RELATIONS BETWEEN THE NODAL EXTERNAL FORCES AND THE NODAL DISPLACEMENTS ON THE BOUNDARY FRAMES OF RECTANGULAR ELASTIC FRAMED SHEAR WALLS PART IV

—Stiffness Matrices and Flexibility Matrices—

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PREFACE

As the pages are limited, the paper of this main title is separated into 5 Parts. The following Parts :

- Part I** Relations between the Nodal External Forces and the Representative Components of Their Fundamental Components (§1, §2)
- Part II** Relations between the Nodal Displacements and the Representative Components of Their Fundamental Components (§3, §4)
- Part III** General Relations between the Nodal External Forces and the Nodal Displacements (§5, §6)

were reported in the previous Transactions No. 237, 238 and 239 published in Nov., Dec. 1975 and Jan. 1976, and the following Part :

Part V A Numerical Example (§10), Conclusions, Notation and References will be published in the following Transactions.

The aim of this paper was described in §1 Introduction in Part I.

§7 Properties of the General Stiffness Matrix for the Nodes of the Boundary Frame of Framed Shear Walls

As discussed in §6, the general stiffness matrix $\bar{K}$ can be expressed in terms of the inverse matrix of the fundamental flexibility matrix F^* whose elements can be given analytically^{1~5)} or must be given experimentally (see Eq. 6.7).

The symbols $k_{m,n}$ of the elements of the general stiffness matrix $\bar{K}$ are defined as shown in Eq. 7.1 indicating $\bar{P}_\delta^* = \bar{K} \bar{\delta}$ (see Eq. 6.9) in detail. When the shape of the shear wall is symmetric with regard to its longitudinal and transversal center lines and the shear wall is an isotropic elastic body, these elements $k_{m,n}$ expressed in terms of the elements of the fundamental stiffness matrix K^* (see Eq. 5.5) are shown in Table 7.1, where the elements of K^* can be expressed in terms of the elements of F^* (see Eqs. 5.2, 5.5 and 5.6) whose elements can be given analytically^{1~5)} or also be given experimentally.

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$$\left\{ \begin{array}{l} X_{\delta i}^* \\ X_{\delta j}^* \\ X_{\delta k}^* \\ X_{\delta l}^* \\ Y_{\delta i}^* \\ Y_{\delta j}^* \\ Y_{\delta k}^* \\ Y_{\delta l}^* \\ M_{\delta i}^*/h \\ M_{\delta j}^*/h \\ M_{\delta k}^*/h \\ M_{\delta l}^*/h \end{array} \right\} = \frac{Et}{4} \left\{ \begin{array}{cccccccccccc} k_{1.1} & k_{1.2} & k_{1.3} & k_{1.4} & k_{1.5} & k_{1.6} & -k_{1.5} & -k_{1.6} & k_{1.9} & k_{1.10} & k_{1.11} & k_{1.12} \\ & k_{1.1} & k_{1.4} & k_{1.3} & -k_{1.6} & -k_{1.5} & k_{1.6} & k_{1.5} & k_{1.10} & k_{1.9} & k_{1.12} & k_{1.11} \\ & & k_{1.1} & k_{1.2} & -k_{1.5} & -k_{1.6} & k_{1.5} & k_{1.6} & -k_{1.11} & -k_{1.12} & -k_{1.9} & -k_{1.10} \\ & & & k_{1.1} & k_{1.6} & k_{1.5} & -k_{1.6} & -k_{1.5} & -k_{1.12} & -k_{1.11} & -k_{1.10} & -k_{1.9} \\ & & & & k_{5.5} & k_{5.6} & k_{5.7} & k_{5.8} & k_{5.9} & k_{5.10} & k_{5.11} & k_{5.12} \\ & & & & & k_{5.5} & k_{5.8} & k_{5.7} & -k_{5.10} & -k_{5.9} & -k_{5.12} & -k_{5.11} \\ & & & & & & k_{5.5} & k_{5.6} & -k_{5.11} & -k_{5.12} & -k_{5.9} & -k_{5.10} \\ & & & & & & & k_{5.5} & k_{5.12} & k_{5.11} & k_{5.10} & k_{5.9} \\ & & & & & & & & k_{9.9} & k_{9.10} & k_{9.11} & k_{9.12} \\ & & & & & & & & & k_{9.9} & k_{9.12} & k_{9.11} \\ & & & & & & & & & & k_{9.9} & k_{9.10} \\ & & & & & & & & & & & k_{9.9} \end{array} \right\} \left\{ \begin{array}{l} \bar{u}_i \\ \bar{u}_j \\ \bar{u}_k \\ \bar{u}_l \\ \bar{v}_i \\ \bar{v}_j \\ \bar{v}_k \\ \bar{v}_l \\ h\bar{\theta}_i \\ h\bar{\theta}_j \\ h\bar{\theta}_k \\ h\bar{\theta}_l \end{array} \right\} \dots\dots (7.1)$$

sym.

where

$k_{m,n}$: The elements of the general stiffness matrix $\bar{K}$ (for the four corner nodes of the boundary frame) of the shear wall (see Table 7.1)

Table 7.1 The elements $k_{m,n}$ of the general stiffness matrix $\bar{K}$ of symmetric shear walls, expressed in terms of the elements of the fundamental stiffness matrix K^* (see Eq. 5.5).

$k_{1.1}$	$\frac{a_{2.2}}{D_i} + \left(\frac{B_{1.1}}{D_{II}} + \frac{d_{2.2}}{D_{IV}} \right)$	$k_{1.2}$	$\frac{a_{2.2}}{D_i} - \left(\frac{B_{1.1}}{D_{II}} + \frac{d_{2.2}}{D_{IV}} \right)$	$k_{1.3}$	$-\frac{a_{2.2}}{D_i} - \left(\frac{B_{1.1}}{D_{II}} - \frac{d_{2.2}}{D_{IV}} \right)$	$k_{1.4}$	$-\frac{a_{2.2}}{D_i} + \left(\frac{B_{1.1}}{D_{II}} - \frac{d_{2.2}}{D_{IV}} \right)$
$k_{1.5}$	$-\frac{a_{1.2}}{D_i} + \frac{B_{1.2}}{D_{II}}$	$k_{1.6}$	$\frac{a_{1.2}}{D_i} + \frac{B_{1.2}}{D_{II}}$				
$k_{1.9}$	$-\frac{1}{2D_i}(a_{2.2} + \lambda a_{1.2}) + \left(\frac{B_{1.3}}{D_{II}} - \frac{d_{1.2}}{D_{IV}} \right)$	$k_{1.10}$	$-\frac{1}{2D_i}(a_{2.2} + \lambda a_{1.2}) - \left(\frac{B_{1.3}}{D_{II}} - \frac{d_{1.2}}{D_{IV}} \right)$	$k_{1.11}$	$-\frac{1}{2D_i}(a_{2.2} + \lambda a_{1.2}) + \left(\frac{B_{1.3}}{D_{II}} + \frac{d_{1.2}}{D_{IV}} \right)$	$k_{1.12}$	$-\frac{1}{2D_i}(a_{2.2} + \lambda a_{1.2}) - \left(\frac{B_{1.3}}{D_{II}} + \frac{d_{1.2}}{D_{IV}} \right)$
$k_{5.5}$	$\frac{a_{1.1}}{D_i} + \left(\frac{B_{2.2}}{D_{II}} + \frac{c_{2.2}}{D_{III}} \right)$	$k_{5.6}$	$-\frac{a_{1.1}}{D_i} + \left(\frac{B_{2.2}}{D_{II}} - \frac{c_{2.2}}{D_{III}} \right)$	$k_{5.7}$	$-\frac{a_{1.1}}{D_i} - \left(\frac{B_{2.2}}{D_{II}} - \frac{c_{2.2}}{D_{III}} \right)$	$k_{5.8}$	$\frac{a_{1.1}}{D_i} - \left(\frac{B_{2.2}}{D_{II}} + \frac{c_{2.2}}{D_{III}} \right)$
$k_{5.9}$	$\frac{1}{2D_i}(a_{1.2} + \lambda a_{1.1}) + \left(\frac{B_{2.3}}{D_{II}} - \frac{c_{1.2}}{D_{III}} \right)$	$k_{5.10}$	$\frac{1}{2D_i}(a_{1.2} + \lambda a_{1.1}) - \left(\frac{B_{2.3}}{D_{II}} + \frac{c_{1.2}}{D_{III}} \right)$	$k_{5.11}$	$\frac{1}{2D_i}(a_{1.2} + \lambda a_{1.1}) + \left(\frac{B_{2.3}}{D_{II}} + \frac{c_{1.2}}{D_{III}} \right)$	$k_{5.12}$	$\frac{1}{2D_i}(a_{1.2} + \lambda a_{1.1}) - \left(\frac{B_{2.3}}{D_{II}} - \frac{c_{1.2}}{D_{III}} \right)$
$k_{9.9}$	$\frac{1}{4D_i}(a_{2.2} + \lambda a_{1.2}) + \frac{\lambda}{4D_i}(a_{1.2} + \lambda a_{1.1}) + \frac{B_{3.3}}{D_{II}} + \left(\frac{c_{1.1}}{D_{III}} + \frac{d_{1.1}}{D_{IV}} \right)$	$k_{9.10}$	$\frac{1}{4D_i}(a_{2.2} + \lambda a_{1.2}) + \frac{\lambda}{4D_i}(a_{1.2} + \lambda a_{1.1}) - \frac{B_{3.3}}{D_{II}} + \left(\frac{c_{1.1}}{D_{III}} - \frac{d_{1.1}}{D_{IV}} \right)$	$k_{9.11}$	$\frac{1}{4D_i}(a_{2.2} + \lambda a_{1.2}) + \frac{\lambda}{4D_i}(a_{1.2} + \lambda a_{1.1}) + \frac{B_{3.3}}{D_{II}} - \left(\frac{c_{1.1}}{D_{III}} + \frac{d_{1.1}}{D_{IV}} \right)$	$k_{9.12}$	$\frac{1}{4D_i}(a_{2.2} + \lambda a_{1.2}) + \frac{\lambda}{4D_i}(a_{1.2} + \lambda a_{1.1}) - \frac{B_{3.3}}{D_{II}} - \left(\frac{c_{1.1}}{D_{III}} - \frac{d_{1.1}}{D_{IV}} \right)$

The general stiffness matrix $\bar{K}$ has the following properties (see the numerical example mentioned in Eq. 10.9).

- 1) Each sum of all elements corresponding to the nodal external balancing forces $\bar{X}_{\delta}^*$ or $\bar{Y}_{\delta}^*$ in any column of $\bar{K}$ is zero, therefore the equilibrium for the forces in the $\bar{x}$ and $\bar{y}$ -directions is identically satisfied. However, the sum of all elements corresponding to the nodal external moments $\bar{M}_{\delta}^*$ in any column of $\bar{K}$ is not zero, because the equilibrium for moment cannot be satisfied unless $\bar{X}_{\delta}^*$ and $\bar{Y}_{\delta}^*$ are taken into consideration. But here, the sum of the elements corresponding to all $\bar{M}_{\delta}^*$, the modified elements $k_{m,n}/2$ corresponding to $\bar{X}_{\delta i}^*$ and $\bar{X}_{\delta j}^*$, the modified elements

$-k_{m,n}/2$ corresponding to $\bar{X}_{\delta k}^*$ and $\bar{X}_{\delta l}^*$, the modified elements $-\lambda k_{m,n}/2$ corresponding to $\bar{Y}_{\delta i}^*$ and $\bar{Y}_{\delta l}^*$, and the modified elements $\lambda k_{m,n}/2$ corresponding to $\bar{Y}_{\delta j}^*$ and $\bar{Y}_{\delta k}^*$, in any column of $\bar{K}$, is zero. Therefore the equilibrium for moment of nodal external balancing forces $\bar{X}_{\delta}^*$, $\bar{Y}_{\delta}^*$, $\bar{M}_{\delta}^*$ is identically satisfied.

- 2) Each sum of all elements corresponding to the nodal displacements $\bar{u}$ or $\bar{v}$ in any row of $\bar{K}$ is zero, therefore the nodal external forces and moments $\bar{X}_{\delta}^*$, $\bar{Y}_{\delta}^*$, $\bar{M}_{\delta}^*$ are not affected by the nodal displacements $\bar{u}$ which equal the linear rigid-body motion u° of the shear wall, or the nodal displacements $\bar{v}$ which equal the linear rigid-body motion v° of the shear wall. However the sum of all the elements corresponding to the nodal rotational angles $\bar{\theta}$ in any row of $\bar{K}$ is not zero, because the nodal displacements :

$$\bar{u}_i = \bar{u}_j = -\bar{u}_k = -\bar{u}_l = \frac{h}{2} \theta^\circ, \quad \bar{v}_i = -\bar{v}_j = -\bar{v}_k = \bar{v}_l = -\frac{l}{2} \theta^\circ$$

together with the nodal rotational angles $\bar{\theta}$ which equal θ° are caused by the small rotational rigid-body motion θ° of the shear wall (see Fig. 3.2). But here, the sum of the elements corresponding to all $\bar{\theta}$, the modified elements $k_{m,n}/2$ corresponding to $\bar{u}_i$ and $\bar{u}_j$, the modified elements $-k_{m,n}/2$ corresponding to $\bar{u}_k$ and $\bar{u}_l$, the modified elements $-\lambda k_{m,n}/2$ corresponding to $\bar{v}_i$ and $\bar{v}_l$, and the modified elements $\lambda k_{m,n}/2$ corresponding to $\bar{v}_j$ and $\bar{v}_k$, in any row of $\bar{K}$, is zero. Here, the rotational rigid-body motion θ° must be small, because this analysis is based on a linear small displacement theory. Therefore $\bar{X}_{\delta}^*$, $\bar{Y}_{\delta}^*$ and $\bar{M}_{\delta}^*$ are not affected by the small rotational rigid-body motion θ° . Hence, the following Eq. 7.2 is induced.

$$\bar{K} \bar{\delta}^\circ = 0 \dots\dots\dots (7.2)$$

- 3) $\bar{K}$ is a symmetric matrix, therefore the reciprocal theorem can be applied if the nodal boundary conditions which definitely decide the nodal displacements are given.
- 4) When the descriptions in 1), 2) and 3) are taken into consideration, three rows (columns), each respectively corresponding to $\bar{X}_{\delta}^*$, $\bar{Y}_{\delta}^*$ and $\bar{M}_{\delta}^*$ ($\bar{u}$, $\bar{v}$ and $\bar{\theta}$) or one corresponding to $\bar{X}_{\delta}^*$ ($\bar{u}$) and two corresponding to $\bar{Y}_{\delta}^*$ ($\bar{v}$) whose lines of action do not coincide or two corresponding to $\bar{X}_{\delta}^*$ ($\bar{u}$) whose lines of action do not coincide and one corresponding to $\bar{Y}_{\delta}^*$ ($\bar{v}$), are dependent rows (columns). Consequently the determinant of the matrix $\bar{K}$ is zero.

$$|\bar{K}| = 0 \dots\dots\dots (7.3)$$

Hence, $\bar{K}$ is singular, and the general flexibility matrix $\bar{F}$ mentioned in §8 cannot be derived by inverting the general stiffness matrix $\bar{K}$. But if stable nodal support conditions which give some nodal displacements are imposed on the shear wall, the conditional flexibility matrix can be given by inverting the conditional stiffness matrix.

§8 Properties of the General Flexibility Matrix for the Nodes of the Boundary Frame of Framed Shear Walls

As discussed in §6, the general flexibility matrix $\bar{F}$ can be expressed in terms of the fundamental flexibility matrix F^* whose elements can be given analytically¹⁻⁵⁾ or must be given experimentally (see Eq. 6.20).

The symbols $f_{m,n}$ of the elements of the general flexibility matrix $\bar{F}$ are defined as shown in Eq. 8.1 indicating $\bar{\delta}_P^* = \bar{F} \bar{P}$ (see Eq. 6.21) in detail. When the shape of the shear wall is symmetric with regard to its longitudinal and transversal center lines and the shear wall is an isotropic elastic body, these elements $f_{m,n}$ expressed in terms of the fundamental flexibility matrix F^* (see Eq. 5.2) are shown in Table 8.1, where the elements of F^* can be given analytically¹⁻⁵⁾ or also be given experimentally.

$$\left\{ \begin{array}{l} \bar{u}_{\rho i}^* \\ \bar{u}_{\rho j}^* \\ \bar{u}_{\rho k}^* \\ \bar{u}_{\rho l}^* \\ \bar{v}_{\rho i}^* \\ \bar{v}_{\rho j}^* \\ \bar{v}_{\rho k}^* \\ \bar{v}_{\rho l}^* \\ h\bar{\theta}_{\rho i}^* \\ h\bar{\theta}_{\rho j}^* \\ h\bar{\theta}_{\rho k}^* \\ h\bar{\theta}_{\rho l}^* \end{array} \right\} = \frac{1}{4Et} \left\{ \begin{array}{cccccccccccc} f_{1.1} & f_{1.2} & f_{1.3} & f_{1.4} & f_{1.5} & f_{1.6} & -f_{1.5} & -f_{1.6} & f_{1.9} & -f_{1.9} & f_{1.11} & -f_{1.11} \\ & f_{1.1} & f_{1.4} & f_{1.3} & -f_{1.6} & -f_{1.5} & f_{1.6} & f_{1.5} & -f_{1.9} & f_{1.9} & -f_{1.11} & f_{1.11} \\ & & f_{1.1} & f_{1.2} & -f_{1.5} & -f_{1.6} & f_{1.5} & f_{1.6} & -f_{1.11} & f_{1.11} & -f_{1.9} & f_{1.9} \\ & & & f_{1.1} & f_{1.6} & f_{1.5} & -f_{1.6} & -f_{1.5} & f_{1.11} & -f_{1.11} & f_{1.9} & -f_{1.9} \\ & & & & f_{5.5} & f_{5.6} & f_{5.7} & f_{5.8} & f_{5.9} & f_{5.10} & -f_{5.10} & -f_{5.9} \\ & & & & & f_{5.5} & f_{5.8} & f_{5.7} & -f_{5.10} & -f_{5.9} & f_{5.9} & f_{5.10} \\ & & & & & & f_{5.5} & f_{5.6} & f_{5.10} & f_{5.9} & -f_{5.9} & -f_{5.10} \\ & & & & & & & f_{5.5} & -f_{5.9} & -f_{5.10} & f_{5.10} & f_{5.9} \\ & & & & & & & & f_{9.9} & f_{9.10} & f_{9.11} & f_{9.12} \\ & & & & & & & & & f_{9.9} & f_{9.12} & f_{9.11} \\ & & & & & & & & & & f_{9.9} & f_{9.10} \\ & & & & & & & & & & & f_{9.9} \end{array} \right\} \left\{ \begin{array}{l} \bar{X}_i \\ \bar{X}_j \\ \bar{X}_k \\ \bar{X}_l \\ \bar{Y}_i \\ \bar{Y}_j \\ \bar{Y}_k \\ \bar{Y}_l \\ \bar{M}_i/h \\ \bar{M}_j/h \\ \bar{M}_k/h \\ \bar{M}_l/h \end{array} \right\} \dots (8.1)$$

where

$f_{m,n}$: The elements of the general flexibility matrix $\bar{F}$ (for the four corner nodes of the boundary frame) of the shear wall (see Table 8.1)

Table 8.1 The elements $f_{m,n}$ of the general flexibility matrix $\bar{F}$ of symmetric shear walls, expressed in terms of the elements of the fundamental flexibility matrix F^* .

$f_{1.1}$	$a_{1.1} + (b_{1.1} + d_{1.1})$	$f_{1.2}$	$a_{1.1} - (b_{1.1} + d_{1.1})$	$f_{1.3}$	$-a_{1.1} - (b_{1.1} - d_{1.1})$	$f_{1.4}$	$-a_{1.1} + (b_{1.1} - d_{1.1})$
$f_{1.5}$	$a_{1.2} + b_{1.2}$	$f_{1.6}$	$-a_{1.2} + b_{1.2}$				
$f_{1.9}$	$b_{1.3} + d_{1.2}$	$f_{1.11}$	$b_{1.3} - d_{1.2}$				
$f_{5.5}$	$a_{2.2} + (b_{2.2} + c_{1.1})$	$f_{5.6}$	$-a_{2.2} + (b_{2.2} - c_{1.1})$	$f_{5.7}$	$-a_{2.2} - (b_{2.2} - c_{1.1})$	$f_{5.8}$	$a_{2.2} - (b_{2.2} + c_{1.1})$
$f_{5.9}$	$b_{2.3} + c_{1.2}$	$f_{5.10}$	$-b_{2.3} + c_{1.2}$				
$f_{9.9}$	$b_{3.3} + (c_{2.2} + d_{2.2})$	$f_{9.10}$	$-b_{3.3} + (c_{2.2} - d_{2.2})$	$f_{9.11}$	$b_{3.3} - (c_{2.2} + d_{2.2})$	$f_{9.12}$	$-b_{3.3} - (c_{2.2} - d_{2.2})$

The general flexibility matrix $\bar{F}$ has the following properties (see the numerical example mentioned in Eq. 10.11).

- 1) Each sum of all the elements corresponding to the nodal external forces $\bar{X}$ or $\bar{Y}$ or $\bar{M}$ in any row of $\bar{F}$ is zero, therefore the nodal displacements $\bar{\delta}_P^*$ due to the deformation of the shear wall subjected to the nodal external balancing components $\bar{P}^*$ are not affected by the nodal unbalancing components $\bar{P}^0$ which equal X^0_{III} or Y^0_{IV} or M^0_I (see Figs. 2.6, 2.8 and 2.3) and which satisfy a balancing force system together with a corresponding component of the resultant of the distributed loads. Hence, the following Eq. 8.2 is induced.

$$\bar{F}\bar{P}^0 = 0 \dots (8.2)$$

- 2) Each sum of all the elements corresponding to the nodal displacements $\bar{u}_P^*$ or $\bar{v}_P^*$ or $\bar{\theta}_P^*$ in any column of $\bar{F}$ is zero, therefore $\sum \bar{u}_P^*$, $\sum \bar{v}_P^*$ and $\sum \bar{\theta}_P^*$ of the nodal displacements due to deformation of the shear wall are identically equal to zero (see §3).
- 3) $\bar{F}$ is a symmetric matrix, therefore the reciprocal theorem can be applied if the nodal boundary conditions which definitely decide the nodal displacements are given.
- 4) When the descriptions in 1), 2) and 3) are taken into consideration, three rows (columns) each respectively corresponding to $\bar{u}_P^*$, $\bar{v}_P^*$ and $\bar{\theta}_P^*$ ($\bar{X}$, $\bar{Y}$ and $\bar{M}$) are dependent rows (columns). Consequently the determinant of the matrix $\bar{F}$ is zero.

$$|\bar{F}| = 0 \dots (8.3)$$

Hence, $\bar{F}$ is singular, and the general stiffness matrix $\bar{K}$ mentioned in §7 cannot be derived by

inverting the general flexibility matrix $\bar{F}$. But if stable nodal support conditions which give some nodal displacements are imposed on the shear wall, the conditional stiffness matrix can be given by inverting the conditional flexibility matrix.

§9 Relations between the Stiffness Matrix and the Flexibility Matrix

When the relations $\bar{P} = \bar{P}^* + \bar{P}^0$ and $\bar{F}\bar{P}^0 = 0$ given by Eqs. 2.1 and 8.2 are taken into consideration, Eq. 6.21 can be rewritten as Eq. 9.1.

$$\bar{\delta}_P^* = \bar{F}\bar{P} = \bar{F}\bar{P}^* \quad (9.1)$$

Hence, the components $\bar{\delta}_P^*$ are defined as the components due to the deformation of the shear wall subjected to the balancing components $\bar{P}^*$ only, and the following Eq. 9.2 is obtained.

$$\bar{P}^* = \bar{K}\bar{\delta}_P^* \quad (9.2)$$

When the relation $\bar{\delta}_P^* = \bar{F}\bar{P}$ given by Eq. 6.21 is substituted into Eq. 9.2 the following Eq. 9.3 is derived.

$$\bar{P}^* = \bar{K}\bar{F}\bar{P} \quad (9.3)$$

If Eq. 9.3 is equated to the relation $\bar{P}^* = {}_1T_{P12}T_{P1}\bar{P}$ given by Eq. 2.12, the product of the general stiffness matrix $\bar{K}$ and general flexibility matrix $\bar{F}$ can be given by Eq. 9.4.

$$\bar{K}\bar{F} = {}_1T_{P12}T_{P1} \quad (9.4)$$

If the relation ${}_1T_{P12}T_{P1} + {}_1T_{P02}T_{P0} = I$ mentioned by Eq. 4.1 is taken into consideration, the Eq. 9.4 can be rewritten as Eq. 9.5.

$$\bar{K}\bar{F} = I - {}_1T_{P02}T_{P0} \quad (9.5)$$

When the relations $\bar{\delta} = \bar{\delta}^* + \bar{\delta}^0$ and $\bar{K}\bar{\delta}^0 = 0$ given by Eqs. 3.1 and 7.2 are taken into consideration, Eq. 6.9 can be rewritten as Eq. 9.6.

$$\bar{P}_\delta^* = \bar{K}\bar{\delta} = \bar{K}\bar{\delta}^* \quad (9.6)$$

Hence, the components $\bar{P}_\delta^*$ are defined as the components which cause the components $\bar{\delta}^*$ due to the deformation of the shear wall, and the following Eq. 9.7 is obtained.

$$\bar{\delta}^* = \bar{F}\bar{P}_\delta^* \quad (9.7)$$

When the relation $\bar{P}_\delta^* = \bar{K}\bar{\delta}$ given by Eq. 6.9 is substituted into Eq. 9.7, the following Eq. 9.8 is derived.

$$\bar{\delta}^* = \bar{F}\bar{K}\bar{\delta} \quad (9.8)$$

If Eq. 9.8 is equated to the relation $\bar{\delta}^* = {}_1T_{\delta12}T_{\delta1}\bar{\delta}$ given by Eq. 3.12, the product of the general flexibility matrix $\bar{F}$ and general stiffness matrix $\bar{K}$ can be given by Eq. 9.9.

$$\bar{F}\bar{K} = {}_1T_{\delta12}T_{\delta1} \quad (9.9)$$

If the relation ${}_1T_{\delta12}T_{\delta1} + {}_1T_{\delta02}T_{\delta0} = I$ mentioned by Eq. 4.9 is taken into consideration, the Eq. 9.9 can be rewritten as Eq. 9.10.

$$\bar{F}\bar{K} = I - {}_1T_{\delta02}T_{\delta0} \quad (9.10)$$

The products ${}_1T_{P12}T_{P1}$ in Eq. 9.4, ${}_1T_{P02}T_{P0}$ in Eq. 9.5, ${}_1T_{\delta12}T_{\delta1}$ in Eq. 9.9 and ${}_1T_{\delta02}T_{\delta0}$ in Eq. 9.10 are not identity matrix (see Eqs. 2.14, 2.15, 3.14 and 3.15), therefore the products $\bar{K}\bar{F}$ and $\bar{F}\bar{K}$ given by Eqs. 9.4, 9.5, 9.9 and 9.10 are neither identity matrix nor null matrix, and $\bar{K}$ and $\bar{F}$ cannot be given as inverse matrix of $\bar{F}$ and $\bar{K}$ whose determinants are zero (see Eqs. 7.3 and 8.3).

If it is taken into consideration that $\bar{K}$ and $\bar{F}$ are symmetric matrix, the Eq. 9.11 is derived.

$$\bar{K}\bar{F} = \bar{K}^t\bar{F}^t = [\bar{F}\bar{K}]^t \quad (9.11)$$

When each side of Eq. 6.20 is premultiplied by the transformation matrix ${}_2T_{\delta1}$ to transform $\bar{\delta}$ into $\bar{\delta}^*$ (see Eqs. 3.8, 3.7 and 3.6) and is postmultiplied by the transformation matrix ${}_1T_{P1}$ to transform $\bar{P}^*$ into $\bar{P}$ (see Eqs. 2.2, 2.4 and 2.5), the following Eq. 9.12 is derived.

$${}_2T_{\delta1}\bar{F}{}_1T_{P1} = {}_2T_{\delta1}{}_1T_{\delta1}\bar{F}{}_1T_{P1}{}_1T_{P1} \quad (9.12)$$

When the relations ${}_2T_{P1}{}_1T_{P1} = I$ and ${}_2T_{\delta1}{}_1T_{\delta1} = I$ given by Eqs. 4.5 and 4.13 are taken into consideration, the Eq. 9.12 can be rewritten as Eq. 9.13.

$$\bar{F}^* = {}_2T_{\delta1}\bar{F}{}_1T_{P1} \quad (9.13)$$

When each side of Eq. 6.7 is premultiplied by the transformation matrix ${}_2T_{P1}$ to transform $\bar{P}$

into P^* (see Eqs. 2.8, 2.6 and 2.7) and is postmultiplied by the transformation matrix ${}_1T_{\delta 1}$ to transform δ^* into $\bar{\delta}^*$ (see Eq. 3.2, 3.4 and 3.5), the following Eq. 9.14 is derived.

$${}_2T_{P1} \bar{K} {}_1T_{\delta 1} = {}_2T_{P1} {}_1T_{P1} K^* {}_2T_{\delta 1} {}_1T_{\delta 1} \quad (9.14)$$

When the relations ${}_2T_{P1} {}_1T_{P1} = I$ and ${}_2T_{\delta 1} {}_1T_{\delta 1} = I$ given by Eqs. 4.5 and 4.13 are taken into consideration, the Eq. 9.14 can be rewritten as Eq. 9.15.

$$K^* = {}_2T_{P1} \bar{K} {}_1T_{\delta 1} \quad (9.15)$$

When F^* given by Eq. 9.13 is substituted into the relation $\bar{K} = {}_1T_{P1} \bar{F}^{*-1} {}_2T_{\delta 1}$ given by Eq. 6.7, the general stiffness matrix $\bar{K}$ expressed in the general flexibility matrix $\bar{F}$ is given by Eq. 9.16.

$$\bar{K} = {}_1T_{P1} [{}_2T_{\delta 1} \bar{F} {}_1T_{P1}]^{-1} {}_2T_{\delta 1} \quad (9.16)$$

When the relation ${}_1T_{P1} = {}_4{}_2T_{\delta 1}$ given by Eq. 4.18 is taken into consideration, the Eq. 9.16 is rewritten as Eqs. 9.17 and 9.18.

$$\bar{K} = {}_2T_{\delta 1}^t [{}_2T_{\delta 1} \bar{F} {}_2T_{\delta 1}^t]^{-1} {}_2T_{\delta 1} \quad (9.17)$$

$$= {}_1T_{P1} [{}_1T_{P1}^t \bar{F} {}_1T_{P1}]^{-1} {}_1T_{P1}^t \quad (9.18)$$

When K^* given by Eq. 9.15 is substituted into the relation $\bar{F} = {}_1T_{\delta 1} K^{*-1} {}_2T_{P1}$ given by Eq. 6.20, the general flexibility matrix $\bar{F}$ expressed in the general stiffness matrix $\bar{K}$ is given by Eq. 9.19.

$$\bar{F} = {}_1T_{\delta 1} [{}_2T_{P1} \bar{K} {}_1T_{\delta 1}]^{-1} {}_2T_{P1} \quad (9.19)$$

When the relation ${}_1T_{\delta 1} = {}_4{}_2T_{P1}$ given by Eq. 4.21 is taken into consideration, the Eq. 9.19 is rewritten as Eqs. 9.20 and 9.21.

$$\bar{F} = {}_2T_{P1}^t [{}_2T_{P1} \bar{K} {}_2T_{P1}^t]^{-1} {}_2T_{P1} \quad (9.20)$$

$$= {}_1T_{\delta 1} [{}_1T_{\delta 1}^t \bar{K} {}_1T_{\delta 1}]^{-1} {}_1T_{\delta 1}^t \quad (9.21)$$

The flow chart of the derivation process of the general stiffness matrix $\bar{K}$ and general flexibility matrix $\bar{F}$, and the numbers of the related equations are mentioned in Fig. 9.1.

Fig. 9.1 The flow chart showing the computation process of $\bar{K}$, $\bar{F}$ and the numbers of the related equations.

CONCLUSIONS

When the shapes of shear walls are symmetric with regard to their longitudinal and transversal center lines and the shear walls are isotropic elastic bodies, the elements of the general stiffness matrix $\bar{K}$ and general flexibility matrix $\bar{F}$ are expressed in terms of the elements of F^* (see Tables 7.1 and 8.1). Thus, the physical meaning of the elements of $\bar{K}$ and $\bar{F}$ are revealed.

The determinants of $\bar{K}$ and $\bar{F}$ are zero, because the framed shear wall whose nodal displacements are not given as the stable nodal support conditions can move freely by its rigid-body motion, and its nodal displacements are indeterminate. Therefore neither of $\bar{K}$ and $\bar{F}$ can be given by inverting the other.

$\bar{K}$ can be expressed in terms of $\bar{F}$ by using the transformation matrix ${}_2T_{\delta 1}$ to transform $\bar{\delta}$ into δ^* or the transformation matrix ${}_1T_{P1}$ to transform P^* into $\bar{P}^*$ (see Eqs. 9.17 and 9.18), where ${}_1T_{P1} = {}_4{}_2T_{\delta 1}^t$ (see Eq. 4.18).

$\bar{F}$ can be expressed in terms of $\bar{K}$ by using the transformation matrix ${}_2T_{P1}$ to transform $\bar{P}$ into P^* or the transformation matrix ${}_1T_{\delta 1}$ to transform δ^* into $\bar{\delta}^*$ (see Eqs. 9.20 and 9.21), where ${}_1T_{\delta 1} = {}_4{}_2T_{P1}^t$ (see Eq. 4.21).

耐震壁の節点外力と節点変位の関係 (第4報)

—Stiffness Matrix および Flexibility Matrix—

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序

紙面の都合で本論文は5編に分割して連載する。

§7 耐震壁の節点に関する一般的な Stiffness Matrix の特性

§6 で述べたように一般的な Stiffness Matrix $\bar{K}$ は、要素が解析^{1)~5)}または実験によって与えられる基本 Flexibility Matrix F^* の逆マトリックスで表わされる (6.7 式参照)。

$\bar{K}$ の要素 $k_{m,n}$ は、 $\bar{P}_\delta^* = \bar{K} \bar{\delta}$ (6.9 式参照) を詳細に示した 7.1 式に示すように定義する。耐震壁の形状が縦横軸に関して対称で、かつその材質が等方性である場合は、 $k_{m,n}$ が基本 Stiffness Matrix K^* の要素 (5.5 式参照) で 7.1 表のように表わされる。なお、 K^* の要素は、解析^{1)~5)}または実験によって与えられる F^* の要素で表わされる (5.2, 5.5, 5.6 式参照)。

$\bar{K}$ には次の特性がある (10.9 式に示した数値計算例参照)。

- 1) $\bar{K}$ のどの列においても $\bar{X}_\delta^*$ および $\bar{Y}_\delta^*$ に対応する要素の和はそれぞれ零である。ゆえに、 $\bar{x}$ および $\bar{y}$ 方向に関する力の釣合は恒等的に満足されている。しかしながら、 $\bar{K}$ のどの列においても $\bar{M}_\delta^*$ に対応する要素の和は零にならない。これは、モーメントの釣合は $\bar{X}_\delta^*$, $\bar{Y}_\delta^*$ を考慮に入れなければならないからである。そこで、 $\bar{M}_\delta^*$ に対応した列要素の和に、 $\bar{X}_{\delta k}^*$ と $\bar{X}_{\delta l}^*$ に対応した要素 $k_{m,n}$ の $1/2$, $\bar{X}_{\delta k}^*$ と $\bar{X}_{\delta l}^*$ に対応した $k_{m,n}$ の $-1/2$, $\bar{Y}_{\delta i}^*$ と $\bar{Y}_{\delta l}^*$ に対応した $k_{m,n}$ の $-1/2$, $\bar{Y}_{\delta j}^*$ と $\bar{Y}_{\delta k}^*$ に対応した $k_{m,n}$ の $1/2$ を加えて、はじめて $\bar{K}$ のどの列も零となる。ゆえに、 $\bar{X}_\delta^*$, $\bar{Y}_\delta^*$ および $\bar{M}_\delta^*$ に関するモーメントの釣合は恒等的に満足されている。
- 2) $\bar{K}$ のどの行においても $\bar{u}$ や $\bar{v}$ に対応する要素の和はそれぞれ零となっている。ゆえに、 $\bar{X}_\delta^*$, $\bar{Y}_\delta^*$ および $\bar{M}_\delta^*$ は耐震壁の剛体運動 u° および v° によって生ずるそれぞれ等しい各節点変位 $\bar{u}$ および $\bar{v}$ によっては影響されない。しかしながら $\bar{K}$ のど

の行においても $\bar{\theta}$ に対応する要素の和は零にならない。これは耐震壁の剛体回転運動 θ° によって θ° に等しい節点回転角 $\bar{\theta}$ のほかに節点変位 $\bar{u}$, $\bar{v}$ が生ずるからである (3.2 図参照)。ゆえに、 $\bar{\theta}$ に対応した行要素の和に、 $\bar{u}_i$ と $\bar{u}_j$ に対応した要素 $k_{m,n}$ の $1/2$, $\bar{u}_k$ と $\bar{u}_l$ に対応した $k_{m,n}$ の $-1/2$, $\bar{v}_i$ と $\bar{v}_l$ に対応した $k_{m,n}$ の $-1/2$ 倍, $\bar{v}_j$ と $\bar{v}_k$ に対応した $k_{m,n}$ の $1/2$ 倍を加えると、はじめて $\bar{K}$ のどの行も零となる。ここに θ° は微小でなければならない。これは本論の解析が線型微小変位理論によって行われているからである。ゆえに、 $\bar{X}_\delta^*$, $\bar{Y}_\delta^*$ および $\bar{M}_\delta^*$ は微小剛体回転運動 θ° によっても影響されない。したがって、7.2 式を得る。

- 3) $\bar{K}$ は対称マトリックスであるので、未知節点変位が一義的に求まるような節点支持条件が与えられると、相反定理が成立する。
- 4) 以上記述した 1), 2), 3) 項を考慮すると、 $\bar{X}_\delta^*$, $\bar{Y}_\delta^*$ および $\bar{M}_\delta^*$ ($\bar{u}$, $\bar{v}$ および $\bar{\theta}$) にそれぞれ対応する3行 (列) や、 $\bar{X}_\delta^*(\bar{u})$ に対応する1行 (列) と力の作用線が一致しない $\bar{Y}_\delta^*(\bar{v})$ に対応する2行 (列)、あるいはまた作用線が一致しない $\bar{X}_\delta^*(\bar{u})$ に対応した2行 (列) と $\bar{Y}_\delta^*(\bar{v})$ に対応した1行 (列) は従属行 (列) である。ゆえに、 $\bar{K}$ の行列式の値が零となり、7.3 式が得られる。また $\bar{K}$ は特異となり、§8 で述べる耐震壁の一般的な Flexibility Matrix $\bar{F}$ は $\bar{K}$ の逆マトリックスとしては与えられない。しかし安定な節点支持条件が与えられると、Flexibility Matrix は Stiffness Matrix の逆マトリックスとして与えられる。

§8 耐震壁の節点に関する一般的な Flexibility Matrix の特性

§6 で述べたように、一般的な Flexibility Matrix $\bar{F}$ は、要素が解析^{1)~5)}または実験によって与えられる基本 Flexibility Matrix F^* で表わされる (6.20 式参照)。

$\bar{F}$ の要素 $f_{m,n}$ は、 $\bar{\delta}_P^* = \bar{F} \bar{P}$ (6.21 式参照) を詳細に示した 8.1 式に示すように定義する。耐震壁の形状が縦横軸に対称で、かつその材質が等方性である場合は $f_{m,n}$ が基本 Flexibility Matrix F^* の要素 (5.2 式参照) で 8.1 表のように表わされる。なお、 F^* の要素

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は解析^{1)~5)}または実験によって与えられる。

$\bar{F}$ には次の特性がある (10.11 式に示した数値計算例参照)。

- 1) $\bar{F}$ のどの行においても, $\bar{X}$, $\bar{Y}$ および $\bar{M}$ に対応する要素の和はそれぞれ零である。ゆえに, 力の釣合を満足する節点外力成分 $\bar{P}^*$ のみ作用する耐震壁の変形に関する節点変位 $\bar{\delta}_{P^*}$ は, 各節点外力が $X^{\circ}_{\text{III}}, Y^{\circ}_{\text{IV}}, M^{\circ}_1$ (2.6, 2.8, 2.3 図参照) に等しく, かつ中間荷重の合力と釣合力系を構成する節点外力の不釣合成分 $\bar{P}^{\circ}$ によっては影響されない。したがって 8.2 式が得られる。
- 2) $\bar{F}$ のどの列においても, $\bar{u}_{P^*}, \bar{v}_{P^*}, \bar{\theta}_{P^*}$ に対応する要素の和はそれぞれ零である。ゆえに, $\sum \bar{u}_{P^*}, \sum \bar{v}_{P^*}$ および $\sum \bar{\theta}_{P^*}$ はそれぞれ恒等的に零に等しい (§3 参照)。
- 3) $\bar{F}$ は対称マトリックスであるので, 未知節点変位が一義的に求まるような節点支持条件が与えられると, 相反定理が成立する。
- 4) 以上述べた 1), 2), 3) 項を考慮すると, $\bar{u}_{P^*}, \bar{v}_{P^*}$ および $\bar{\theta}_{P^*}(\bar{X}, \bar{Y}$ および $\bar{M})$ にそれぞれ対応する 3 行 (列) は従属行 (列) である。ゆえに, $\bar{F}$ の行列式の値は零となり, 8.3 式が得られる。また $\bar{F}$ は特異となり, §7 で述べた $\bar{K}$ は $\bar{F}$ の逆マトリックスとしては与えられない。しかし安定な節点支持条件が与えられると Stiffness Matrix は Flexibility Matrix の逆マトリックスとして与えられる。

§9 Stiffness Matrix と Flexibility Matrix の関係

2.1 および 8.2 式を考慮すると, 6.21 式が 9.1 式のように書き直される。ゆえに, $\bar{\delta}_{P^*}$ は $\bar{P}^*$ のみが作用する耐震壁の弾性変形による節点変位成分として定義され, 9.2 式が得られる。6.21 式を 9.2 式に代入すると, 9.3 式が得られる。9.3 式を 2.12 式に等値すると $\bar{K}$ と $\bar{F}$ の積が 9.4 式で与えられる。

4.1 式を考慮すると, 9.4 式が 9.5 式のように書き直され, 3.1 および 7.2 式を考慮すると, 6.9 式が 9.6 式のように書き直される。ゆえに, $\bar{P}_{\delta^*}$ は耐震壁の弾性変形による節点変位成分 $\bar{\delta}^*$ を生じさせる節点外力成分として定義され, 9.7 式が得られる。6.9 式を 9.7 式に代入すると, 9.8 式が誘導される。

9.8 式を 3.12 式と等値すると, $\bar{F}$ と $\bar{K}$ の積が 9.9 式で与えられる。4.9 式を考慮すると, 9.9 式が 9.10 式のように書き直される。

9.4 式の ${}_1T_{P1}{}_2T_{P1}$, 9.5 式の ${}_1T_{P0}{}_2T_{P0}$, 9.9 式の ${}_1T_{\delta1}{}_2T_{\delta1}$ および 9.10 式の ${}_1T_{\delta0}{}_2T_{\delta0}$ が単位マトリックスでないことを考慮すると (2.14, 2.15, 3.14, 3.15 式参照), 9.4, 9.5, 9.9 および 9.10 式で与えたマトリ

ックス積 $\bar{K}\bar{F}$ と $\bar{F}\bar{K}$ は, 単位マトリックスでも零マトリックスでもない。ゆえに, $\bar{K}$ と $\bar{F}$ は, 行列式の値が零である $\bar{F}$ と $\bar{K}$ の逆マトリックスとして与えられない (7.3, 8.3 式参照)。

$\bar{K}$ と $\bar{F}$ が対称マトリックスであることを考慮すると, 9.11 式が誘導される。

$\bar{\delta}$ を δ^* に変換する変換マトリックス ${}_2T_{\delta1}$ (3.8, 3.7, 3.6 式参照) を 6.20 式の両辺の左から乗じ, 次に P^* を $\bar{P}^*$ に変換する変換マトリックス ${}_1T_{P1}$ (2.2, 2.4, 2.5 式参照) を 6.20 式の両辺の右から乗ずると, 9.12 式が得られる。4.5 および 4.13 式を考慮すると, 9.12 式が 9.13 式のように書き直される。

$\bar{P}$ を P^* に変換する変換マトリックス ${}_2T_{P1}$ (2.8, 2.6, 2.7 式参照) を 6.7 式の両辺の左から乗じ, 次に δ^* を $\bar{\delta}^*$ に変換する変換マトリックス ${}_1T_{\delta1}$ (3.2, 3.4, 3.5 式参照) を 6.7 式の両辺の右から乗ずると, 9.14 式が得られる。4.5 および 4.13 式を考慮すると, 9.14 式が 9.15 式のように書き直される。

9.13 式を 6.7 式に代入すると, 一般的 Flexibility Matrix $\bar{F}$ から一般的 Stiffness Matrix $\bar{K}$ を求める 9.16 式が誘導される。4.18 式を考慮すると, 9.16 式は 9.17 および 9.18 式のように書き直される。

9.15 式を 6.20 式に代入すると, 一般的 Stiffness Matrix $\bar{K}$ から一般的 Flexibility Matrix $\bar{F}$ を求める 9.19 式が誘導される。4.21 式を考慮すると, 9.19 式は 9.20 および 9.21 式のように書き直される。

$\bar{K}$ と $\bar{F}$ の誘導過程の流れ図とその関係式番号を 9.1 図に示す。

結 論

耐震壁の形状が縦横軸に関して対称で, かつその材質が等方性である場合, $\bar{K}$ および $\bar{F}$ の要素は F^* の要素で表わされる (7.1, 8.1 式参照)。

$\bar{K}$ と $\bar{F}$ の要素の物理的意味も明らかにした。

$\bar{K}$ と $\bar{F}$ の行列式の値は零である。なぜなら, 耐震壁の節点変位が安定な節点支持条件として与えられない限り, 自由な剛体運動が可能で, 節点外力に対して節点変位が一義的に定まらないからである。したがって, $\bar{K}$ と $\bar{F}$ はお互いにその逆マトリックスとして与えられない。

${}_1T_{P1}=4{}_2T_{\delta1}^t$ (4.18 式参照) を考慮すると, $\bar{\delta}$ を δ^* に変換する変換マトリックス ${}_2T_{\delta1}$, または P^* を $\bar{P}^*$ に変換する変換マトリックス ${}_1T_{P1}$ を使って, $\bar{F}$ から $\bar{K}$ を求めることができる (9.17, 9.18 式参照)。

${}_1T_{\delta1}=4{}_2T_{P1}^t$ (4.21 式参照) を考慮すると, $\bar{P}$ を P^* に変換する変換マトリックス ${}_2T_{P1}$, または $\bar{\delta}^*$ を δ^* に変換する変換マトリックス ${}_1T_{\delta1}$ を使って, $\bar{K}$ から $\bar{F}$ を求めることができる (9.20, 9.21 式参照)。