^{-\lambda_1 x} + (1-\theta) \lambda_2 e^{-\lambda_2 x} \quad \lambda_1 > \lambda_2 \dots\dots\dots (8)$$

and the corresponding distribution function by the equation,

$$F(x) = \theta e^{-\lambda_1 x} + (1-\theta) e^{-\lambda_2 x} \dots\dots\dots (9)$$

The Laplace transforms of these two functions are respectively,

$$f^*(s) = \theta \left(\frac{\lambda_1}{\lambda_1 + s} \right) + (1-\theta) \left(\frac{\lambda_2}{\lambda_2 + s} \right) \dots\dots\dots (10)$$

and

$$F^*(s) = \frac{\theta}{\lambda_1 + s} + \frac{1-\theta}{\lambda_2 + s} \dots\dots\dots (11)$$

Except for the sign of s , the Laplace transform is the moment generating function. Thus by the Taylor expansion of (10) and (11), the mean of $f(x)$ and the mean and variance of $F(x)$, M and V , are derived as follows:

$$L = \frac{\theta}{\lambda_1} + \frac{1-\theta}{\lambda_2} \dots\dots\dots (12)$$

$$M = \frac{1}{L} \left(\frac{\theta}{\lambda_1^2} + \frac{1-\theta}{\lambda_2^2} \right) \dots\dots\dots (13)$$

$$V = \frac{2}{L} \left(\frac{\theta}{\lambda_1^3} + \frac{1-\theta}{\lambda_2^3} \right) - M^2 \dots\dots\dots (14)$$

The simultaneous equations (12), (13) and (14) can be solved explicitly. First we substitute $1/\lambda_1$ with p , and $1/\lambda_2$ with q and transform these equations to

$$a = \theta(p-q) + q = L \dots\dots\dots (15)$$

$$b = \theta(p^2 - q^2) + q^2 = LM \dots\dots\dots (16)$$

$$c = \theta(p^3 - q^3) + q^3 = \frac{L}{2}(M^2 + V) \dots\dots\dots (17)$$

Writing $p+q=u$, and $pq=v$, these equations can be solved, giving,

$$u = \frac{c-ab}{b-a^2} = \frac{M^2 + V - 2LM}{2(M-L)} \dots\dots\dots (18)$$

$$v = \frac{ac-b^2}{b-a^2} = \frac{L(V-M^2)}{2(M-L)} \dots\dots\dots (19)$$

We then derive the values p and q by solving the quadratic equation

$$X^2 - uX + v = 0 \dots\dots\dots (20)$$

The value of θ is then obtained by equation (15). This method yields uniquely the values for the three parameters of distribution by step-by-step calculation.

b) Percentage-point Method

The above mean-variance method is not applicable when the available data is severely truncated at the upper tail. For these cases, the percentage-point method produces a better result, although the solution may not be unique depending on the choice of points used for estimation.⁶ In this method

5. H.O. Hartley, "The Modified Gauss-Newton Method for Fitting of Non-Linear Regression Functions by Least Squares". *Technometrics*, Vol. 3, No. 2, May 1961.

6. D.J. Bartholomew, "Note on the Measurement and Prediction of Labor Turnover", *Journal of the Royal Statistical Society*, A. 122, 1959. pp. 232-239.

we first obtain from the observed data the values of $F(x)$ at three points t_1, t_2, t_3 so that

$$t_2 = 2t_1, t_3 = 6t_1 \dots\dots\dots (21)$$

Let $p = e^{-\lambda_1 t_1}, q = e^{-\lambda_2 t_1}$

then we have the equations

$$F(t_1) = \theta(p - q) + q \dots\dots\dots (22)$$

$$F(t_2) = \theta(p^2 - q^2) + q^2 \dots\dots\dots (23)$$

$$F(t_3) = \theta(p^6 - q^6) + q^6 \dots\dots\dots (24)$$

Writing $p + q = u$ and $pq = v$ these equations can be solved explicitly, giving

$$\theta = \frac{F(t_1) - q}{p - q} \dots\dots\dots (25)$$

$$v = F(t_1)u - F(t_2) \dots\dots\dots (26)$$

$$u^4 - 2F(t_1)u^3 + 3F(t_2)u^2 = \frac{F(t_3) - F(t_2)^3}{F(t_2) - F(t_1)^2} \dots\dots\dots (27)$$

Equation (27) can be solved by the Newton-Raphson method starting with unity for a value of u . In our experience, the best result is obtained when t_1 is taken in the neighborhood of the average length of stay. In this respect, it is suggested here that the data should cover the range at least six times longer than the expected average length of stay. The third point need not be at six times of t_1 , but any integer $k=3, 4, 5$, times t_1 with a corresponding change in equation (27).

2.3 Erlangian Model

In cases where the coefficient of variability is less than unity, that is, the distribution is less than pure random, Erlangian distributions often prove to be satisfactory. We can visualize the Erlangian model as an arrangement in series of exponential channels of identical means. The number of phases or channels in series is usually denoted by k and determines the shape of distributions curve. The Laplace transform, $f(x)$ and $F(x)$ of Erlangian distributions are as follows,

$$f^*(s) = \left(\frac{\rho}{\rho + s} \right)^k \dots\dots\dots (28)$$

$$f(x) = \rho(\rho x)^{k-1} \frac{e^{-\rho x}}{(k-1)!} \quad k=1, 2, 3, \dots \quad r=0, 1, 2, \dots \dots\dots (29)$$

$$F(x) = \sum_{r=0}^{k-1} \frac{e^{-\rho x} (\rho x)^r}{r!} \dots\dots\dots (30)$$

where ρ is the rate parameter of the exponential curve of each of the phases. The choice of k can be made by comparing successively the theoretical $F(x)$ curves calculated for various integer values $k=1, 2, 3, \dots$ with the observed curve of $F(x)$. The parameter ρ then is obtained by multiplying the reciprocal of the total length of stay by k , that is

$$\rho = \frac{k}{L} \dots\dots\dots (13)$$

The standard deviation and the coefficient of variability of this distribution are

$$\text{S.D.} = \frac{\sqrt{R}}{\rho}, \quad \text{C.V.} = \frac{1}{\sqrt{R}} \dots\dots\dots (32)$$

2.4 Markov Process Model

So far we have considered only the arrangement of channels in parallel, that is, mixed exponential, and the one in series, that is, Erlangian. The physical interpretation of these arrangements are shown in Figure II. A more complex model of any combination of these arrangements can be made by using the theory of Markov process. Loosely speaking, there are two conditions for its application. First, the definition of phases or steps should be mutually exclusive and unique. In other words, at any particular moment, a patient must occupy one and only one of these or states as they are called in the theory of Markov process. Second, the transition probabilities governing the change are determined entirely by the state occupied at the moment and not by his previous history. These two conditions are implicit also in the three types of models we have discussed.

A general formulation of the theory of Markov process is quite complex and cannot be made in this short paper⁷. We consider therefore a special combination of mixed-exponential and Erlangian, depicted as d) in Figure II. This is a four state Markov process in continuous time.

Suppose at any moment a patient can be uniquely classified to one of the following states:

State 1: Acute treatment stage subsequent to admission,

State 2: Short-term recuperative stage after treatment,

State 3: Long-term (chronic) recuperative stage after treatment,

State 4: Discharged either by recovery or by death.

If we denote the transition rate from State i to State j by r_{ij} the process is defined by the transition rate matrix Q such that

$$Q = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{pmatrix} \dots \dots \dots (33)$$

For the sake of simplicity, we assume that certain transitions cannot occur and use the following rate matrix for our study. It is reminiscent of the Erlangian model that the elements r_{11} and r_{22} are assumed to be equal.

$$Q = \begin{pmatrix} -\lambda_1 & \theta\lambda_1 & (1-\theta)\lambda_1 & 0 \\ 0 & -\lambda_1 & 0 & \lambda_1 \\ 0 & 0 & -\lambda_2 & \lambda_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \dots \dots \dots (34)$$

The duration of time patients spend from the entrance to State 1 to the entrance to State 4 is what we call the length of stay. This quantity is usually known as the first passage time in the theory of Markov process. In this model, there are two possible routes to proceed from State 1 to State 4. The route passing through State 2 can be considered a special Erlangian channel with $k=2$ having the Laplace transform of the probability density function of required passage time,

$$f_1^*(s) = \left(\frac{\lambda_1}{\lambda_1 + s} \right)^2 \dots \dots \dots (35)$$

On the other hand, the other route through State 3 is a generalized Erlangian channel. The corresponding Laplace transform of density function is

$$f_2^*(s) = \left(\frac{\lambda_1}{\lambda_1 + s} \right) \left(\frac{\lambda_2}{\lambda_2 + s} \right) \dots \dots \dots (36)$$

Considering the ratio of these two groups of patients, we then have the Laplace transform of the density function of the total length of stay distribution in the form

$$f^*(s) = \theta \left(\frac{\lambda_1}{\lambda_1 + s} \right)^2 + (1-\theta) \left(\frac{\lambda_1}{\lambda_1 + s} \right) \left(\frac{\lambda_2}{\lambda_2 + s} \right) \dots \dots \dots (37)$$

On inversion, we obtain the density function and the distribution function of the length of stay such that

$$f(x) = \theta \lambda_1^2 x e^{-\lambda_1 x} + (1-\theta) \frac{\lambda_1 \lambda_2 (e^{-\lambda_2 x} - e^{-\lambda_1 x})}{\lambda_1 - \lambda_2} \dots \dots \dots (38)$$

$$F(x) = \theta (1 + \lambda_1 x) e^{-\lambda_1 x} + (1-\theta) \frac{\lambda_1 e^{-\lambda_2 x} - \lambda_2 e^{-\lambda_1 x}}{\lambda_1 - \lambda_2} \dots \dots \dots (39)$$

The average length of stay of this model, obtained by integrating equation (39) expressed in the form

$$L = \frac{2\theta}{\lambda_1} + (1-\theta) \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} \dots \dots \dots (40)$$

In the following chapter, the models we have discussed will be tested against the actual data, with

Fig.-II Service Channel Arrangement

7. For example, D.R. Cox and H.D. Miller. The Theory of Stochastic Processes. (Methuen: London, 1965).

the exception of the exponential model. The estimation of the exponential model may be considered as a special case of the Erlangian model with $k=1$.

3. APPLICATION

3.0 Mixed-exponential Model

The two proposed methods of estimation have been used to test the mixed-exponential model against the data collected by Dr. Matsumoto. The data covers a total of 902 patients staying at three Workmen's Accident Compensation Hospitals, known as Rosai Byoin, in Tohoku, Kyushu and Ehime on a survey day in 1966.⁸ Since the information on the exact daily variation of the number of admissions was not included in the original study, the average number of 385 admissions per 50 days was used to calculate the observed values in Table I.

a) Mean-variance method

The average length of stay, mean and variance of the probability distribution function are calculated from the unpooled original data as follows:

$$L=336.7, M=535.6,$$

$$V=286,852.4 \dots\dots\dots(41)$$

By using equations (15) to (20), the parameters of functions are calculated as shown in Table I.

b) Percentage-Point Method

From the pooled data as shown in the second column of Table I, three readings at 75, 150 and 450 days are selected estimation. Thus it is not accidental that the calculated values at the three points are identical to the observed.

The results of the two methods are shown in Table I, together with the parameters given by Matsumoto in his paper for the purpose of comparison. Two sets of chi-square values are calculated from the result

Table I Observed and Calculated Probability Distributions of Mixed-exponential model (Workmen's Compensation Hospitals, 1966)

Time (days)	Observed	Mean-Variance Method	Percentage point method	Matsumoto
25	0.735	0.717	0.710	0.758
75	0.382	0.382	0.382	0.440
150	0.183	0.168	0.183	0.203
250	0.095	0.079	0.094	0.082
350	0.068	0.052	0.059	0.041
450	0.039	0.041	0.039	0.028
550	0.038	0.034	0.027	0.022
650	0.017	0.029	0.018	0.019
750	0.021	0.025	0.012	0.017
850	0.017	0.022	0.008	0.015
950	0.018	0.019	0.006	0.013
1 100	0.013	0.015	0.003	0.011
1 300	0.007	0.011	0.002	0.009
1 500	0.008	0.008	0.0007	0.007
1 700	0.009	0.006	0.0003	0.006
1 900	0.004	0.004	0.0001	0.005
2 100	0.004	0.003	—	0.004
2 300	0.004	0.002	—	0.003
2 500	0.004	0.002	—	0.003
θ	—	0.921	0.787	0.964
$1/\lambda_2$	—	69.1	59.2	87.0
$1/\lambda_1$	—	660.7	263.6	950.0
L (days)	116.1	116.1	102.8	118.1
Chi-squares up to 350	—	10.0	2.26	28.5
Total	—	24.9	∞	51.1

to compare the degree of fitness of the three curves. It should be noted that the statistical significance of these values is trivial as they are arbitrarily dependent upon the pooling intervals. They are used here just for comparative purposes only. The first set of chi-squares cover observations up to 350 days. It is clear the percentage point method yields better fit in this range, although its fitness at the upper tail declines. The second set of chi-squares cover the total observation. Here, the mean-variance method shows the best fit of the three. It is interesting to note the variation in the average length of stay of the long-term curve $1/\lambda_2$, suggesting a danger of giving too much confidence to this parameter.

3.1 Erlangian model

This and Markov process models are tested against the data obtained from Tokyo T.B. Hospital.⁹ The data includes the dates of admission of 1,200 patients staying at the hospital on a survey day in

8. A. Matsumoto, "Length of Stay and the Ward Composition of Workmen's Accident Compensation Hospitals", Transactions of the Architectural Institute of Japan, No. 124, August 1966. The author owes gratitude to Dr. Matsumoto for the permission to use his data in this paper.
9. The author wishes to express gratitude for the kind cooperation from Professor Y. Yoshitake, Mr. Isomura, and Mr. Yokobori with respect to the collection of the data.

Fig.-III Mixed-exponential Probability Distributions

Fig.-IV Erlangian and Markov Process Probability Distributions

October 1970 and also the number of daily admissions as far back as October 1963. The admission figures beyond the date are assumed to be the average of 1963 and 1964, namely, 1,000 per annum.

The estimation of this model is relatively simple. First, the scatter diagram of the observed data is compared to specimen curves in a textbook and the value 2 is chosen as the best fit. The average length of stay is obtained by equation (5). As shown in Table II the upper tail of this model reduces too fast, suggesting the following Markov process model.

3.2 Markov Process Model

As the form the distribution function of this model as expressed by equations (38) and (39) is rather complex, we have proceeded with its estimation inductively relying on the findings of previous studies, and arrived at a satisfactory result. It was reported by Yoshitake that he identified two homogeneous groups of T.B. patients, one having an average length of stay of about a year

and the other having the one of five years.¹⁰ Thus we have assigned the average of one year to the passage through State 2 and the one of five years to those who proceed through State 3. Since in our model we assume that the length of stay in State 1 is the same as the one in State 2, we have the average length of stay of half a year (180 days) for States 1 and 2, four and a half years (1,620 days) for State 3. The ratio parameter is then chosen to yield the observed average length of stay by using equation (40). The curve obtained by the function with these parameters follows the observed distribution quite well, as shown in the last column of Table II, suggesting a closeness of this model to the reality.

The observed and calculated values of probabilities in Tables I and II are illustrated in semi-logarithmic scale in Figures III and IV respectively.

4. CONCLUSION

To summarize, the study has found that

Table II Observed and Calculated Probability Distributions of Enlang ($k=2$) and Markov Process models (Tokyo T.B. Hospital, 1970)

Time (30 days)	Observed	Erlangian Model	Markov Process Model
1.5	0.981	0.986	0.976
4.5	0.852	0.898	0.841
7.5	0.786	0.774	0.674
10.5	0.600	0.643	0.520
13.5	0.474	0.521	0.394
16.5	0.231	0.414	0.297
19.5	0.268	0.324	0.225
22.5	0.210	0.251	0.174
27.0	0.116	0.168	0.122
33.0	0.066	0.096	0.084
42.0	0.052	0.040	0.058
54.0	0.028	0.012	0.042
66.0	0.045	0.003	0.033
78.0	0.022	0.0009	0.026
90.0	0.010	0.0003	0.021
180.0	0.003	—	0.004
L (30 days)	16.75	16.75	16.75
Parameters		$k=2$ $1/\rho=8.375$	$\theta=0.901$ $1/\lambda_1=6.0$ $1/\lambda_2=54.0$
Chi-squares up to 22.5	—	107.0	77.6
Total	—	∞	98.0

10. Yoshitake, *op. cit.*

- a) The frequency distribution of "time after admission" can be regarded as the probability distribution function with a slight modification in data collection based on a probabilistic interpretation of the service process:
 - b) Yoshitake's mixed exponential model may be generalized as a special arrangement of service channel in parallel:
 - c) Three parameters of the mixed exponential curve can be analytically calculated by using 1) Mean-variance method or 2) Percentage point method:
 - d) More complex model of service process can be made by using the theory of Markov process, and that
 - e) These models can be fitted to the actual data for a practical application.
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病院入院期間の確率論的考察(梗概)

正会員 谷 村 秀 彦*

1. ま え が き

1.0 病院計画における入院期間の意義

社会福祉の発展に伴い医療施設の整備が、社会全体の公共的責任として認識されてきている。従来の病院の私的性格に基いて設けられた病院計画の諸指標は、この観点から再検討されなければならない。入院期間は、このように再検討を必要としている指標の一つである。

入院期間(入院日から退院日までの日数、入院日退院日とも1日と数える) L と、病床数 B 、平均1日当り新入院数 A 、病床利用率 u との間には次の関係があることはよく知られている。

$$B=A \times L / u \quad \cdots \cdots (1)$$

病院計画の規模算定には、常にこの3つの指標を同時に考慮しなければならないことは、この式に明らかである。

1.1 研究の目的と方法

本研究の目的は、病院の入院期間を分析する実用的な方法を開発することにある。この分野における研究は東大吉武教授によって進められ既に1960年に発表がなされているが、この方法は、①新入院患者の経日的変動の影響を受けやすい、②分布曲線の近似に主観的判断が入る、など改良の余地があった。この研究は、入院期間を確率論的に捉えることによってこの方法を改善し、マルコフ過程の理論を適用することによって、理論を一步前進させようとする試みである。

2. 理 論

2.0 序 論

ある調査日において、 u 日前に新入院し調査日にまだ在院している患者数を、 u 日前のその日に新入院した全患者数で割った数値は、このグループの入院期間が u 日以上である確率と考えられる。この数値の分布を $F(x)$ 、入院期間の確率分布を $f(x)$ と表わすと、両者の間には

$$F(x) = \int_x^{\infty} f(x) dx, \int_0^{\infty} f(x) dx = 1 \quad \cdots \cdots (2)$$

の関係がある。関数 $f(x)$ は従って確率密度関数、関数 $F(x)$ は同一現象の確率分布関数である。 $F(x)$ を確率と考えることによって、新入院患者数の経日的変動の影響は除くことができる。平均入院期間は、Fig.-I に示

す通り、 $F(x)$ 曲線の下に面積に等しい。在院患者数が変動している時には、この方法が最も安定した平均入院期間の定義と考えられる。

$$L = \int_0^{\infty} x f(x) dx = \int_0^{\infty} F(x) dx \quad \cdots \cdots (3)$$

2.1 指数関数モデル

退院する確率が全くランダムである場合は、入院期間は次の分布を持つ。

$$f(x) = \lambda e^{-\lambda x}, F(x) = e^{-\lambda x}, L = \frac{1}{\lambda} \quad \cdots \cdots (4)$$

平均入院期間は、パラメーター λ の逆数である。いうまでもなく、指数関数はセミログ方眼紙で直線になる。

2.2 混合指数関数モデル

ある母集団の中に2つ以上の性格の異なるグループが存在する時には、その分布が指数関数の和で表わされることがある。このモデルは、Fig.-II b) に示すように、サービス、チャネルの平行配置と考えられる。曲線の近似は、ガラス、ニュートンの最小二乗法を用いることもできるが、曲線のすその分布が安定していない時はこの方法では解答が収束しない。代案として次の二通りの方法が考えられる。

a) 平均-分散法

混合指数関数モデルは次の分布を持つ。

$$\left. \begin{aligned} f(x) &= \theta \lambda_1 e^{-\lambda_1 x} + (1-\theta) \lambda_2 e^{-\lambda_2 x} \\ F(x) &= \theta e^{-\lambda_1 x} + (1-\theta) e^{-\lambda_2 x} \end{aligned} \right\} \quad \cdots \cdots (5)$$

観察された $F(x)$ の分布から、平均入院期間 L 、 $F(x)$ の平均値 M 、 $F(x)$ の分散、 V を計算する。これらの数値には、次の様な関係がある。

$$\left. \begin{aligned} L &= \frac{\theta}{\lambda_1} + \frac{(1-\theta)}{\lambda_2} \\ M &= \frac{1}{L} \left(\frac{\theta}{\lambda_1^2} + \frac{1-\theta}{\lambda_2^2} \right) \\ V &= \frac{2}{L} \left(\frac{\theta}{\lambda_1^3} + \frac{1-\theta}{\lambda_2^3} \right) - M^2 \end{aligned} \right\} \lambda_1 > \lambda_2 \quad \cdots \cdots (6)$$

この連立方程式は解があって、 $1/\lambda_1, 1/\lambda_2$ は、二次方程式

$$X^2 - \frac{M^2 + V - 2LM}{2(M-L)} X + \frac{L(V-M^2)}{2(M-L)} = 0 \quad \cdots (7)$$

の2根である。再分布の割合 θ は、従って

$$\theta = \left(L - \frac{1}{\lambda_2} \right) / \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \quad \cdots \cdots (8)$$

b) パーセント、ポイント法

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任意に t_1 を選び、3 点 $t_1, t_2=2t_1, t_3=6t_1$ における観察値を読み、この 3 点に適合する曲線のパラメーターを推定する。 $p=e^{-\lambda_1 t_1}, q=e^{-\lambda_2 t_1}$ とおくと

$$\left. \begin{aligned} F(t_1) &= \theta(p-q) + q \\ F(t_2) &= \theta(p^2 - q^2) + q^2 \\ F(t_3) &= \theta(p^6 - q^6) + q^6 \end{aligned} \right\} \dots\dots\dots (9)$$

さらに、 $u=p+q, v=pq$ とおくと、連立式 (9) は解があって、それは次の様に表わされる。

$$\left. \begin{aligned} \theta &= \frac{F(t_1) - q}{p - q} \\ v &= F(t_1)u - F(t_2) \\ u^4 - 2F(t_1)u^3 + 3F(t_2)u^2 &= \frac{F(t_3) - F(t_2)^3}{F(t_2) - F(t_1)^2} \end{aligned} \right\} \dots\dots\dots (10)$$

2.3 エアラング関数モデル

分布の偏差が全くのランダム以下である場合の分布は、エアラング関数で表わされることがある。このモデルは、Fig.-II c) のようにサービス、チャネルの直列配置と考えられる。チャネルの数を整数 $k=1, 2, 3$ とすると、この分布は

$$\left. \begin{aligned} f(x) &= \rho(\rho x)^{k-1} \frac{e^{-\rho x}}{(k-1)!} \\ F(x) &= \sum_{r=0}^{k-1} \frac{e^{-\rho x}(\rho x)^r}{r!} \\ L &= \frac{k}{\rho} \end{aligned} \right\} \dots\dots\dots (11)$$

で表わされる。

2.4 マルコフ過程モデル

上記の混合指数関数モデル、エアラング関数モデルは、単純指数モデルの 1 次元的な配置—平列と直列—であるが、マルコフ過程の理論を用いてもっと複雑な多次元的配置を考えることが可能である。ここでは、Fig.-II d) に示す 4-States のマルコフ過程をモデルとする。4 つの状態は次の様に定義される。

1. 入院直後の積極的治療
2. 1. の後の短期療養
3. 1. の後の長期療養
4. 死亡または退院

State i から State j への Transition rate を r_{ij} とすると、このマルコフ過程はマトリックス

$$Q = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ 0 & r_{22} & 0 & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \dots\dots\dots (12)$$

で定義される。特別の場合として $r_{11} = -\lambda_1, r_{12} = \theta\lambda_1, r_{13} = (1-\theta)\lambda_1, r_{22} = -\lambda_1, r_{24} = \lambda_1, r_{33} = -\lambda_2, r_{34} = \lambda_2$ を以下考察する。(行の総和は常に 0, また $\lambda_1 \neq \lambda_2$)

1, 2, 4 と進むルートの入院期間分布のラプラス変換は、

$$f_1^*(s) = \left(\frac{\lambda_1}{\lambda_1 + s} \right)^2 \dots\dots\dots (13)$$

同じく、1, 3, 4 と進むルートのラプラス変換は

$$f_2^*(s) = \left(\frac{\lambda_1}{\lambda_1 + s} \right) \left(\frac{\lambda_2}{\lambda_2 + s} \right) \dots\dots\dots (14)$$

従って、全体入院期間の密度分布のラプラス変換は

$$f^*(s) = \theta \left(\frac{\lambda_1}{\lambda_1 + s} \right)^2 + (1-\theta) \left(\frac{\lambda_1}{\lambda_1 + s} \right) \left(\frac{\lambda_2}{\lambda_2 + s} \right) \dots\dots (15)$$

すなわち、入院期間の分布は次の通りとなる。

$$\left. \begin{aligned} f(x) &= \theta\lambda_1^2 x e^{-\lambda_1 x} + (1-\theta) \frac{\lambda_1 \lambda_2 (e^{-\lambda_2 x} - e^{-\lambda_1 x})}{\lambda_1 - \lambda_2} \\ F(x) &= \theta(1 + \lambda_1 x) e^{-\lambda_1 x} + (1-\theta) \frac{\lambda_1 e^{-\lambda_2 x} - \lambda_2 e^{-\lambda_1 x}}{\lambda_1 - \lambda_2} \\ L &= \frac{2\theta}{\lambda_1} + (1-\theta) \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} \end{aligned} \right\} \dots\dots\dots (16)$$

3. 適 用

3.0 混合指数関数モデル

松本啓俊氏の好意により、1966 年に氏が発表された労災病院の在院期間分布の原資料 (学論 No. 124) が利用できたので、先に述べた 2 つの推定法を適用した。

a) 平均-分散法

原資料から $L=336.7, M=535.6, V=286, 852.4$ を計算し、式 (6), (7) を用いて Table I に示される数値を得た。新入院患者数変動は原資料に含まれていないので、平均値 385 人/50 日を用いた。

b) パーセント・ポイント法

75 日, 150 日, 450 日の 3 点の数値を読み、この 3 点に合う曲線を式 (9), (10) から求めた。

適合度を比較するために、350 日までと、全体の 2 種のカイ自乗を表に示すように算定した。パーセント・ポイント法は、もちろん、選択した点の近くでよく適合し、平均-分散法は全体としてよく適合している。松本氏の推定を含めて Table I, Fig.-III に結果を示した。

3.1 エアラング関数モデル

東京病院 (元東療清瀬) の 1970 年 10 月 21 日における在院患者 1,200 人の入院日と、1963 年までの新入院患者数変動が、吉武教授および磯村、横堀両氏の御好意により入手できたので、このモデルの適合性をテストした。 $k=2$ とし、平均値をあわせた分布を Table-II, Fig.-IV に示す。入院期間 3 年のあたりから、エアラング関数は観察値と著るしく違ってくる。

3.2 マルコフ過程モデル

上記の結果より、この資料はエアラングと混合指数のコンビネーションとみられそうなのでマルコフ過程モデルを適用した。吉武教授の 1962 年の発表にあるように (学論 No. 71), 短期 1 年, 長期 5 年の 2 グループを考え、それぞれ 6 か月, 6 か月, 54 か月の平均滞在期間を State 1, State 2, State 3 に対して想定し、平均入院期間が観察値と合致するよう θ を定め、数値を計算した。結果は、Table-II, Fig.-IV に示すごとく、非常に良好でこのモデルが極めて実際現象に近いことを示す。

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COST STUDY OF STRUCTURAL FRAME FOR SCHOOL BUILDINGS

(Part II)

—Prefabricated School Buildings—

by **AKIO MATSUMOTO**, Member of A.I.J.

A comparative study of the economy and the labor on-site of ALC (Autoclaved Lightweight Concrete) curtain wall system, HPC (Hshaped steel and precast concrete pannel) system and PC (precast concrete) pannel system, which are suitable for the prefabricated school systems, are made with brief structural calculations and cost estimates.

If 4-story schools are compared, the PC pannel system is 20% more expensive in comparison to the conventional concrete construction. And the ALC curtain wall system and the HPC system are approximately twice as much as the conventional method. The unit of labor on site, however, will be less in an order the conventional method, the ALC curtain wall system, the HPC system which is most prefabricated; therefore saves 35% labor on site in comparison to the conventional method.

With the change of education system which is apparant, the space and the use of classrooms will eventually change from the present standard. Rather than designing fixed classrooms whith the antiseismic wall like Type I, free open classrooms like Type II will be more adaptable to the future education.

From the educational and structural points of view, it is worse push forward the application of the ALC curtain wall system and the HPC systems of Type II as the future school building structures rather than cheaper PC pannel system of Type I.

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A PROBABILISTIC ANALYSIS OF LENGTH OF STAY IN HOSPITAL

(See Page 75)

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U.D.C. 64.01

A STUDY ON MAN-SPACE SYSTEM

by **RIYOZO NAKAMURA, HIROSHI WATANABE, HIROYUKI
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We have been studying human-space systems and have already conducted surveys, analyzed our data, and made predictions. The purpose of this part of our study, the fourth, is to attempt to deal with problems of systems evaluation.

The study has been conducted to identify inherent human factors affecting behavior in the space of the T Amusement Park; to establish some standard of value for these factors in our system through quantification; to thence evaluate human behavior; and finally, by feeding our findings back into the spatial plan, to make the human-space relationship a moer human as well as a more scientific one.