PART 1 : RELATIONSHIP BETWEEN INTEGRATION TIME INTERVAL AND RESPONSE STABILITY IN PSEUDO DYNAMIC TESTING

(Stability and accuracy behavior of pseudo dynamic response)

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1. Introduction

It was 1974 when Takanashi et al.⁽¹⁾ of the Institute of Industrial Science, the University of Tokyo, first reported on a simulation technique of earthquake response of building structures by means of a combined experiment and numerical analysis. In this technique, the dynamic response of a structure is computed by solving the equations of motion of a discrete system representing the structure by direct integration with respect to the time domain. Because of the versatile capability of this technique to directly simulate the earthquake response behavior of structural systems, this technique has attracted many research bodies of various fields in both Japan and overseas.⁽²⁻⁵⁾ In their studies, this simulation method was designated in various ways such as the computer actuator on-line system,⁽¹⁾ the on-line hybrid experiment,⁽²⁾ or the pseudo dynamic testing method.⁽³⁻⁵⁾ Throughout this paper, the method will be referred to as the pseudo dynamic (PSD) test method.

Since the outset of the PSD test method, the central difference method (CDM) has been used exclusively for the computation of equations of motion. CDM is one of explicit integration methods. Since no iterative procedure is required even for the inelastic response computation, this integration method has been found suitable for PSD test. This method, however, is conditionally stable unlike most implicit integration methods. Therefore, the time interval used for the computation of equations of motion should be smaller than a certain value, called the critical interval, in order to maintain the numerical stability of the solution. This stability condition, however, is a condition which guarantees that the solution does not diverge, but not a condition which promises the accurate solution. In the numerical analysis, the solution becomes more accurate as the integration time interval approaches zero, but never the same as the true analytic solution as long as the interval is finite. It is now clear that one should carefully select the integration time interval in reference to both the stability and accuracy conditions of CDM when employing the PSD test.

As an alternative of CDM, the explicit Newmark method ($\beta=0$) is often employed in PSD Test. Two comments are to be given with respect to the relative effectiveness in PSD application between the two methods. First, it should be noted that the explicit Newmark method is identical with CDM in the computation algorithm⁽⁶⁾. Second, in the real implementation of PSD test, errors are inevitably involved because of the finite accuracies of the displacement and force sensors as well as the actuator control capacity. According to the study made by Shing and Mahin⁽⁷⁾, the explicit Newmark method is better in the error control than CDM. The study, however, indicated that the two methods are identical even in terms of the error control if the computed displacement is used in place of the measured displacement for solving the equations of motion, and also commented that better response results can be obtained with the use of the computed displacement. In this respect, use of CDM combined with the computed displacement can be said no worse than the explicit Newmark method.

There is a classical study done by Leech et al.⁽⁸⁾, in which the stability condition of CDM was derived mathematically, and a suggestion was made on the time interval insuring the accurate solution. As a matter of fact, many later works on PSD test referred to the results of this Leech study. In this study, however, the stability condition was derived only for undamped systems (no velocity term included in the equations of motion).

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Furthermore, their suggestion on the accurate solution, i. e. that the time interval be smaller than one sixth (1/6) of the critical time interval, was rather vague ; they neither indicated how much error could be expected if this interval was used nor stated how much the solution would be aggravated if, say, an interval of twice of this interval was used.

Considering the incompleteness of information on the stability and accuracy of CDM applied to PSD test, the objective of this study is to investigate in detail the stability and accuracy characteristics of CDM in terms of the integration time interval and propose guidelines for the selection of the time interval in PSD test. This paper (Part 1) focuses its investigation on the fundamental stability condition of CDM and the accuracy characteristics of displacement response when CDM is employed for systems subjected to free vibration. In the companion paper (Part 2), the accuracy characteristics of displacement, velocity, and acceleration responses of systems subjected to general external forces are examined in detail, and practical comments on the selection of the time interval in one's PSD test are provided. This part also includes numerical experimentation in which the validity of the major findings of this study is demonstrated. Throughout this study, investigation is given only for linear single degree of freedom (SDOF) systems since response of multi degrees of freedom (MDOF) systems can be treated as an assembly of SDOF systems if the concept of modal analysis is employed. When a MDOF system has an arbitrary form of viscous damping matrix, the modal decoupling may not be achieved with respect to the real domain. The investigation here, however, is made assuming that the viscous damping matrix has a form such that the system would be decoupled with respect to the real domain as commonly made in both numerical analysis and previous PSD tests.

2. Stability Condition of Numerical Methods

Fundamental stability condition of numerical integration methods is reviewed here. The equation of motion of a linear elastic SDOF system is given as :

$$m \cdot \ddot{x}(t) + c \cdot \dot{x}(t) + k \cdot x(t) = r(t) \quad (1)$$

where $m, c, k, x(t)$, and $r(t)$, are the mass, viscous damping, stiffness, displacement, and external force respectively. In the numerical computation of transient response of the system, the displacements are discretized with respect to the time domain, and the differential equation is approximated to a difference equation. Stability condition of difference equations has been studied in various papers.⁽⁹⁻¹³⁾ The essence of stability condition is that the error introduced into the approximate solution by a particular difference method remains uniformly bounded as $n \rightarrow \infty$ with n as the solution time step. This condition can also be interpreted such that the solution does not grow without bound for arbitrary initial conditions.⁽¹²⁾ For clear understanding of this condition, it is helpful to write the difference equation in recursive form as :

$$X_{n+1} = A \cdot X_n + L \cdot r(n\Delta t) \quad (2)$$

in which Δt is the integration time interval.

By repeated use of Eq.2, we can obtain :

$$X_n = A^n \cdot X_0 + \sum_{i=1}^n A^{n-i} \cdot L \cdot r[\Delta t(i-1)] \quad (3)$$

In the above equation, X_{n+1} and X_n are vectors storing the solution quantities such as the displacement and velocity at time $t = (n+1)\Delta t$ and $t = n\Delta t$. Matrix A is called the amplification matrix, and L the load operator. Since the stability is evaluated by examining the behavior of the numerical solution for any initial conditions, consideration is given regarding that no external force is specified. Then, Eq.3 is reduced to :

$$X_n = A^n \cdot X_0 \quad (4)$$

This equation indicates that the boundness of the solution is a function of the amplification matrix A . Matrix A^n can be decomposed to :

$$A^n = \phi \cdot J^n \cdot \phi^{-1} \quad (5)$$

Where ϕ is the matrix of eigenvectors of A , and $J = \text{diag}(\lambda_i)$ with λ_i as the i -th eigen value of A . This matrix is known as the Jordan form of A . Introducing the spectral radius of A as :

$$\rho(A) = \max |\lambda_i| \quad (6)$$

one can find J^n and A^n , (and accordingly the solution of Eq.4) is bounded for $n \rightarrow \infty$ if $\rho(A) \leq 1$. This is the condition of stability.

3. Stability of CDM

Stability of CDM for the equation of motion of a linear undamped system was studied by Leech et al.⁽⁸⁾ Their

derivation, however, did not follow the procedure described in the last section. Here, stability of CDM is examined by use of the recursive form of Eq.4. This approach enables us to evaluate not only the stability but also the numerical accuracy of the solution. First, consideration is given to the undamped system. If no external force is applied (free vibration), the undamped equation of motion of a SDOF system is given as :

$$\ddot{x}(t) + \omega^2 \cdot x(t) = 0 \quad \dots\dots\dots (7)$$

where ω^2 is k/m .

This equation is now converted to the difference form by CDM. Considering :

$$\ddot{d}_n = \frac{d_{n+1} - 2d_n + d_{n-1}}{\Delta t^2} \quad \dots\dots\dots (8)$$

in which d_n and $\ddot{d}_n$ are the displacement and acceleration at time $t = n\Delta t$ specified by CDM, and inserting this expression in Eq.7, we can obtain :

$$d_{n+1} + (\omega^2 \Delta t^2 - 2) \cdot d_n + d_{n-1} = 0 \quad \dots\dots\dots (9)$$

This equation is now rearranged to recursive form as :

$$\begin{Bmatrix} d_{n+1} \\ d_n \end{Bmatrix} = \begin{bmatrix} (2 - \omega^2 \Delta t^2) & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} d_n \\ d_{n-1} \end{Bmatrix} \quad \dots\dots\dots (10)$$

The eigen values of the amplification matrix is :

$$\lambda^2 + (\omega^2 \Delta t^2 - 2) \cdot \lambda + 1 = 0 \quad \dots\dots\dots (11)$$

and

$$\lambda_{1,2} = \frac{(2 - \omega^2 \Delta t^2)}{2} \pm i \cdot \sqrt{1 - \frac{(2 - \omega^2 \Delta t^2)^2}{4}} \quad \dots\dots\dots (12)$$

Then, we can obtain for the condition of stability : $\rho(A) \leq 1$:

$$\omega \Delta t \leq 2 \quad \dots\dots\dots (13)$$

This condition is identical with what was proven by Leech et al.⁽⁸⁾ Identity of Eq.11 with Eq.9 except λ 's instead of d_n 's leads us to the following expression for $\omega \Delta t$'s smaller than 2 :

$$d_n = c_1 \cdot \lambda_1^n + c_2 \cdot \lambda_2^n \quad \dots\dots\dots (14)$$

Where c_1 and c_2 are the integration coefficients to be determined from initial conditions. Suppose that $\lambda_{1,2} = A \pm i \cdot B$

$$d_n = \exp(-\bar{h} \bar{\omega} t_n) \cdot (c_1 \cdot \cos \bar{\omega} t_n + c_2 \cdot \sin \bar{\omega} t_n) \quad \dots\dots\dots (15)$$

Where : $\theta = \tan^{-1}(B/A)$

$$\bar{h} = -\ln(A^2 + B^2)/2\theta$$

$$\bar{\omega} = \theta/\Delta t$$

This is the analytic expression representing the numerical solution. In the above equation, $\bar{h}$ is defined as the coefficient of numerical damping introduced into the difference approximation, and $\omega/\bar{\omega}$ as the numerical period distortion. One can understand without difficulty those terminologies, reminded that the numerical solution (Eq.15) approximates the undamped free vibration of a SDOF system (Eq.7). Substituting the eigen values of Eq.12 into Eq.15 leads to :

$$\bar{h} = 0 \quad \dots\dots\dots (16)$$

This equation means that the numerical approximation does not contain any inherent numerical damping. In Fig.1, $\omega/\bar{\omega}$ is plotted against $\omega \Delta t$ in the stability range. It is explained from this figure that the period computed by CDM is shorter than the true period, while the difference between the two decreases with the decrease of the time interval and finally becomes zero in the limit $\omega \Delta t = 0$. This period contraction is characterized as the approximation introduced in CDM. The period contraction behavior of the explicit Newmark method (identical with CDM) was also investigated in Ref.14. Figures 2(a) to (c) show the time history of Eq.7 computed by CDM for various values of $\omega \Delta t$. The employed initial conditions are :

$$x(0) = 1.0 \text{ \& } \dot{x}(0) = 0.0 \quad \dots\dots\dots (17)$$

In those figures, also plotted are d_n 's (in Eq.15) having the corresponding $\omega \Delta t$. It is seen that the numerical response linearly traces the points on the d_n curve (Eq.15) at every specified time interval Δt . The figures also indicate that no

 Fig.1 Period Distortion in CDM (if $h=0$)

 Fig.2 Numerical Free Vibration Response (CDM vs. d_n in Eq.15)

In a region: $\omega\Delta t \geq 2\sqrt{1-h^2}$, both eigen values are real. After some algebra onto Eq.20, we find that λ_2 continuously decreases as $\omega\Delta t$ increases and $\lambda_2 = -1.0$ at $\omega\Delta t = 2$, while λ_1 remains less than unity in $\omega\Delta t = 2$. This change in λ 's is illustrated in Fig.3. When $\omega\Delta t$ is more than 2, then the absolute value of λ_2 is larger than unity, which leads to instability. The stability criteria for the damped system can now be summarized as :

- Condition 1 : $\omega\Delta t > 2$ Unstable
 Condition 2 : $\omega\Delta t < 2\sqrt{1-h^2}$ Stable (two complex conjugates)
 Condition 3 : $2\sqrt{1-h^2} \leq \omega\Delta t \leq 2$ Stable (two Reals)

 Fig.3 Real λ 's in Eq.20 ($h \neq 0$)

amplitude decay occurs in the response ($\bar{h}=0$), and the numerical period becomes shorter as $\omega\Delta t$ increases.

4. Stability of Damped System

In the foregoing section, CDM was discussed for its stability when applied to the undamped SDOF system. The next step is to extend the discussion to general damped systems. Analogously to the undamped system, stability is examined in terms of the free vibration condition. Then, the equation of motion of a damped SDOF system is :

$$\ddot{x}(t) + 2h\omega \dot{x}(t) + \omega^2 x(t) = 0 \quad (18)$$

where h is the viscous damping coefficient. Following the procedures given in Eqs.7 to 10, we obtain for the eigen values of the amplification matrix :

$$\lambda^2 + \frac{\omega^2 \Delta t^2 - 2}{1 + h\omega \Delta t} \lambda + \frac{1 - h\omega \Delta t}{1 + h\omega \Delta t} = 0 \quad (19)$$

and

$$\lambda_{1,2} = \frac{1}{2(1 + h\omega \Delta t)} \{ (2 - \omega^2 \Delta t^2) \pm i \omega \Delta t \cdot \sqrt{4(1 - h^2) - \omega^2 \Delta t^2} \} \quad (20)$$

Unlike the undamped system, here the condition that the eigen values are a pair of complex conjugates is not identical to the condition of the spectrum radius being less than unity. Note that, in the undamped system, the eigen values are a pair of complex conjugates as long as the integration time interval is smaller than the critical interval. For the values to be complex in the damped system :

$$\omega\Delta t < 2\sqrt{1-h^2} \quad (21)$$

Fig.4 Period Distortion in CDM

Condition 2 is now investigated in further detail. The numerical solution can be written in a form of Eq.15. In Fig.4, the ratio of numerical to exact natural periods ($\bar{T}/T$) of the damped system is plotted against $\omega\Delta t$ for various h 's. For h 's smaller than about 0.7 (Fig.4(a)), the numerical natural period is always smaller than the true period. The numerical natural period increases constantly as $\omega\Delta t$ decreases, reaching the true period in the limit of $\omega\Delta t=0$. For h 's greater than about 0.7 (Fig.4(b)), on the other hand, the numerical natural period is infinite (indicating no oscillation) in the limit : $\omega\Delta t=2\sqrt{1-h^2}$ and decreases precipitously to a level of the true period for the smaller values of $\omega\Delta t$. This critical point of h is given as :

$$h=\sqrt{0.5}\doteq 0.707 \dots\dots\dots (22)$$

which is derived from the condition that $\lambda_{1,2}=0$ (equal root) at the limit : $\omega\Delta t=2\sqrt{1-h^2}$. This critical point (Eq.22) can be explained as follows : When the eigen values of Eq.20 provide an equal root, the analytic expression of the solution d_n is given as follows⁽¹⁵⁾ :

$$d_n=c_1\cdot\lambda_1^n+c_2\cdot n\cdot\lambda_1^n \dots\dots\dots (23)$$

For $h^2<0.5$, the equal root is negative. Since d_n is computed as the power of n , d_n changes its sign as each time step is advanced. In this situation, the numerical natural period is twice the time increment. If $h^2>0.5$, on the other hand, the equal root is positive, indicating that d_n decreases to zero without changing its sign. This means that the numerical natural period is infinite.

In Fig.5, the numerical displacement response is shown for $h=0.1$. The employed initial conditions are those given in Eq.17. It is seen in this figure that the numerical period is smaller than the true period, gradually approaching the true one as $\omega\Delta t$ decreases. The relationship between the numerical and true damping ratios is illustrated in Fig.6. Considering that the true solution of the damped system is :

Fig.5 Numerical Free Vibration Response for $h=0.1$
(Analytical Solution, CDM, and d_n in Eq.15)

Fig.6 Numerical Damping in CDM ($h\neq 0$)

$$x(t)=\exp(-h\omega t)\cdot(c_1\cdot\cos \omega_D t+c_2\cdot\sin \omega_D t) \dots\dots\dots (24)$$

in which $\omega_D=\omega\sqrt{1-h^2}$

and comparing this with the numerical solution (Eq.15), the numerical damping ratio is adjusted to :

$$\bar{h}_e=\bar{h}\cdot\frac{\omega}{\omega_D} \dots\dots\dots (25)$$

As evidenced from this figure, the numerical damping ratio is very close to the true damping ratio in the entire stability range of $\omega\Delta t$ if h is small. As h increases, the error in the damping ratio increases. Further, for each h , the discrepancy between the numerical and true damping ratios becomes smaller with the decrease of $\omega\Delta t$.

5. Concluding Remarks

The important findings obtained in this part of the study can be summarized as follows.

- 1) The stability limit of CDM when applied to the damped system is identical with that for the undamped system. In damped systems, $\omega_b \Delta t = 2 \cdot \sqrt{1-h^2}$ is a critical point. For ω 's smaller than ω_b , the numerical solution can be expressed as Eq.15 and therefore ensures oscillatory solution.
- 2) The error caused by employing CDM in PSD test is characterized as the period distortion (Figs.1 and 4). For undamped systems, no amplitude decay (numerical damping) is introduced in the computation. In damped systems, on the other hand, the numerical viscous damping is larger than the true viscous damping. The discrepancy between the numerical and true dampings is as shown in Fig.6.

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仮動的実験における積分時間刻みと応答の安定の関係

仮動的実験応答の安定と精度

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1. 序

1974 年東京大学生産技術研究所高梨他（文献 1）が数値解析と実験を併用した新しい地震応答解析手法を発表した。本手法は構造物の地震時応答を直接再現できる極めて魅力的な手法として内外の注目を浴びている（文献 2-5）。本手法は種々な名前では呼ばれているが（文献 1-5）本論では仮動的実験手法と称する（文献 3-5）。

本手法では数値計算に中央差分法が用いられる。これは本手法の性格から直接積分の解法が陽的でなければならないこと、またあるステップから次のステップへの計算過程においてひずみの戻りを含む試行錯誤を許容しないことによる。積分時間刻みを十分に小さく採れば中央差分法は十分な精度を保証するが、弱点は積分時間刻みに対して無条件には安定ではないことである。すなわち時間刻みがある限界を越えると解が発散してしまい仮動的実験が逐行できなくなる。したがって仮動的実験を成功させ、また得られた結果を意味のあるものにするためには、積分時間刻みと解の安定・精度の条件の関係を正しく評価する必要がある。

仮動的実験では中央差分法の代わりに陽的ニューマーク法 ($\beta=0$) が用いられることもある。まず陽的ニューマーク法はその基本アルゴリズムにおいて中央差分法と同一である（文献 6）。しかし両方法は実験上の誤差の集積に違いが見られることもある。誤差の集積に関する研究（文献 7）では、運動方程式を解く時に計測された変位を用いると中央差分法の方が誤差の集積が大きい事、また計算された変位を用いると両方法による誤差集積は等しい事、そして計算変位を用いる方が良い応答結果が得られる事、を報告している。すなわち計算変位を基準にして仮動的実験を行う限り、中央差分法と陽的ニューマーク法は同一である。

Leech 等（文献 8）の研究によって中央差分法の解の安定条件が導かれ、また彼らは精度の良い解を得るために安定限界に対応する積分時間刻みの $1/6$ 以下の積分時間刻みを選ぶことを推奨している。しかしながら彼らの

研究では 1) 安定条件を非減衰系についてのみしか導いておらずまた、2) $1/6$ の根拠が示されていない（すなわち $1/6$ ならどれ程の精度が保証され、また $1/3$ ならどれ程精度が落ちるのかについての定量的評価がない）。

本研究の目的は仮動的実験で中央差分法を適用する時に得られる解の安定と精度の条件を調べ、仮動的実験における積分時間刻みの選び方に対する指標を与えることである。本論（その 1）では安定条件と自由振動時の変位応答の精度を論じ、（その 2）で一般的な外力に対する変位・速度・加速度応答の精度について述べる。また（その 2）では解析的な検討結果が妥当であることを数値実験によっても検証している。なお本研究では弾性一質点系を対象として議論を進めている。多質点系についてもモード分解の考え方を導入すれば一質点系についての議論が適用できる。なお非比例減衰においては実数領域でモード分解ができないが、本論では数値解析や過去の仮動的実験において頻繁に用いられるように粘性減衰は比例減衰型であると限定する。

2. 数値解法の安定条件

ここでは数値解析法の安定を論ずる一般的方法を紹介する。

弾性一質点系の運動方程式は式（1）で表される。数値解析ではこの式を時間軸に対して離散化することにより解くが、ここで解の安定とは n を解の時間ステップとして n が無限大になっても誤差が発散しないことと定義される（文献 9-13）。またこの条件は任意の初期条件に対して解が発散しないこととも解釈される（文献 12）。積分解法の安定条件はその数値解法を漸化式（式（3））で表現することによって検討できる。発散は任意の初期条件における数値解の挙動から調べられるので今自由振動状態を考える。この時式（3）は式（4）と簡略化される。この式から安定はマトリックス A （アンプリフィケーションマトリックスと呼ばれる）の特性に依存することが判る。さらに A が式（5）のように分解されることを考慮すると、 A の特性根の絶対値が 1 を越えないことが安定の条件となる。

3. 中央差分法の安定条件

本節では前節で示した方法を中央差分法に適用して中

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中央差分法の安定限界を導く。

まず無減衰系(式(7))を考える。式(7)に中央差分法で表される加速度(式(8))を代入して整理すると式(10)に示す漸化式が得られる。この式のマトリックスの特性根が1を越えないという条件から式(13)を得る。この条件はLeech等によって導かれたものと同じである。(彼らは漸化式表現を用いずに安定条件を導びいた)。また特性根を用いて数値解を式(15)のように表すことができる。この式は数値解の解析的表現であり、減衰系の自由振動解に似た形となっている。ここで $\bar{h}$ は数値解に生じる減衰(数値的減衰)と、また $\bar{\omega}$ は数値解の円振動数と理解できる。式(12)を式(15)に代入すると式(16)が得られる。つまり数値解も同様に非減衰系であることを意味する。図-1は数値解の円振動数に対する系の円振動数の比($\omega/\bar{\omega}$)を積分時間刻みに対してプロットしたものである。この図から数値解析で得られる系の振動周期は真の周期より小さい。すなわち数値解では見かけの固有周期が縮むことが判る。このことは文献14にも記されている。図-2(a)~(c)は式(17)に与えられた初期条件下での自由振動について、中央差分法を適用して得た解と式(15)を比較したものである。中央差分法によって得られる自由振動応答は式(15)を積分時間刻みごとに直線で結んだ折れ線になっていることが判る。

4. 減衰系の安定条件

減衰系(式(18))の安定条件についても、前節と同様の手順を適用すると式(20)の特性根が得られる。この特性根が1を越えず、さらに共役複素数であるための条件は式(21)で与えられる。なお非減衰系では特性根が1より小さい場合、その根は自動的に共役複素数であった。式(21)が満足されない場合特性根は実数になる。図-3はこのような実数根の値を $\omega\Delta t$ に対してプロットしたものであり $\omega\Delta t$ が2に至る時一方の特性根は-1となる。

式(21)が満足される時は数値解を式(15)の形で表現できる。図-4には数値解から得られる見かけの周期のずれを種々の粘性減衰定数(h)に対して示している。 h が約0.7以下の時には(図-4(a))見かけの周期は常に真の周期よりも短かく $\omega\Delta t$ の増大に伴ってそのずれは大きくなる。一方で h が約0.7より大きい時には(図-4(b))式(21)が満足される(すなわち特性根が共役複素数である)限界において見かけの周期は無量大となり $\omega\Delta t$ が小さくなるにつれて見かけの周期は真の周期に近づいて行く。限界において見かけの周期が有限で真の周期より短い場合(図-4(a))と、限界において見かけの周期が無量大になる場合(図-4(b))との臨界に位置する h は式(22)で与えられる。この式は解が共役複素数でなくなる臨界状態で特性根(この時特性根は重根となる。)が0になるという条件から導かれる。

図-5は $h=0.1$ の系が式(17)で表現される初期条件で自由振動している状態を示している。 $\omega\Delta t$ が小さくなる程数値解の周期が延び真の周期に近づいて行く様子が顕著である。

図-6は数値的減衰($\bar{h}$)の真の粘性減衰に対する比を示す。 h が小さい時は $\omega\Delta t$ に関わらず数値的減衰は真の減衰にほぼ等しいが、 h が小さくない場合は $\omega\Delta t$ の増大に従って数値的減衰が真の粘性減衰に比べて相当大きくなることが判る。

5. 結 論

(1) 中央差分法を減衰系に適用した時の解の安定条件は非減衰系と同じ式(13)で与えられる。また式(21)が満足される時特性根は共役複素数となり、数値解は式(15)として解析的に表現され、数値解が振動系であることを保証する。

(2) 中央差分法を適用した場合、系の自由振動解の真の解に対する誤差は、見かけの周期のずれと、見かけの粘性減衰の増大として現れる。周期のずれは図-4で、粘性減衰の増大は図-6でそれぞれ与えられる。