PAPER

Calculation of eigenray with equi-sound-speed division of sound speed profile

Nobuaki Takahashi,* Kohji Futa,* Toshio Tsuchiya,** and Toshiaki Kikuchi*

**National Defense Academy (NDA),
1-10-20, Hashirimizu, Yokosuka, 239-8686 Japan*

***Japan Marine Science and Technology Center (JAMSTEC),
2-15, Natsushima-cho, Yokosuka, 237-0061 Japan*

(Received 16 June 1999)

A new method, in which a sound speed profile in water is divided into layers at the points of equal sound speed above and below the SOFAR (sound fixing and ranging) axis, is proposed to calculate eigenrays for ocean acoustic tomography under range-independent sound speed profiles. The proposed method yields the following advantages over the conventional shooting method: first, it provides us with an eigenray in an analytical formula, which therefore makes the calculation time much shorter, with less quasi-eigenrays due to split beam rays at layer boundaries. Second, it allows us to easily calculate the number of eigenray cycles and to evaluate the effect caused by the discrepancy in depth between the sound source/receiver and the axis. Moreover, it clarifies how a small change in the structure of a sound speed profile above the axis relates to a structural change in a received pulse group. We can thus say that the proposed method is a promising means for identifying the received pulses. Based on a computer simulation that was performed, the three-pulse structure observed in the experiment was found to depend strongly on the relative arrangement between the source and receiver around the axis.

Keywords: Sound speed profile, Eigen ray, Tomography, Ray trace, OAT

PACS number: 43.30.Cq, 43.30.Pc

1. INTRODUCTION

In the analysis of ocean acoustic tomography (OAT), the identification of an eigenray, *i.e.*, the determination of the ray path that the observed pulses have traced, plays an important role.^{1,2)} Of late, both the ray trace method and the wave theory have been primarily used in analyzing the propagation of a sound wave. Recently, many researchers have worked on the wave theory because it gives quantitatively accurate results in the calculation of an acoustic field.³⁾ Since the wave theory requires a few orders more time in calculation than the ray trace method, however, it is not practical to apply it to OAT, which is expected to provide the identification of an eigenray eventually in real time.

In calculating an eigenray with the conventional

method of ray tracing, one needed to have a number of divisions in the radiation angle to obtain the sound ray that reaches the receiver (called the shooting method).⁴⁾ One of the problems associated with the shooting method is that the calculation time becomes longer in proportion to the propagation range of the sound wave. In addition, if one wants higher accuracy in the radiation angle of an eigenray using this method, further division of the angle is needed, which then results in a further increase in the calculation time. Since in the conventional method the division of the sound speed profile is based on the sound speed gradient change, there may appear a quasi-eigenray due to discontinuity of the division points.

In this paper, we propose a new method for range-independent forward problems⁵⁾ in which we regard

the minimum speed point as a reference axis and divide the sound speed profile at the same speed point above and below the SOFAR axis. By using the new method, it is possible to suppress the occurrence of a quasi-eigenray. Moreover, an eigenray can be conveniently expressed by an analytical formula and one can thereby complete the calculation in much less time compared to the conventional method of division. Referring to the experimental results obtained from the field measurements, we compare our new method for calculating an eigenray to the conventional method that is used in combination with the shooting method. Based on the result deduced from the above results, we show the advantages of the new method compared to the shooting method; it is easier to predict the influence of the number of the sound ray cycles, the depth of the transceiver, and the level of received pulses. Moreover, there is a distinct relationship between the sound-speed profile and the ray path, and finally it is a superior sound ray method in identifying received pulses in OAT observation.

2. EIGENRAYS IN BI-LINEAR PROFILES

In the calculation of a sound ray path, conventionally, one assumes a layered structure for the ocean instead of assuming a continuous inhomogeneous medium, *i.e.*, one divides into several tens of layers the sound speed profile, which is the distribution of the sound speed in the vertical direc-

tion, and calculates the path in each layer iteratively. At the moment, the method of division is not clearly defined and the only criteria is to minimize the sound speed gradient change in each layer. Since not only the sound ray path but also the eigenray depend on the method of division, however, we shall clarify its properties by proposing a new method, *i.e.*, the equi-sound-speed division method of the profile. The fundamental feature of the proposed method will be first made clear with the simplest example of a bi-linear profile.^{6,7)}

Consider a bi-linear profile for the sound speed distribution as shown in Fig. 1. In the conventional method, the gradient of sound speed above the SOFAR axis, g_0 is treated as negative quantity and that below the axis, G_0 is treated as positive, since the vertical coordinate origin ($z=0$) is placed at the ocean surface. If we place the origin on the SOFAR axis and take only the magnitude of the gradients into account, the analysis is expected to be simplified because the sound ray path behaves periodically within the SOFAR channel; in the following analysis the ray path behaviors are assumed to be independent of each other in the interval above and below the axis of $z=0$ and both g_0 and G_0 are regarded as positive. The radii of curvature of the upper ray path and lower ray path, R_u and R_d are therefore given by

$$R_u = \frac{c_0}{g_0 \cos \theta_0}, \quad R_d = \frac{c_0}{G_0 \cos \theta_0}, \quad (1)$$

respectively, where c_0 is the sound speed at the sound source and θ_0 is the radiation angle measured from the axis on which the sound source is placed and is defined always positive on either side of the axis. Providing that the receiver is placed on the axis as well as the sound source, the propagation pattern of the sound ray from the source to receiver, *i.e.*, the so-called eigenray, is then limited. The following four forms of eigenrays can be realized. The first form involves eigenrays in which the number of half periods above the axis is larger than that below the axis by one, which is called cycle mode 1. The second form consists of eigenrays that are propagated above the axis and have an equal number of half periods above and below the axis, called cycle mode 2. The third form, cycle mode 3, is the same as mode 2 except that the ray begins to propagate below the axis. The last form, cycle mode 4, is the counterpart of mode 1, *i.e.*, eigenrays that have a greater number of half periods below the axis than

Fig. 1 Model of bi-linear sound speed profile. The origin of the z -axis is placed on the SOFAR axis and the depth is defined as the distance from the axis and is positive both above and below the axis. An example of an eigenray is shown in the figure.

above the axis by one. Letting the number of half periods in modes 2 and 3 be n , the eigenrays in modes 1, 2 (and 3) and 4 satisfy the equations

$$\begin{aligned} 2nR_u \sin \theta_0 + 2(n-1)R_d \sin \theta_0 &= D, \\ 2nR_u \sin \theta_0 + 2nR_d \sin \theta_0 &= D, \\ 2nR_u \sin \theta_0 + 2(n+1)R_d \sin \theta_0 &= D, \end{aligned} \quad (2)$$

respectively. Here D is the horizontal range between the sound source and receiver. Substituting Eq. (1) into the second formula of Eq. (2), we obtain the radiation angle of the eigenray in cycle modes 2 and 3, θ_0 , as follows:

$$\theta_0 = \tan^{-1} \left\{ \frac{D}{2nc_0(1/g_0 + 1/G_0)} \right\}. \quad (3)$$

Similarly, the radiation angles of the eigenrays in modes 1 and 4 are readily obtained.

The travel time of the sound ray is given by integrating the reciprocal of the sound speed along the ray path. Converting it to the angle using Snell's law, we have

$$t = \int_0^s \frac{ds}{c(z)} = \int_0^\theta \frac{d\theta}{c'(z) \cos \theta}, \quad (4)$$

where the negative sign is involved in the sound speed gradient $c'(z)$. In a general case that the sound speed gradient is an arbitrary function of the depth z , it is difficult to carry out the above integral Eq. (4) and a numerical method is then required to evaluate the integral. If it is a linear function of the depth as in the bi-linear profile, it is easy to solve the integral analytically:

$$t = \frac{1}{g} \int_{\theta_0}^\theta \frac{d\theta}{\cos \theta} = \frac{1}{g} \cdot \log \frac{(1 - \sin \theta)/\cos \theta}{(1 - \sin \theta_0)/\cos \theta_0}, \quad (5)$$

where θ is an angle of a sound ray at the depth reached by the ray. The travel time in the half period above the axis is equal to twice the time spent by the ray from the starting point to the turning point at which $\theta=0$.

Using the gradients g_0 and G_0 above and below the axis in the equation, the travel time of the eigenray for whole n cycles is given by

$$\begin{aligned} t &= 2n \left(\frac{1}{g_0} + \frac{1}{G_0} \right) T, \\ T &\equiv \log \frac{2nc_0(g_0 + G_0)}{\sqrt{(2nc_0(g_0 + G_0))^2 + (Dg_0G_0)^2} - Dg_0G_0}. \end{aligned} \quad (6)$$

Although the above equation does not contain explicitly the angle θ_0 and is a function of g_0 , G_0 and n alone, the corresponding angle θ_0 is readily calculated using Eq. (3) once the cycle number n is determined. A similar procedure is applicable to the determination of the travel time in modes 1

Fig. 2 Horizontal range correction according to the depths of the source/receiver.

and 4.

In a general case of an OAT experiment, the depth of the receiver is not necessarily equal to that of the SOFAR axis, and there is usually some discrepancy between them. Figure 2 illustrates the range correction due to the depth discrepancy. If the receiver is located at a position higher than the axis, the horizontal range in cycle mode 1 is reduced by ΔD_r given by

$$\Delta D_r = R(\sin \theta_r - \sin \theta_0), \quad (7)$$

where θ_r is the angle of the sound ray at the receiving point. Replacement of D by $(D \pm \Delta D_r)$ in Eq. (2) therefore provides us with the equation of the eigenray even if the receiver position is above the axis. Considering the path of the sound rays, the sign in $\pm \Delta D_r$ is negative in modes 1 and 3, and is positive in modes 2 and 4. Similarly, the correction is also made in cases where the sound source is located off the axis. The correction of the travel time can be readily made by adding the time for the propagation through the range $\pm \Delta D_s$.

The relationship between the positions of the sound source and receiver in the SOFAR structure in an actual OAT observation is always categorized into one of the above cycle modes. All the eigenrays and their travel times are found to be analytically obtained by using the above equations instead of the shooting method. In order to obtain the eigenray of the propagation with the shooting method as has been done conventionally, one has to examine every possible ray path while varying the radiation angle in incremental steps. By using the proposed method that gives an analytical solution for the eigenray on the other hand, we can express the eigenray in terms of a fundamental formula once

the depth and the distance between the sound source and the receiver are given, which thereby allows us to significantly reduce the calculation time.

3. EIGENRAYS IN MULTILAYER PROFILES

In order to extend the method developed in the previous section to the actual sound speed profile, we shall next investigate a multilayer structure of the profile. As Fig. 3 shows, the origin of the z -axis is placed on the SOFAR axis and z is positive for both the upper and lower direction. Because of the pressure effect, the sound speed increases monotonically with increasing z below the SOFAR axis. It also increases uniformly above the axis as a whole, although it may have some decrease in a localized area above the axis. We assume here for simplicity that it increases monotonically both below and above the axis, since we are interested in understanding the overall behavior of a sound ray.

Division into a multilayer structure is not carried out arbitrarily as in the conventional method, but is done based on the equi-sound-speed division method as shown in the figure. In general, the sound speed change is more complicated above the axis compared to that below the axis. Therefore the sound speed profile is at first divided into small layers above the axis so that the gradient of the sound speed within each layer can be treated as

Fig. 3 The equi-sound-speed division of sound speed profile.

constant, which is the same as that in the conventional method, and then it is divided below the axis at the point where the speed is the same as at the division point above the axis rather than at an arbitrarily chosen point as in the conventional method.

Providing that there are L layers both above and below the axis, the eigenrays in cycle modes 2 and 3 satisfy the following equation :

$$\sum_{k=0}^{L-1} \left[(R_{u,k} + R_{d,k}) (\sin \theta_k - \sin \theta_{k+1}) \right] + (R_{u,L} + R_{d,L}) \sin \theta_L = \frac{D}{2n}, \quad (8)$$

where θ_k is the angle of the ray incident on the k -th layer, and $R_{u,k}$ and $R_{d,k}$ are the radii of curvature of the ray in the k -th layer above and below the axis, which are given by

$$R_{u,k} = \frac{c_k}{g_k \cos \theta_k} = \frac{c_0}{g_k \cos \theta_0}, \quad R_{d,k} = \frac{c_k}{G_k \cos \theta_k} = \frac{c_0}{G_k \cos \theta_0}, \quad (9)$$

where c_k is the sound speed in the k -th layer interface. g_k and G_k are the sound speed gradients in the k -th layer above and below the axis, respectively. Use of Snell's law allows the incident angle in each layer, θ_k , to be expressed in terms of the radiation angle θ_0 and the sound speed at the source, c_0 . Thus if we substitute Eq. (9) into (8) and using Snell's law, the eigenrays in cycle modes 2 and 3 satisfy the equation

$$\left(\frac{1}{g_0} + \frac{1}{G_0} \right) \sin \theta_0 + \sum_{k=1}^L \left(\frac{1}{g_k} - \frac{1}{g_{k-1}} \right) \sin \theta_k + \sum_{k=1}^L \left(\frac{1}{G_k} - \frac{1}{G_{k-1}} \right) \sin \theta_k = \frac{D \cos \theta_0}{2 n c_0} \quad (10)$$

The equations satisfied by the eigenrays in cycle modes 1 and 4 are obtained simply by replacing n in the equations for modes 2 and 3 by $n-1$ and $n+1$, respectively. The travel times for these eigenrays are obtained with a method similar to that used for obtaining Eq. (6) and is given as follows :

$$t = 2n \left\{ \sum_{k=0}^{L-1} \left[\left(\frac{1}{g_k} + \frac{1}{G_k} \right) T_k \right] + \left[\left(\frac{1}{g_L} + \frac{1}{G_L} \right) T \right] \right\}, \quad (11)$$

$$T_k \equiv \log \frac{(1 - \sin \theta_{k+1}) / \cos \theta_{k+1}}{(1 - \sin \theta_k) / \cos \theta_k} \quad (12)$$

$$T \equiv \log \frac{\cos \theta_L}{1 - \sin \theta_L}, \quad (13)$$

$$\cos \theta_k = \frac{c_k}{c_0} \cos \theta_0, \quad (14)$$

$$\sin \theta_k = \sqrt{1 - \cos^2 \theta_k}. \quad (15)$$

4. NUMERICAL RESULTS

4.1 Bi-linear Profile

A typical sound speed profile in the intermediate latitude areas is conveniently expressed by Munk's equation.^{5,8)} When the depth of the axis is 1000 m, Munk's equation gives the sound speed profile shown in Fig. 4. Approximating this profile by bi-linear lines, the sound speed at the ocean surface, on the SOFAR axis and at a depth of 2913 m are 1529, 1492 and 1529 m/s, respectively. Under these conditions, the magnitude of the gradients g_0 and G_0 are 0.037 and 0.0127/s, respectively. We have assumed that the sound source and the receiver are located at a depth of 1000 m and the horizontal travel distance is 1000 km, which is a typical range commonly used in OAT. We first obtain the number of cycles n and the radiation angle θ_0 by using Eq. (2) and then the travel time of the eigenray with Eq. (6). We finally calculate the propagation loss from the depth difference between the eigenray and its adjacent ray, whose radiation angle is 0.02-degree larger than that of the eigenray.

The numerical results are shown in Fig. 5 (a), where the horizontal axis corresponds to the travel time and the vertical to the transmission loss and the radiation angle marked by circles. The pulses in

Fig. 4 Munk's sound speed profile and its approximated bi-linear profile used in the numerical calculation.

the figure are the eigenrays corresponding to all the cycle modes within the radiation angle of ± 12 degrees. Figure 5 (b) is the same as Fig. 5 (a) with the time scale expanded.

As the time range from 665.2 until 665.6 seconds in Fig. 5 (b) shows, the pulses in cycle modes 2 and 3, which have the radiation angles 11 and -11 degrees, respectively, arrive at the receiver approximately at the same time. Three pulses can be observed in the figure, though four pulses actually exist.

The pulses in cycle modes 1 and 4 reach the receiver after and before the pulses in modes 2 and 3 arrive, respectively. Accordingly, it can be said that there are pulse groups for every different cycle n

Fig. 5 Relation between travel time and relative received level of eigenrays and radiation angles of rays. Results for all eigenrays are shown in the upper plot, while an expanded view of the time period from 665 to 666 seconds is represented in the lower plot.

Fig. 6 Travel time difference between an n -cycle pulse and an $(n+1)$ -cycle pulse. The range from the source to the receiver is 1000 km.

consisting of three pulses.

Although the bi-linear profile of the sound speed offers qualitatively a good explanation for the phenomenon, it quantitatively differs somewhat from the actual profile, especially above the axis. Here we thus investigate the influence of the sound speed gradient on the characteristics of the eigenray, such as the relative travel time difference between the pulse groups of three-pulse structure. Figure 6 shows the travel time differences between the pulse groups in the cycle n and that in the cycle $n+1$ as a function of the radiation angle of the eigenray for three values of the sound speed gradient. The solid, broken and dotted broken lines show the results for the gradient $g_0=0.027$, 0.037 and 0.047 , respectively. As the figure shows, the time intervals of pulse groups of three-pulse structure vary significantly with the sound speed gradient above the axis.

In order to clarify the three-pulse structure in the pulse group, the travel time differences of the pulses of cycle modes 1 and 4 from the pulses of cycle mode 2 are shown as a function of the radiation angle in Fig. 7. We can see from the figure that as the radiation angle increases, the differences become larger and therefore the three-pulse structure tends to be distinctively recognized. This is because for the larger value of the radiation angle, the travel distance is longer and the travel time difference caused by the difference of the sound speed gradient also becomes larger above and below the SOFAR axis.

The above-mentioned travel time differences are shown as a function of the number of the cycles n in

Fig. 7 Travel time difference of pulses in cycle mode 1 and 4 from a pulse in cycle mode 2 as a function of radiation angle at the sound source.

Fig. 8 Travel time difference of pulses in cycle mode 1 and 4 from a pulse in cycle mode 2 as a function of the number of eigenray cycles.

Fig. 8. One can see that when the radiation angle is large, *i.e.*, when the eigenray propagates near the ocean surface, which is a case of great importance in OAT, the pulse interval in the eigenray depends strongly on the magnitude of the gradient g_0 . For an eigenray pulse of many cycles, which propagates near the SOFAR axis, the sound speed gradient does not have significant influence on the pulse interval.

Knowledge of the eigenray characteristics described in the above discussion is expected to help us to discriminate eigenrays in the observed signals.

4.2 Comparison between Shooting and Equi-Sound-Speed Division Methods

4.2.1 Sound speed profile for simulation

The data under our investigation with a model simulation were obtained when a 200 Hz transceiver was tested in the ocean in December, 1995. The

Fig. 9 Measured sound speed profiles. The profiles are measured at four points along the line from the sound source to the receiver at 50-km intervals.

Fig. 11 Travel time of rays as a function of the radiation angle.

Fig. 10 Division methods of sound speed profiles for numerical calculation. (a) corresponds to the conventional method, and (b) to the equi-sound-speed division method.

horizontal range between the two transceivers is 200.65 km and their installed depths are 992 and 1114 m. The sound speed profile was measured every 50 km along a propagation path of 200 km. All the profiles at the four points are shown in Fig. 9. These show typical SOFAR structures in the intermediate-latitude ocean area and the depth of the SOFAR axis is approximately 1000 m from the surface. Since the difference between the sound speed profiles measured at the four points is small, we have regarded the average of those profiles as the

sound speed in the area to carry out the simulation. Figure 10 (a) shows the division points of the profile based on the conventional shooting method, in which there are more divisions in the area of large gradient change and fewer divisions in the area of small gradient change. The total number of the divisions is 26. Figure 10(b) shows the divisions based on the equi-sound-speed division method, in which the division is made at the same speed point above and below the SOFAR axis. In this example, the numbers of the divisions above and below the axis are respectively 13 and the total is 26, as in the conventional method.

4.2.2 Comparison in eigenray characteristics

The time spent by a sound ray that radiates from the source within an angle of ± 12 degrees and propagates to the receiver 200 km away from the source is shown in Fig. 11. The dark line corresponds to the equi-sound-speed method and the light line to the conventional method. It is clear from the figure that both lines show a similar tendency as a whole, but the dark line shows finer structures than the light line. This is presumably because when using the conventional method, the number of layers that the ray passes above the axis is different from that below the axis, and a kind of averaging is done during the propagation of the ray. As Fig. 12 shows, if the ray is refracted near the layer boundary above the axis, in the proposed method, it is refracted near the corresponding layer boundary below the axis as well. In the conventional method of division on the other hand, different rays are

Fig. 12 Ray paths calculated by the equi-sound-speed division method.

refracted near the layer boundary above and below the axis and a discontinuity appears in the variation of the sound ray flux as a function of the radiation angle.

In the ray theory, the propagation loss is inversely proportional to the distance between adjacent rays. The discontinuity of the sound ray flux and the depth difference between adjacent rays are likely to cause some ghosts and may give rise to a serious error in identifying an eigenray. The energy radiated from the sound source is constant within each sound ray flux. Since the ray passes through layers that have different sound speeds above and below the axis, in the conventional method, the sound ray flux may easily suffer an extra diffusion like split beam rays.⁴⁾ On the other hand, in the proposed method, the ray always passes through layers of the same sound speed above and below the axis, the sound ray flux does not suffer an extra diffusion and thereby the occurrence of a ghost pulse is reduced.

4.3 Comparison with Experiment

A comparison between the experimental result and theory is shown in Fig. 13. The fact that there are not many eigenrays observed in this case can be understood because (1) the depth of the sound source is different from that of the receiver and (2) the propagation distance of 200 km is relatively

short. Figure 13 (a) shows the received pulses and (b) and (c) show the theoretical predictions based on the conventional and proposed method, respectively. The magnitude of pulse B in Fig. 13 (b) is smaller than the observed pulse A in (a) by approximately 15 dB. In addition, there is a ghost pulse C that appears in the theoretical prediction but is not observed in the experiment. Since this pulse has the receiver incident angle of -14 degrees and reaches the receiver at the earliest time, it is usually recognized as an important pulse in identifying the sound ray and there is a significant risk of it giving rise to an error. On the other hand, in an analysis based on the proposed method, coincidence between the theoretical prediction and experimental result is fairly good and no ghost is seen, which shows the validity of the proposed method. However, a precise investigation regarding the slowest pulse such as the pulse D in (a), which propagates in the neighborhood of the SOFAR axis, shows such a discrepancy between the theoretical prediction and experimental result that the theoretical prediction yields a fairly large number of sound rays near E even with the calculation based on the proposed method, though the observed data consists of only one eigenray. This is not due to the method of division, but to the method of approximation for the sound speed profile, which should then be investigated in

Fig. 13 Temporal structure of received pulses. (a): experimental data, (b): calculated by the conventional method. (c): calculated by the equi-sound-speed division method.

future work.

5. SUMMARY

Since the observation area of OAT will grow even larger from now on, the conventional division method of a sound speed profile in combination with the shooting method is extremely limited in its validity to identify eigenrays efficiently and accurately. As an alternative approach, we have proposed the equi-sound-speed division method, with which it is possible to obtain an eigenray in analytical form, with less quasi-eigenrays due to split beam rays at the layer boundaries. In the proposed method, first the origin of the depth coordinate is placed on the SOFAR axis and a sound speed profile is divided into line segments at the points of equal sound speed above and below the axis: in each segment the sound speed gradient is

assumed to be constant. Next, the cycle mode of an eigenray is determined depending on the relative arrangement between the sound source/receiver and the SOFAR axis, and the ray path is obtained analytically. Comparing to the experimental result, the proposed method is found to have the following advantages over the conventional method:

(1) Since a sound speed profile is divided at the equi-sound-speed points above and below the SOFAR axis, the propagation of a sound ray is symmetrical with respect to the sound speed above and below the axis and the degree of influence due to the division of a sound speed profile is reduced to half of that in the conventional method, which results in the reduction of ghosts in the calculation.

(2) It is possible to predict the intensity level of an eigenray pulse with high accuracy, which is difficult to do with the shooting method.

(3) Since various parameters associated with the ray path of each sound ray can be expressed in terms of analytical formulae, the calculation time is significantly reduced and it is expected to open up the possibility of real-time analysis of an eigenray.

REFERENCES

- 1) W. H. Munk and C. A. Wunsch, "Ocean acoustic tomography: A scheme for large scale monitoring," *Deep-Sea Res.*, 123-161 (1979).
- 2) R. C. Spindel and P. F. Worcester, "Ocean acoustic tomography programs: Accomplishments and plans," *Proc. OCEANS '90* (1990).
- 3) A. Tolstoy, D. H. Berman, and E. R. Franchi, "Ray theory versus the parabolic equation in a long-range ducted environment," *J. Acoust. Soc. Am.* **78**, 176-189 (1985).
- 4) C. B. Officer, *Introduction to the Theory of Sound Transmission with Application to the Ocean* (McGraw-Hill, New York, 1958).
- 5) W. Munk, P. Worcester, and C. Wunsch, *Ocean Acoustic Tomography* (Cambridge University Press, New York, 1995), pp. 30-58.
- 6) G. V. Frisk, *Ocean and Seabed Acoustics—A Theory of Wave Propagation* (Prentice-Hall Inc., Englewood Cliffs, 1994), p. 227.
- 7) I. Tolstoy and C. S. Clay, *OCEAN ACOUSTICS—Theory and Experiment in Underwater Sound* (AIP Inc., New York, 1987), p. 148.
- 8) W. H. Munk, "Sound channel in an exponentially stratified ocean, with application to SOFAR," *J. Acoust. Soc. Am.* **55**, 220-226.