Extension of sampling theorem for wave equations

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1. Introduction

The numerical solution of ordinary differential equations has been widely used in many fields including wave propagation analysis. To represent a continuous function in terms of its discrete sampled values in a sequence, it should satisfy the sampling theorem. However, in conventional wave propagation analysis, the experiential finite difference technique has generally been used. In this paper, the proposed sampling extension which converges more rapidly than in the case of classical cardinal series is used to obtain the sampled values of the spatial differential of wave fields. In the present method, the spatial differential equations are solved by a simple sampling extension method whereas the temporal iteration is performed by a finite difference technique. The applicability of the sampling extension method is shown by simulating both one-dimensional and two-dimensional propagation of an impulsive incident wave. Numerical results also indicate that the sampling extension method is much accurate than that of the finite difference method.

Various sampling theorems [1] provide the justification for the use of sampled data in data processing, communication, and control. The best known of these is associated with Shannon [2]. This theorem states that if a deterministic signal $f(t)$ is low-pass band-limited to W cycles per second, then the signal can be represented exactly by an infinite number of equispaced sampling points. In this case, the representation of $f(t)$ is given by a well-known cardinal series expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin 2\pi W(t - nT)}{2\pi W(t - nT)}, \quad (1)$$

where $T = 1/2W$ is the sampling rate [2–4].

It is possible to obtain sampling extensions where the expansion converges more rapidly than in the case of the cardinal series. The extension for the band-limited functions $f(t)$ can be represented by a series of exponential functions in terms of the parameter σ .

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin 2\pi W(t - nT)}{2\pi W(t - nT)} \exp\left\{-\frac{\sigma^2}{4}(t - nT)^2\right\}. \quad (2)$$

This result can be generalized to include the sampled values of derivatives of the signal of all orders, as well as the sampled values of integration of the signal.

In practical application, the right-hand side of Eq. (2) usually differs from $f(t)$. The difference is called sampling

error and may arise from several sources. This includes the extension and aliasing errors which are caused by violating the band-limitedness of the signal, the truncation error which results when only a finite number of samples are used instead of the infinite number of samples required for the sampling representation, and the round-off error which arises in digital recording of the sampled values. A comprehensive treatment of some of these errors have been reported [5].

2. Two-dimensional wave equation

The wave equation in two-dimensional spatial coordinates x, y is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}, \quad (3)$$

where $\phi(x, y, t)$ represents a wave moving with velocity c . In order to specify a wave, the equation is subject to boundary conditions

$$\left. \begin{aligned} \phi(0, y, t) &= 0 \\ \phi(L_x, y, t) &= 0 \\ \phi(x, 0, t) &= 0 \\ \phi(x, L_y, t) &= 0 \end{aligned} \right\} (0 < t < \infty),$$

and initial conditions

$$\left. \begin{aligned} \phi(x, y, 0) &= g(x, y) \\ \frac{\partial \phi(x, y, 0)}{\partial t} &= 0 \end{aligned} \right\} \begin{aligned} &(0 \leq x \leq L_x), \\ &(0 \leq y \leq L_y), \end{aligned}$$

where, L_x and L_y are lengths in x and y directions, respectively. The general solution can be solved as a Fourier series as [6]

$$\phi(x, y, t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} B_{pq} \cos(\omega_{pq} t) \sin\left(p\pi \frac{x}{L_x}\right) \sin\left(q\pi \frac{y}{L_y}\right), \quad (4)$$

where ω is defined by

$$\omega_{pq} = \sqrt{\left(\frac{p\pi c}{L_x}\right)^2 + \left(\frac{q\pi c}{L_y}\right)^2},$$

and

$$B_{pq} = \frac{2}{L_x} \frac{2}{L_y} \int_0^{L_x} \int_0^{L_y} g(x, y) \sin\left(p\pi \frac{x}{L_x}\right) \sin\left(q\pi \frac{y}{L_y}\right) dx dy.$$

2.1. Finite difference time-domain solution

The second derivative can be approximated by

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$$\frac{\partial^2 \phi}{\partial x^2} \simeq \frac{\phi(x + \Delta x) - 2\phi(x) + \phi(x - \Delta x)}{(\Delta x)^2}. \quad (5)$$

Then, the solution for finite difference time-domain method (FDTD) for the special case where $\Delta x = \Delta y$ becomes

$$\begin{aligned} \phi_{i,j}^{k+1} = & \gamma^2 (\phi_{i+1,j}^k + \phi_{i-1,j}^k + \phi_{i,j+1}^k + \phi_{i,j-1}^k) \\ & + 2(1 - 2\gamma^2) \phi_{i,j}^k - \phi_{i,j}^{k-1}, \end{aligned} \quad (6)$$

where $\phi_{i,j}^k = \phi(x_i, y_j, t_k)$, $x_i = x_0 + i\Delta x$, $y_j = y_0 + j\Delta y$, $t_k = t_0 + k\Delta t$, and

$$\gamma^2 \equiv c^2 \frac{(\Delta t)^2}{(\Delta x)^2},$$

which is called the Courant-Freidrichs-Lewy number, or CFL number [7]. The FDTD scheme is stable for $\gamma < 1$.

2.2. Derivative filter solution

The second derivative with respect to x and y can be found using proposed derivative filter by simply derivating Eq. (2) twice with respect to x and y , respectively. Denoting the left-hand side of Eq. (3) with $\phi_{xx,yy}$, then Eq. (3) becomes

$$\frac{\phi_{i,j}^{k+1} - 2\phi_{i,j}^k + \phi_{i,j}^{k-1}}{(\Delta t)^2} = c^2 (\phi_{xx,yy})_{i,j}^k. \quad (7)$$

Rearranging this equation, then

$$\phi_{i,j}^{k+1} = \gamma^2 (\Delta x)^2 (\phi_{xx,yy})_{i,j}^k + 2\phi_{i,j}^k - \phi_{i,j}^{k-1}. \quad (8)$$

3. Accuracy tests

In order to evaluate the accuracy of our proposed differential filter solution, $L_x = L_y = 200$ mm, $\Delta x = \Delta y = 1$ mm, $c = 1,500$ m/s, and $\text{CFL} = 0.25$ were used. Initial conditions were chosen to specify a pulsative Gaussian incident wave as shown in Fig. 1. An example of derivative

Fig. 1 Initial condition field for a 200×200 mm plane.

Fig. 2 The solution of wave motion by derivative filter at $t = 66.5 \mu\text{s}$. Wave fields are plotted using a bipolar logarithmic scale with a dynamic range of 60 dB.

Fig. 3 The solution of wave motion by FDTD at $t = 66.5 \mu\text{s}$. Wave fields are plotted using a bipolar logarithmic scale with a dynamic range of 60 dB.

filter computation and FDTD computation of the wavefield impinging at time $t = 66.5 \mu\text{s}$ are shown in Figs. 2 and 3, respectively. To verify these calculations, the fields are compared to those obtained using the given exact solution, where the iterations terminate when the successive value is within the tolerance of 10^{-6} . Figures 4 and 5 show the error

Fig. 4 Accuracy of the derivative filter solution compared with the exact solution.

Fig. 5 Accuracy of the FDTD solution compared with the exact solution.

for derivative filter and FDTD solutions, respectively. The y-axis of both graphs reveals the maximum error, where the error for FDTD solution is about 5 times as compared to the error for derivative filter solution. These results demonstrate the accuracy of our proposed solution.

4. Conclusion

The present sampling extension has been shown to solve the ordinary wave equation numerically. Numerical results also indicate that the derivative filter solution is much accurate than that of the FDTD approach.

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