THEORETICAL STUDIES OF THE BUCKLING AND NONLINEAR DEFLECTION BEHAVIOR OF CLAMPED PARTIAL CIRCULAR CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE

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1. Introduction

Many circular cylindrical roofs have collapsed during the snow in Japan, and most of them seem to have buckled generally¹⁾. Thus the buckling criteria of partial circular cylindrical shells have required for the designs. In this paper, to make clear the basic general buckling behaviors of cylindrical shell roofs, we shall take the case of partial cylindrical shells submitted to the action of uniform normal pressure.

For complete cylindrical shells under external pressure, the buckling behavior was fully explored toward the beginning of the 1970 s²⁾. On the other hand, for the partial ones, it does not seem to have been fully looked into theoretically in particular. The strongest reason might be that the partial cylindrical shells have not only the curved edges but also the straight edges.

Uchiyama³⁾ experimentally clarified the buckling behavior and expressed an approximate method of estimation of the buckling stress for the design. Singer *et al*⁴⁾ obtained the buckling loads of simple-supported partial cylindrical shells based on Donnell's linear equations neglected the prebuckling deflection in a wide range of the geometrical parameters. Yokoo and Nakamura⁵⁾ theoretically obtained some examples of the symmetrical nonlinear deflection behavior of clamped partial cylindrical shells using the finite differential method. And they⁶⁾ carried out experiments on the pre-and post-buckling behavior for the shells with free edges and represented an empirical formula for the buckling loads. Kokawa⁷⁾ theoretically analyzed the symmetrical and asymmetrical general buckling behavior of reticulated cylindrical roofs with arbitrary generator edges. Suzuki and Ogawa⁸⁾ also analyzed that behavior using a simplified method which dealt bending rigidity as rotational spring stiffness of member joints. The authors^{9), 10)} showed the experimental data of the overall distribution of initial geometrical imperfections and deflections for many accurate specimens of clamped partial cylindrical shells, and discussed the buckling behavior.

The object of this paper is to clarify the buckling loads and the nonlinear deflection behavior of clamped partial cylindrical shells under external pressure in a wide range of the Batdorf parameter Z and of the aspect ratio a systematically. We obtain not only the symmetrical snapthrough buckling loads but also the asymmetrical bifurcation ones applying the Galerkin procedure to Donnell's nonlinear equations. The possibility of asymmetrical buckling, which have not experimentally been established, is indicated for some shells. The effects of the parameters on the deflection behavior and the buckling modes are also shown, and the characteristics are in good agreement with the authors' experimental results.

2. Method of Analysis

Let us consider a thin-walled partial circular cylindrical shell of longitudinal length L , wall thickness h , radius R and central angle ϕ , which is made of homogeneous, isotropic elastic material with Young's modulus E and Poisson's ratio ν . The middle surface of the shell is referred to cylindrical coordinates as shown in Fig. 1. Distances

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from the middle surface are measured by a coordinate z , positive inward. Displacement components in the x , y , and z directions, respectively, are denoted by u , v , and w . The shell is considered to be subjected to uniform external pressure q (positive inward). On the basis of the Donnell nonlinear theory of the cylindrical shell, the basic equations are given as follows²⁾⁽¹¹⁾:

$$\left. \begin{aligned} \frac{\partial n_x}{\partial x} + \frac{\partial n_{xy}}{\partial y} &= 0 \\ \frac{\partial n_{xy}}{\partial x} + \frac{\partial n_y}{\partial y} &= 0 \\ -D \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 w + \frac{n_y}{R} + n_x \frac{\partial^2 w}{\partial x^2} + 2n_{xy} \frac{\partial^2 w}{\partial x \partial y} + n_y \frac{\partial^2 w}{\partial y^2} + q &= 0 \end{aligned} \right\} \quad (1)$$

$$\text{where } D = \frac{Eh^3}{12(1-\nu^2)}$$

In equation (1), n_x , n_y , and n_{xy} are the stress components per unit length, and

$$\left\{ \begin{array}{l} n_x \\ n_y \\ n_{xy} \end{array} \right\} = \frac{Eh}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array} \right] \left\{ \begin{array}{l} \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{array} \right\} \quad (2)$$

The following nondimensional forms are introduced:

$$\left. \begin{aligned} \xi &= x/L, \quad \eta = y/R\phi, \\ U &= Ru/Lh, \quad V = v/\phi h, \quad W = w/h, \\ Z &= \sqrt{1-\nu^2} L^2/Rh \quad (\text{Batdorf parameter}^{12}) \\ \alpha &= L/R\phi \quad (\text{aspect ratio}) \\ Q &= RL^2 q / \pi^2 D \quad (\text{load parameter}) \end{aligned} \right\} \quad (3)$$

The nondimensional basic equations are derived by the introduction of equations (2) and (3) into equation (1), as follows:

$$\left. \begin{aligned} L_x &= \beta \left\{ 2 \frac{\partial^2 U}{\partial \xi^2} + (1-\nu)\alpha \frac{\partial^2 U}{\partial \eta^2} + (1+\nu) \frac{\partial^2 V}{\partial \xi \partial \eta} - 2\nu \frac{\partial W}{\partial \xi} \right\} \\ &\quad + \frac{\partial W}{\partial \xi} \left\{ 2 \frac{\partial^2 W}{\partial \xi^2} + (1-\nu)\alpha \frac{\partial^2 W}{\partial \eta^2} \right\} + (1+\nu)\alpha \frac{\partial W}{\partial \eta} \frac{\partial^2 W}{\partial \xi \partial \eta} \\ &= 0 \\ L_y &= \beta \left\{ 2 \frac{\partial^2 V}{\partial \eta^2} + \frac{(1-\nu)}{\alpha} \frac{\partial^2 V}{\partial \xi^2} + (1+\nu) \frac{\partial^2 U}{\partial \xi \partial \eta} - 2 \frac{\partial W}{\partial \eta} \right\} \\ &\quad + \frac{\partial W}{\partial \eta} \left\{ 2\alpha \frac{\partial^2 W}{\partial \eta^2} + (1-\nu) \frac{\partial^2 W}{\partial \xi^2} \right\} + (1+\nu) \frac{\partial W}{\partial \xi} \frac{\partial^2 W}{\partial \xi \partial \eta} \\ &= 0 \\ L_z &= \frac{\pi^2}{3} \beta Q + 4\beta^2 \left(\nu \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} - W \right) - \frac{1}{3} \left(\frac{\partial^2}{\partial \xi^2} + \alpha \frac{\partial^2}{\partial \eta^2} \right)^2 W \\ &\quad + 4\beta \left[\left\{ \frac{\partial^2 W}{\partial \xi^2} + \nu\alpha \frac{\partial^2 W}{\partial \eta^2} \right\} \frac{\partial U}{\partial \xi} + (1-\nu)\alpha \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial U}{\partial \eta} + \frac{\nu}{2} \left(\frac{\partial W}{\partial \xi} \right)^2 - \nu W \frac{\partial^2 W}{\partial \xi^2} \right. \\ &\quad \left. + \left\{ \nu \frac{\partial^2 W}{\partial \xi^2} + \alpha \frac{\partial^2 W}{\partial \eta^2} \right\} \frac{\partial V}{\partial \eta} + (1-\nu) \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial V}{\partial \xi} + \frac{\alpha}{2} \left(\frac{\partial W}{\partial \eta} \right)^2 - \alpha W \frac{\partial^2 W}{\partial \eta^2} \right] \\ &\quad + 2 \left[\left\{ \frac{\partial^2 W}{\partial \xi^2} + \nu\alpha \frac{\partial^2 W}{\partial \eta^2} \right\} \left(\frac{\partial W}{\partial \xi} \right)^2 + 2(1-\nu)\alpha \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial W}{\partial \xi} \frac{\partial W}{\partial \eta} \right. \\ &\quad \left. + \alpha \left\{ \nu \frac{\partial^2 W}{\partial \xi^2} + \alpha \frac{\partial^2 W}{\partial \eta^2} \right\} \left(\frac{\partial W}{\partial \eta} \right)^2 \right] \\ &= 0 \end{aligned} \right\} \quad (4)$$

Fig.1 Sign convention and shell geometry.

where $\alpha = a^2$ and $\beta = Z/\sqrt{1-\nu^2}$

The independent parameters for the shell geometry in equation (4) are the Batdorf parameter Z and the aspect ratio a instead of the four geometrical values R , h , L and ϕ in equation (1).

The boundary conditions are

$$\left. \begin{aligned} W = \frac{\partial W}{\partial \xi} = U = V = 0, \quad \text{at } \xi = 0, 1 \\ W = \frac{\partial W}{\partial \eta} = U = V = 0, \quad \text{at } \eta = 0, 1 \end{aligned} \right\} \dots\dots\dots (5)$$

The displacement functions U , V , and W are assumed in the form of a linear combination of functions multiplied by unknown coefficients A_{ij} , B_{ij} , and C_{ij} which are not only independent of each other but also satisfy equation (5), and they are written as

$$\left. \begin{aligned} U &= \sum_i \sum_j A_{ij} U_{ij} = \sum_i \sum_j A_{ij} \sin m_i \pi \xi \sin n_j \pi \eta \\ V &= \sum_i \sum_j B_{ij} V_{ij} = \sum_i \sum_j B_{ij} \sin m_i \pi \xi \sin n_j \pi \eta \\ W &= \sum_i \sum_j C_{ij} W_{ij} = \sum_i \sum_j C_{ij} \sin m_i \pi \xi \sin n_j \pi \eta \sin \pi \xi \sin \pi \eta \end{aligned} \right\} \dots\dots\dots (6)$$

where m_i and n_j are the longitudinal wave number and the circumferential wave number, respectively. By application of Galerkin's method to equation (4), the Galerkin system of equations is given by

$$\int_0^1 \int_0^1 L_x U_{ij} d\xi d\eta = \int_0^1 \int_0^1 L_y V_{ij} d\xi d\eta = \int_0^1 \int_0^1 L_z W_{ij} d\xi d\eta = 0 \dots\dots\dots (7)$$

By substituting equation (6) into equation (7), the nonlinear algebraic equations, involving the load parameter Q , for the unknown coefficients A_{ij} , B_{ij} , and C_{ij} are obtained¹³⁾. The numerical solutions of the algebraic equations are procured step-by-step by both a linear incremental method and the Newton-Raphson method. The bifurcation point for an asymmetrical deflection mode is obtained as the point at which the determinant of coefficient matrix in the linear simultaneous algebraic equations for the supposed asymmetrical mode is zero, or regarded as zero¹⁴⁾. The computation was performed on a digital computer (ACOS 99-NEAC 1000) at Tohoku University in double precision.

3. Numerical Results

First, we checked our numerical results with some previous theoretical results. In ref. [15], the comparison with the results of Yokoo *et al.* obtained by using the finite difference method⁵⁾ was described. Figure 2 also shows the good agreement with Crisfield's results obtained by using the finite element method¹⁶⁾ as to nonlinear large deflection behavior.

In the computation, parameters m_i , and n_j in equation (6) are given as follows :

for symmetrical mode ;

$$\left. \begin{aligned} U &\left\{ \begin{array}{l} m_i = 2, 2, 2, 2, 2, 4, 4, 4, 4, 6, 6, 6, 8, 8, 10 \\ n_j = 1, 3, 5, 7, 9, 1, 3, 5, 7, 1, 3, 5, 1, 3, 1 \end{array} \right. \\ V &\left\{ \begin{array}{l} m_i = 1, 3, 5, 7, 9, 1, 3, 5, 7, 1, 3, 5, 1, 3, 1 \\ n_j = 2, 2, 2, 2, 2, 4, 4, 4, 4, 6, 6, 6, 8, 8, 10 \end{array} \right. \\ W &\left\{ \begin{array}{l} m_i = 1, 1, 1, 1, 1, 3, 3, 3, 3, 5, 5, 5, 7, 7, 9 \\ n_j = 1, 3, 5, 7, 9, 1, 3, 5, 7, 1, 3, 5, 1, 3, 1 \end{array} \right. \end{aligned} \right\} (N_s = 3 \times 15)$$

for asymmetrical mode ;

$$\left. \begin{aligned} U &\left\{ \begin{array}{l} m_i = 2, 2, 2, 2, 4, 4, 6, 8 \\ n_j = 2, 4, 6, 8, n, 2n, n, n \end{array} \right. \\ V &\left\{ \begin{array}{l} m_i = 1, 1, 1, 1, 3, 3, 5, 5 \\ n_j = 1, 3, 5, 7, n-1, n+1, n-1, n+1 \end{array} \right. \\ W &\left\{ \begin{array}{l} m_i = 1, 1, 1, 1, 3, 3, 5, 7 \\ n_j = 2, 4, 6, 8, n, 2n, n, n \end{array} \right. \end{aligned} \right\} (N_a = 3 \times 8)$$

Fig. 2 Comparison of load Q versus center deflection W_c curves with Crisfield's results.

Fig. 3 Example of pre-and post-buckling deflection behavior for the shell having $a=1.0$ and $Z=200$.
 (a) Circumferential distributions of deflection W along the center arch at various load levels.
 (b) Load Q versus center deflection W_c curves.

where N_s and N_a are the total numbers of terms for symmetrical and asymmetrical displacement modes, respectively, and n is an even number of circumferential half-waves of a supposed asymmetrical deflection mode. On the basis of the previous experimental results^{(9), (10), (17)}, asymmetrical deflection modes in the longitudinal direction are slighted in this study. The difference of symmetrical snap-through buckling loads calculated by two adoptions which were $N_s=3 \times 15$ and $N_s=3 \times 21$ was about 0.5 % in the case of the shell having $Z=300$ and $a=1.0$.

Poisson's ratio ν was taken as 0.3 in this study.

Next, we shall elucidate the variation of the distribution of pre-and post-buckling deflection with applied pressure Q for clamped partial cylindrical shells. Because of space consideration, the results for only the shell having $Z=200$ and $a=1.0$ are shown in Fig. 3 (a). In this figure, solid and dotted lines correspond to the circumferential distributions of deflection W at pre-and post-buckling equilibrium points a, b, c, d, and e in Fig. 3 (b). The nonlinear deflection behavior is remarkably recognizable not only after buckling but also before buckling, and it is found that the effect of prebuckling nonlinearity can not be omitted in this problem.

4. Effects of the Geometrical Parameter Z

The geometrical parameter Z is widely known as the Batdorf parameter for a complete circular cylindrical shell⁽¹²⁾ and participates the aspect ratio a in the independent geometrical parameters for a partial circular cylindrical shell as shown in equation (4). Calculations were carried out for $Z \leq 1000$.

Table 1 shows the comparison between the present buckling loads $Q_{cr} = RL^2 q_{cr} / \pi^2 D$ and Yamaki's results of complete cylindrical shells subjected to lateral pressure⁽¹⁸⁾. The buckling behavior of partial cylindrical shells seems to differ from that of the complete ones without straight edges. However, it is considered that the dominant component of buckling stress is the circumferential membrane component for both the partial ones and the complete ones, so the similarity is anticipated when a is small enough. The buckling loads Q_{cr} for $a=1.0$ and $a=2.0$ lie 16-29 % and 62-91 % higher than those of the complete shells, respectively.

The variation of Q_{cr} with Z is also shown in Fig. 4, where n_{cr} denotes the buckling mode number which is defined as the circumferential half-wave number of the dominant component of the symmetrical or asymmetrical deflection at the buckling point in the case of the symmetrical or asymmetrical buckling, respectively. The buckling load Q_{cr} and the buckling mode number n_{cr} become large as Z increases. In the case of $a=1.0$

Table 1 Buckling loads for partial circular cylindrical shells.

Z	PRESENT RESULTS		YAMAKI'S Q_0 (COMPLETE)
	$a = 1.0$	$a = 2.0$	
40	-----	-----	10.63
50	13.72	-----	11.63
70	17.04	23.13	13.44
100	20.30	29.42	15.77
150	23.87	36.40	19.05
200	26.92	41.46	21.86
300	32.38	49.23	26.63
400	35.77	53.62	30.68
500	41.15	57.63	34.28
700	47.27	69.60	40.58
1000	56.27	78.62	48.56

Fig. 4 Effects of the geometrical parameter Z on the buckling loads Q_{cr} .

Fig. 7 Longitudinal distributions of deflection W at the buckling points in the case of $a=1.0$

Fig. 5 Load Q versus volume change V_c curves in the case of $a=1.0$

Fig. 6 (a, b) Load Q versus center deflection W_c curves in the cases of $a=1.0$ and $a=2.0$, respectively.

the shells for $Z < 45$ do not buckle, and the non-buckling range expands into $Z < 60$ in the case of $a=2.0$. The discrepancy between the authors' experiment^{9),10)} denoted by the triangle symbols and the present theory might be caused by the initial imperfections¹⁹⁾.

Figures 5, 6, 7 and 8 are the variation of the deflection behavior with Z . In Fig. 5, the initial slope of the load Q

Fig. 8 (a, b) Variations of circumferential distributions of deflection along the center arch with Z in the cases of $a=1.0$ and $a=2.0$, respectively.

versus the volume change V_c curve increases as Z increases, where, from equations (3) and (6),

$$V_c = v_c / hLR\phi = \int_0^1 \int_0^1 W d\xi d\eta = C_{11}/4 \quad (8)$$

The circles and the triangles represent the symmetrical buckling points and the asymmetrical bifurcation points, respectively. In this figure, the even numbers accompanied by the triangles are the bifurcation mode numbers. When a shell has an asymmetrical bifurcation point before symmetrical snapthrough buckling, it is common for the load after buckling to decrease in the same manner as the curve ce in Fig. 3(b). However, in the case of the shell with $Z=40$ and $a=1.0$ which does not have a symmetrical buckling point, the secondary equilibrium path with $n=2$ has a positive slope. Therefore we understood that it does not buckle.

According to the results of the calculation, asymmetrical buckling ($n_{cr}=\text{even}$) occurs for $Z=(80-400)$ and $Z=(100-250, 700-1000)$ in the cases of $a=1.0$ and $a=2.0$, respectively. And it should be more emphasized that the asymmetrical buckling points on the load-deflection curves are very close to the symmetrical buckling points in all cases as shown in Figs. 5 and 6.

Figure 7 shows the distributions of W in the longitudinal direction through the maximum points at the buckling loads Q_{cr} . For all values of the geometrical parameters, the dominant component of the longitudinal distributions of W was one half wave mode in this study. It is in good agreement with the previous experimental results^{(9), (10), (17)}. On the other hand, the distribution of W in the circumferential direction at $Q=Q_{cr}$ becomes of a high order as Z increases as shown in Fig. 8. Figure 9 indicates that the distribution of asymmetrical buckling deflection in the circumferential direction becomes of a high order as Z keeps increasing.

5. Effects of the Geometrical Parameter a

In this chapter we shall discuss the effects of the geometrical parameter a , which is defined as the aspect ratio

Fig. 9 Examples of asymmetrical buckling modes for $a=2.0$

Fig. 10 Effects of the geometrical parameter a on the buckling load Q_{cr} in the case of $Z=200$.

Table 2 Comparison between the authors' experimental results and the present theoretical results for the buckling loads and modes.

a	EXPERIMENT				THEORY		Q _{cr} EX/TH %
	ϕ deg.	Q _{cr} KPa	Q _{cr}	n _{cr}	Q _{cr}	n _{cr}	
0.72	80	1.17	29.0	6	36.7	5	79
0.82	70	1.27	31.5	5	37.2	5	85
0.95	60	1.27	31.5	5	39.4	5	80
1.15	50	1.26	31.3	4	42.2	4	74
1.43	40	1.63	40.3	3	47.6	4	85
1.91	30	1.99	49.3	2	54.6	3	90

Z = 477, h = 0.3 mm, R = L = 150 mm, E = 5.6 GPa.

Fig. 12 Load Q versus volume change V_c curves in the case of $Z=200$.

Fig. 11 Load Q versus center deflection W_c curves in the case of $Z=200$.

Fig. 13 Variation of circumferential distributions of deflection along the center arch with a in the case of $Z=200$.

Fig. 14 Examples of asymmetrical buckling modes for $Z=200$.

$L/R\phi$ in equation (3), on the buckling load and the deflection behavior on the basis of the results for $Z=200$ and $Z=477$. In this study, a was varied from 0.6 to 4.0.

Figure 10 shows the variation of the buckling load with a . The ordinate is the ratio of the buckling load for partial cylindrical shells Q_{cr} to those for complete cylindrical shells Q_0 obtained by Yamaki¹⁸⁾. The buckling load Q_{cr} increases with an increase in a , and that of $a=4.0$ is about three times as large as Q_0 . As a keeps decreasing, Q_{cr} decreases like approaches Q_0 , and the buckling mode shows more and more waves in the circumferential direction.

In Table 2, the present results are compared with the previous experimental results⁹⁾ in the case of $Z=477$. The present buckling loads are 10-26 % higher than the experimental ones. However, the agreement between the theory and the experiment as to the effects of the parameter a on the buckling loads and modes is obtained.

The load-deflection behavior is shown in Fig. 11. In the initial loading stage the nonlinearities are not very recognizable, however, these are remarkably recognizable in the vicinity of the buckling load. The initial slopes of $Q \sim V_c$ curve in Fig. 12 have a constant gradient of 294/1 when $Z=200$, and this value is larger than the gradient 260/1 (denoted by a broken line) for the complete cylindrical shell. The difference is caused by the clamped longitudinal straight edges of the partial cylindrical shells. The results of Figs. 5 and 12 indicate that the initial slope of $Q \sim V_c$ curve depends not on the parameter a but on the parameter Z .

Figure 13 shows that the symmetrical deflection mode in the circumferential direction has more and more waves as a keeps decreasing. The circumferential distribution of asymmetrical buckling deflection becomes of a high order as a keeps decreasing as shown in Fig. 14.

6. Conclusions

Applying the Galerkin procedure to Donnell's nonlinear equations, the buckling and nonlinear deflection behavior of partial circular cylindrical shells under external pressure is theoretically clarified. We treat the fully clamped boundary conditions so as to compare our results with the previous experimental results. Main contents are summarized as follows :

- (1) The load-deflection relationships in the initial loading stage are almost linear. However, the nonlinearities are remarkably recognizable in the vicinity of the buckling point.
- (2) The nondimensional buckling load Q_{cr} increases with an increase in the Batdorf parameter Z and with an increase in the aspect ratio a . This is in good agreement with the previous experimental results.
- (3) It is possible to buckle asymmetrically as well as symmetrically. However, when the shell has an asymmetrical bifurcation point before symmetrical snap-through buckling, the bifurcation point is very close to the symmetrical buckling point.
- (4) The initial slopes of the nondimensional load Q versus the nondimensional volume change V_c curves are almost constant irrespective of a if Z is fixed.
- (5) The dominant component of the longitudinal distribution of deflection is one half wave mode in each case. The circumferential distribution of buckling deflection has more and more waves with an increase in Z and with a decrease in a . These are also in good agreement with the previous experimental results.

This study shows that, by using this analytical method, reasonably accurate solutions can be systematically obtained for the buckling loads and the deflection behavior of partial cylindrical shells, in the range of values of Z and a analyzed in this paper. The present parametrical results may be available for the design of cylindrical shell roofs.

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外圧を受ける周辺固定部分円筒殻の座屈と変形性状に 関する理論解析的研究 (梗概)

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1. 序

従来、円筒形屋根の積雪による倒壊災害例は非常に多い¹⁾。これらの倒壊の多くは屋根面全体の座屈破壊と考えられ、設計に当たって全体座屈に対する criteria が必要とされている。本文では、円筒形屋根の基礎的な全体座屈性状を明らかにするため、荷重は面にノーマルな一様分布の部分円筒殻の場合を取り扱うこととする。

外圧を受ける完全円筒殻の座屈性状については、1970年代初頭には十分に解明がなされている²⁾。しかし、部分円筒殻については、境界条件の扱いがより困難なこともあって、特に理論面で不十分な状況といえよう。

内山³⁾は、一様外圧を受ける部分円筒殻の座屈実験によってその座屈性状を明らかにし、設計用座屈応力を求めるための近似的手法を提案した。Singer ら⁴⁾は、周辺単純支持の部分円筒殻の座屈荷重を、座屈前変形を無視した Donnell 型の線形座屈条件式を解くことで、形状係数の広い範囲で系統的に求めた。横尾・中村⁵⁾は、周辺固定部分円筒殻の飛移り座屈付近の非線形変形性状を、差分法を用いて数値解析を行った。さらに彼ら⁶⁾は、母線縁自由の部分円筒殻の座屈実験を行い、座屈荷重に関する実験式を求めた。粉川⁷⁾は、任意母線縁境界をもつ円筒形網目シェル屋根の全体座屈についての理論解析を行い、対称ならびに非対称座屈性状を明らかにした。鈴木・小河⁸⁾は、円筒形トラスシェル屋根の座屈性状を、トラス部材の剛接度をバネ剛性で評価し有限要素法で解析した。筆者ら^{9), 10)}は、周辺固定部分円筒殻の座屈実験を行い、種々の形状係数を有する試験体について、初期不整と座屈前後の変位分布を殻全面にわたって詳細に測定し座屈性状を明らかにした。

本論文は、外圧を受ける周辺固定部分円筒殻の座屈荷重と非線形変形性状を、Batdorf の係数 Z と辺長比 a の広い範囲で系統的に解明することを目的としている。良く知られている Donnell 型の非線形方程式に Galerkin 法を適用することにより、対称飛移り座屈値

ならびに非対称分岐座屈値をも求めている。さらに、これらの形状係数が変形性状ならびに座屈モードに及ぼす効果を明らかとし、その特性が筆者らの実験結果と良く対応することを示している。

2. 解析方法

母線長さ L 、殻厚 t 、曲率半径 R 、開角 ϕ なる等方性弾性体の薄肉部分円筒殻を考える。 x, y, z 方向の変位を各々 u, v, w とする (図—1)。一様外圧 q は内向きを正とした。円筒殻に関する Donnell¹¹⁾ の非線形理論に基づいて、式 (1) の平衡方程式を基礎式として用いた^{2), 11)}。式 (2) は応力・変位関係式である。式 (3) なる無次元量を用いて式 (1) を整理すると、無次元化された基礎式 (式 (4)) が得られる。すると、形状に関する独立なパラメータは、Batdorf¹²⁾ の係数 Z と辺長比 a の二つであることがわかる。境界条件を式 (5) とする。変位係数 U, V, W には、式 (5) を満足するように式 (6) を仮定した。ここで、 m_i と n_j は各々母線方向と周方向の波数を表す。Galerkin 条件式は式 (7) で与えられる。式 (6) を式 (7) に代入し積分を実行すれば、変位関数の未定係数 A_{ij}, B_{ij}, C_{ij} に関する (荷重 Q を含む) 非線形連立代数方程式が得られる¹³⁾。この代数方程式の数値解は、増分法と Newton-Raphson 法を併用することで逐次求められる。非対称分岐座屈点は、仮定した非対称変形モードに対する線形同次代数方程式において、その係数マトリックスの行列式の符号が変化する点として得られる¹⁴⁾。数値計算は、東北大学大型計算機センター ACOS 99-NEAC 1000 を使用し、倍精度で行った。

3. 数値解析結果

まず、既往の数値解析結果との比較を行った。前報¹⁵⁾では、差分法による横尾らの結果⁵⁾との比較的良好な対応を示した。図—2 は、有限要素法による Crisfield¹⁶⁾の結果との極めて良い一致を示したものである。本研究では、数値解析に際して、式 (6) の仮定変位関数の項数は、対称項 $N_s=3 \times 15$ 、非対称 $N_a=3 \times 8$ としている。なお、既往の実験結果^{9), 10), 17)}を参考として、母線方向に沿う非対称なたわみモードについては考慮しないこととした。 $N_s=3 \times 15$ と 3×21 との差は、($Z=300, a=1.0$) の対

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称座屈値で約0.5%であった。ポアソン比 ν は本研究においては0.3とした。

次に、荷重の増加に伴う座屈前後のたわみ分布の変化を明らかにする。一例として、($Z=200$, $a=1.0$)の殻についての結果を図—3(a)として示す。この図における周方向のたわみ W の分布は、図—3(b)のa, b, c, d, eの各平衡状態のものである。非線形変形状は座屈後だけでなく座屈前でも極めて顕著であり、座屈前の非線形性が本問題では無視できないことがわかる。

4. 形状係数 Z の効果

形状係数 Z は、完全円筒殻におけるBatdorfの係数として良く知られており²⁾、部分円筒殻では式(4)でわかるように辺長比 a とともに形状に関する独立なパラメータの一つである。本研究では、 $Z \leq 1000$ の範囲を対象に解析を行った。

表—1は、本解析による座屈値を、側圧を受ける完全円筒殻についてのYamaki¹⁸⁾による結果との比較も行えるように整理したものである。部分円筒殻と完全円筒殻では、母線縁境界の有無の差より、その座屈性状は本質的に異なるが、両者とも周方向面内応力が座屈時において支配的であると考えられ両者の類似性も予想される。そこで両者を比較すると、本結果は完全円筒殻の値よりも、 $a=1.0$ の場合で16~29%、 $a=2.0$ の場合で62~91%も高くなっている。図—4のように、座屈値 Q_{cr} と周方向座屈半波数 n_{cr} は、 Z が大きくなる程、増加している。 $a=1.0$ の殻では $Z < 45$ で、 $a=2.0$ の殻では $Z < 60$ で、それぞれ座屈を生じない。図中の三角印で示した実験値^{9), 10)}と本解析値との差は、主に初期不整の影響と考えられる¹⁹⁾。

荷重 Q と体積変化 V_c の初期こう配は Z の増加に伴って増大する(図—5)。式(3)と式(6)から体積変化 V_c は式(8)として書ける。図中の丸印は対称座屈点を、三角印は非対称分岐点を示す。三角印に付記した偶数番号は分岐座屈モードの半波数である。対称飛移り座屈を生ずる以前に非対称分岐点が存在する場合は、図—3(b)の曲線ceのような分岐座屈後の荷重低下現象がみられた。しかし、($Z=40$, $a=1.0$)の場合、対称座屈点は存在せず、分岐座屈後の平衡曲線も荷重の低下を伴うことはなかったので、この殻は座屈を生じないと判断した。

本解析によれば、非対称座屈は、 $a=1.0$ では $Z=(80 \sim 400)$ で、 $a=2.0$ では $Z=(100 \sim 250, 700 \sim 1000)$ で生ずる。しかしより強調されることは、非対称座屈を生ずるすべての場合において、その非対称座屈点が対称座屈点に極めて近接して存在していることである(図—5, 6)。

母線方向に沿ったたわみ分布は、形状寸法に無関係に、半波数1のモードが卓越している(図—7)。これは、

既往の実験結果と良く一致する傾向である。一方、周方向に沿った分布(図—8)は、 Z の増加とともに高次となっている。非対称分岐座屈モードの W 分布(図—9)も、 Z の増加する程、高次の分布となることが分かる。

5. 辺長比 a の効果

辺長比 a が座屈値ならびにたわみ性状に及ぼす効果を、主に $Z=200$ の殻に対する結果をもとに述べる。 a の範囲は0.6~4.0とした。

座屈値に対する効果(図—10)については、 a を大きくする程 Q_{cr} は増加し、 $a=4.0$ では完全円筒殻の座屈値 Q_0 の約3倍に達している。一方、 a を小さくすると、周方向座屈半波数の増加を伴いながら、座屈値 Q_{cr} は Q_0 に漸近する形で減少することが明らかとなった。表—2は、 $Z=477$ の殻に対する座屈値と座屈モードについて、文献(9)の実験結果と比較したものである。座屈値については10~26%の差はあるものの、両者の傾向は極めて良く一致している。荷重~たわみ曲線(図—11)では、初期の荷重過程では直線的であるが、座屈直前には非線形性が著しくみとめられる。図—12の $Q \sim V_c$ 曲線の初期こう配は、294/1と a の値に無関係に一定で、完全円筒殻の場合の260/1よりも大きくなっている。この差は、部分円筒殻が母線縁でも支持されているためであると考えられる。図—5と図—12から、 $Q \sim V_c$ 曲線の初期こう配は、 a ではなく Z によって決まるものであることがわかる。

周方向に沿った座屈前対称変形モードは、 a が小さくなる程高次の分布となる(図—13)。非対称分岐座屈モードも、 a が小さくなる程、高次の分布となっている(図—14)。

5. 結 論

部分円筒殻の外圧による座屈値と非線型変形状を理論解析的に明らかにし、既往の実験結果との対応を検討して、次の結論を得た。

- (1) 座屈前の荷重・変形関係は初期では直線的であるが、座屈直前では非線形性が極めて著しい。
- (2) 無次元座屈値 Q_{cr} は、Batdorfの係数 Z が大きい程かつ辺長比 a が大きい程増加し、その傾向は既往の実験結果と良く一致している。
- (3) 非対称座屈の可能性はあるものの、その非対称分岐点と対称座屈点とは極めて近接している。
- (4) Z を一定とした場合、無次元荷重 Q ~無次元体積変化 V_c 曲線の初期こう配は、 a に無関係にほぼ一定である。
- (5) 母線方向に沿ったたわみ分布は半波数1のモードが卓越した形である。一方、周方向に沿った分布では、 Z が大きい程かつ a が小さい程、より多い波数成分が卓越するようになる。これらは既往の実験結果と良く一致している。