Elliptically curved acoustic lens for emitting strongly focused finite-amplitude beams: Application of the spheroidal beam equation model to the theoretical prediction

Masahiko Akiyama* and Tomoo Kamakura

The University of Electro-Communications,
1–5–1 Chofugaoka, Chofu, 182–8585 Japan

(Received 12 October 2004, Accepted for publication 26 November 2004)

Abstract: Strong focusing of a plane progressive ultrasound wave by a plano-concave lens with a widely opening aperture, which is made of acrylic resin and is submerged in water, is investigated theoretically and experimentally. To eliminate spherical aberration, an elliptic surface lens is introduced. The amplitude of the sound emitted from the lens increases abruptly with propagation toward the focus, then nonlinear harmonic generation in the beam becomes active significantly. The spheroidal beam equation (SBE), which has been previously proposed to be more amendable to the analysis of a highly focused nonlinear beam, is used to predict the first three harmonic components in the beam. To make sure of the effectiveness of the present theory, experiment is executed in water using an elliptic surface lens of eccentricity 0.544 which is attached tightly to a 1.7-MHz planar transducer with a circular aperture of 75-mm in diameter. It has been shown that the experimental data are overall in excellent agreement with the theoretical prediction. Especially, remarkable suppression of side-lobe levels in beam patterns of higher harmonics has been demonstrated successfully.

Keywords: Acoustic lens, Elliptic surface, SBE, Nonlinearity, Harmonic generation

PACS number: 43.58.Ls, 43.25.Cb, 43.25.Jh

[DOI: 10.1250/ast.26.279]

1. INTRODUCTION

Focused sound beams have many applications such as medical diagnosis and non-destructive testing of material. Especially, a strongly focusing source is frequently used to achieve high lateral resolution in ultrasonic imaging systems and high ultrasonic power at a localized spot, the focus, in industrial technology. Since the amplitude of the sound radiated from such a focusing source increases abruptly with propagation toward the focus, waveform distortion may intrinsically occur due to the inherent nonlinearity of the medium, even if the source pressure level is relatively low.

Focusing of a sound beam is often achieved by combination of a planar transducer and an acoustic lens. The lens that is ordinarily made of plastic or solid material propagates sound at a higher speed than that at a surrounding medium such as water and body tissues, then the configuration of the lens should be concave. A spherically curved surface lens is widely used because of theoretical simplicity for describing sound beams emitted from it and of its easy fabrication. It is known, however, that the aberration of the spherical lens occurs significantly, in particular for a widely opening aperture lens and becomes responsible for image blurring. To reduce the spherical aberration, the present authors have proposed an elliptic surface lens whose eccentricity is equal to the ratio of sound speed in a surrounding fluid to that in the lens [1]. Using a 1.7-MHz ultrasound beam, they have experimentally demonstrated that side-lobe levels of the sound pressure in the focal plane are about 8-dB lower than those by a spherical surface lens.

Incidentally, quite a number of reports have been published on theoretical and experimental investigations of focused nonlinear sound beams. Almost all the work begin with the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation that has been widely used as a model equation of finite-amplitude sound beams propagating in a viscous fluid. The KZK equation is derived under the paraxial approximation. Therefore, its applicability to focused beams is limited to small opening angles of aperture. In fact, it is pointed out that it is reliable for aperture half-angles of the sources that are 16° or less [2]. For more widely opening aperture sources, Kamakura et al. proposed

*e-mail: akiyama@ew3.ee.uec.ac.jp
another model equation, the spheroidal beam equation (SBE), whose upper limit of the applicability is at least 40° for the half-aperture angle [3]. Usefulness of the SBE has been reported so far for describing strongly focused nonlinear beams [4–6].

The SBE model resorts to the theoretical hypothesis that an ultrasound source whose aperture shape is a spherical cap or bowl radiates sound waves with equiphase over the surface. In this case, the pressure amplitude reaches its maximum (i.e., acoustic focus) at a position near the center of the bowl curvature. For a concave lens, however, the acoustic focus is generally positioned away from the curvature center because sound speed in the lens is finite and the phase over the lens surface is delayed from the curvature center because sound speed in the lens is finite and the phase over the lens surface is delayed from the curvature center because sound speed in the lens is finite and the phase over the lens surface is delayed from the curvature center because sound speed in the lens is finite. However, spherical aberration occurs simultaneously and the gain does not increase as can be monotonously. Therefore, spherical aberration occurs simultaneously and the gain does not increase as can be expected. As a means of eliminating the aberration, geometrical optics suggests that the eccentricity of an elliptic surface lens \( \varepsilon = \sqrt{a^2 - b^2} / a \), a and b are the lengths of the semi-major axis and of the semi-minor axis, respectively) is equalized to the velocity ratio \( n_1 \); i.e. [7],

\[
\varepsilon = n_1. \tag{2}
\]

When a plastic lens made of acrylic resin is submerged in water, for example, we have \( c_1 = 2,720 \, \text{m/s} \) and \( c_0 = 1,480 \, \text{m/s} \) at 20°C, then \( n_1 \) or the eccentricity takes a value of 0.544, smaller than unity [1].

To make sure whether the above hypothesis is valid in acoustics, we analyze a focused linear sound field emitted from an axisymmetric elliptic surface lens. The geometry of the lens is shown in Fig. 1. A plane ultrasonic wave is assumed to propagate along the positive z axis. Let the coordinates of the position Q of a point sound source on the surface be \((z_0, r_0)\). These two spatial variables are

\[
z_0 = -R_0(\theta) \cos \theta, \quad r_0 = R_0(\theta) \sin \theta, \tag{3}
\]

where \( R_0 \) is the distance from the origin O to Q, \( \theta \) is the angle between \( \overline{OQ} \) and the \( -z \) axis. Substituting \( z_0 \) and \( r_0 \) into the elliptic equation of \( z_0^2 / a^2 + r_0^2 / b^2 = 1 \), we obtain

\[
R_0(\theta) = \frac{ab}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/2}}. \tag{4}
\]

Therefore, \( z_0 \) and \( r_0 \) are given as follows:

\[
z_0 = -\frac{ab \cos \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}, \tag{5}
\]

\[
r_0 = \frac{ab \sin \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}. \tag{6}
\]

The line element \( dl \) on the surface is

\[
dl = \frac{ab}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \sqrt{a^4 \sin^2 \theta + b^4 \cos^2 \theta} \, d\theta, \tag{7}
\]

then the area element \( dS \) is expressed by \( dl \times r_0 d\varphi \), where \( d\varphi \) is an increment in the azimuth \( \varphi \) from the \( r-\)z plane. We thus obtain

\[
dS = \frac{R_0^4}{a^2 b^2} \sqrt{a^4 \sin^2 \theta + b^4 \cos^2 \theta} \times \sin \theta \, d\theta d\varphi. \tag{8}
\]

Additionally, the distance between the observation point \( P(z, r) \) and Q gives

\[
R = \sqrt{(z - z_0)^2 + r^2 - 2r_0 \cos \varphi + r_0^2}. \tag{9}
\]
When a planar ultrasound wave with vibrating velocity amplitude \( v_z \) is propagating in the lens in parallel with the \( z \)-axis, the normal component of the velocity on the interface between the lens and water takes the following approximate form:

\[
v_n = v_z \frac{2Z}{Z + Z_0} (k \cdot n) e^{-\left(\alpha + jk(z_0 + a)\right)},
\]

(10)

where \( Z \) and \( Z_0 \) are the acoustic impedances of the lens and water, respectively, \( k_l = \omega/c_l \) is the wavenumber of the longitudinal wave in the lens, \( \alpha \) is its sound absorption coefficient, \( k \) is the unit vector along the \( z \)-axis and \( n \) is the outer normal unit vector on the surface. Between the latter two unit vectors, there is the angle relationship of \( k \cdot n = \cos \theta/\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \), which is called directional cosine. Since \( v_z 2Z/(Z + Z_0) \) is independent of the spatial variables, we replace it with \( v_z \) again. Then, we finally obtain the velocity potential in the sound field by the Rayleigh integral [1]:

\[
\phi = \frac{v_z}{\pi \alpha a} \int_0^\alpha \int_0^{\theta_m} e^{-\left(\alpha + jk(z_3 + a)\right)} \frac{e^{-jkR}}{R} \times R_0^a \cos \theta \sin \theta d\theta d\phi.
\]

(11)

Especially, when the observation point exists on the \( z \)-axis, the integration with respect to \( \varphi \) in Eq. (11) is readily performed to be

\[
\phi = \frac{2v_z}{\alpha} \frac{1}{a} \int_0^{\theta_m} e^{-\left(\alpha + jk(z_3 + a)\right)} \frac{e^{-jkR}}{R} \times R_0^a \cos \theta \sin \theta d\theta.
\]

(12)

In Eqs. (11) and (12), the upper limit of the integration becomes

\[
\theta_m = \sin^{-1} \left[ \frac{1}{\sqrt{(a/a_0)^2 + 1 - (a/b)^2}} \right],
\]

where \( a_0 \) is the radius of the lens aperture.

### 2.2. Boundary Conditions Appropriate to SBE

To apply successfully the SBE model to the field analysis of a strongly focused sound beam emitted from a concave lens, it is primarily needed to specify the geometrical focus. Figure 2 shows the geometry and notations used in the present lens model. The actual lens surface \( S_1 \) is depicted in a solid line, which indicates an ellipse with semi-major length \( a \) and semi-minor length \( b \). The origin \( O \) is the center of the ellipse. A sound beam passing through the lens attains its maximum pressure amplitude near the geometrical focus \( O' \). The more the speed in the lens becomes fast, the more the position of \( O' \) approaches the origin \( O \). From the preceding section, the distance of \( O' \) from the lens center is given \( z_1 = a + \sqrt{a^2 - b^2} \). The dot-dashed line in the figure is a spherical equiphase surface \( S_2 \) with curvature radius \( z_f \), where the beam seems to be hypothetically emitted from. The idea we utilize here is based on geometrical acoustics, which projects the sound field at the elliptically curved surface \( S_1 \) back to the spherically curved surface \( S_2 \) on the condition that the linear dimensions of the lens are much greater than the wavelength [8]. This follows approximately that except for the region near the focus the spherically converging field by the elliptic lens coincides in magnitude with the spherically spreading field of a point source located at the focus \( O' \). The magnitude of the field is expressed by \( A/r' \), where \( r' \) is the distance from \( O' \) to an arbitrary point, and \( A \) is a constant. Hence the normal component of the particle velocity \( v_{z_2} \) on \( S_2 \) is given by \( (R_1/z_1)v_{z_1} \) from two field equations \( v_{z_1} = A/R_1 \) and \( v_{z_2} = A/z_f \), in which \( v_{z_1} \) is the normal component of the velocity on \( S_1 \), and \( R_1 \) is the distance from \( O' \) to a point \( Q \) on \( S_2 \). Additionally, the directional cosine between the normal vector on \( S_2 \) and \( k \) is not \( k \cdot n \), but \( \cos \theta_0 \). Further attention must be paid for determining boundary conditions. The wave propagating in the lens decays inevitably due to energy dissipation in the lens. From all the above discussion, the boundary conditions for the initial pressure distribution on the translated surface \( S_2 \) yield

\[
F(\psi) = \begin{cases} 
\frac{R_1(\theta_0)}{z_f} \cos \theta_0 \exp[-\alpha D(\theta_0)] & (0 \leq \psi \leq \psi_m), \\
0 & (\psi > \psi_m)
\end{cases}
\]

(14)

where \( \psi \) is the angular variable in the oblate spheroidal coordinates, and is related with \( \theta_0 \) by \( \tan \theta_0 = \sqrt{1 + \alpha^2} \tan \psi \). \( \alpha \) in the equation is the radial variable in the spheroidal coordinates. \( D(\theta_0) \) is the distance between \( Q \) and \( Q' \) in Fig. 2 and is given by \( z_0 + a \). Furthermore, \( \psi_m \) is the maximum opening half-angle of the lens at \( Q \).
Imposing the boundary conditions given in Eq. (14), we can readily obtain the solution of SBE via a finite-difference scheme. Numerical parameters such as step sizes in discretization are not described here because those have been already discussed in detail in References [3] and [4].

To examine that the present theory predicts well a focused sound field, numerical demonstrations are presented of sound pressure curves along and across the beam axis by comparing with the prediction obtained from the Rayleigh integral of Eq. (12). After an accompanying paper [1], the longitudinal sound speeds in the lens and in water are assigned as \( c_l = 2,720 \text{ m/s} \) and \( c_0 = 1,480 \text{ m/s} \), respectively. It is also assigned for the lens sizes as follows: the curvature radius of an initial spherical lens is \( d = 100 \text{ mm} \), then the geometric focus is located from the lens center at \( z_f = d/(1 - n_l) = 219 \text{ mm} \) owing to the speed ratio of \( n_l = 0.544 \). The semi-major and semi-minor lengths of the ellipse are \( a = z_f/(1 + n_l) = 144 \text{ mm} \) and \( b = z_f\sqrt{(1 - n_l)/(1 + n_l)} = 120 \text{ mm} \), respectively. Let the aperture radius of the lens \( a_0 \) be 86.6 mm, which corresponds to 60° for the aperture half angle of the initial spherical lens by \( \tan^{-1}(a_0/d) = 60^\circ \).

Shown in Fig. 3 are propagation curves for a 1-MHz ultrasound beam. For simplicity, sound absorptions in the lens and in water as well are not taken into account. The solid curve is the SBE model associated with the appropriate boundary conditions and dotted curve is the solution obtained from Eq. (12). The abscissa is an axial distance from the lens center and the ordinate is an amplitude of the dimensionless pressure \( p/p_0 \), where \( p \) is the pressure amplitude and \( p_0 \) is the initial amplitude at the center of the lens.

The pressure amplitude, as might be expected, attains its maximum at almost the same position as the geometric focus 219 mm, and takes the focusing gain 78. In both the pre- and post-focal regions, oscillation with peaks and dips in amplitude appears clearly owing to diffraction of direct and edge waves, just like a focused beam from a bowl ultrasound source [4]. However, it should be noted that in this case the amplitude has nonzero values on the axis, in contrast to the bowl source case where the vibration is normal to the surface [9]. Overall, the two solutions based on the present SBE model and the Rayleigh integral are in excellent agreement within the framework of the present numerical range even for a widely opening aperture lens.

Another interesting comparison is given in Fig. 4 for beam patterns in the focal plane at \( z = 219 \text{ mm} \). The frequency of ultrasound beams is 1 MHz. The lens sizes are the same as in Fig. 3. Solid curve denotes the SBE model solution and dotted curve the Rayleigh integral solution.

Fig. 4 Beam patterns in the focal plane at \( z = 219 \text{ mm} \). The frequency of ultrasound beams is 1 MHz. The lens sizes are the same as in Fig. 3. Solid curve denotes the SBE model solution and dotted curve the Rayleigh integral solution.

3. EXPERIMENTS

The configuration of a plano-concave lens with one flat surface and one elliptic surface is downsized by about a
half, as shown in Fig. 5. Since \( a = 61.5 \text{ mm} \) and \( b = 51.6 \text{ mm} \), the geometric focus is expected to exist at \( z_f = a + \sqrt{a^2 - b^2} = 94.9 \text{ mm} \). The lens with a circular aperture in radius \( a_0 = 37.5 \text{ mm} \) is attached tightly to an ultrasonic planar transducer with the same aperture radius. They were immersed in fresh and degassed water. The transducer driven by a power amplifier radiates a tone-burst sinusoidal wave of 1.7-MHz with about 40 cycles duration to avoid reflection from the walls of a rectangular water bath. The absorption coefficient in the lens is 24 Np/m in this experimental situation. To pick up sound pressure as locally as possible, a card-type PVDF hydrophone with a very small sensing area \( 0.2 \times 0.2 \text{ mm} \) is utilized, and is mounted on a translation stage driven by stepping motors. Output signals from the hydrophone are captured using a digital storage oscilloscope. The stepping motors and oscilloscope are both controlled by a personal computer.

Figure 6 shows sound pressure curves along the beam axis. Even if initial sound pressure level is low enough the amplitude becomes high near the focus, especially in high focusing systems, generating harmonics in the beam due to the inherent nonlinearity of the water. Only the first three harmonics are shown for the experimental data in symbols and for the theoretical prediction in solid curves. The initial pressure \( p_0 \), which is needed for the numerical solution of the SBE model with the appropriate boundary conditions, was theoretically determined by a best fit of on-axis first harmonic pressure curve between the measured data and the theory, and was predicted to be 150 kPa.

As a whole, experimental data agree well with the present theoretical prediction. However, the positions of the peaks and dips in the fundamental harmonics are quite different in the pre-focal region: the spatial period in the theory is shorter than that in the experiment. Additionally, the theoretical dips deepen more than the measured ones in the post-focal region. One of possible sources for such discrepancies is probably attributed to multiple reflection in the lens, especially near the center of the lens with thickness 2 mm. To make sure how much the multiple reflection may occur, auxiliary experiment was done using an acrylic resin plate of thickness 3 mm in place of the lens and using a short pulse of 2.5 MHz. The second pulse with a delay of about 2.2 \( \mu \text{s} \) after the first pulse was observed on the beam axis away from the transducer. The amplitude of the second pulse was about 20% of the first pulse amplitude. The multiple reflection can produce patterns of constructive and destructive interference of the incident and reflected waves. Then the distributions of the velocity amplitude and phase over the lens face differ from the theoretical boundary conditions Eq. (14). Actually the lens face is curved, then the multiple reflection is restricted to the center region of the lens even if it occurs significantly. Further investigation is going on for the multiple reflection effect on focused sound fields.

The most interesting findings are shown in the curves of the higher harmonics. The pressure amplitudes with less oscillation increase monotonically during propagation before the focus, attains the maximum at the focus, and then decreases monotonically after the focus. It can be found evidently in the third harmonic that there are almost no oscillatory peaks and dips. This is just like a Gaussian beam.

Figure 7 shows beam patterns of the three harmonics at \( z = 94 \text{ mm} \), in the vicinity of the focal plane. The experimentally obtained first side-lobe level relative to the main-lobe level is \(-25 \text{ dB} \) for the first harmonic, and is about 2 dB smaller than the theory. The beam patterns of the higher harmonics are also given in the figure. Additional side-lobes in the higher harmonics, which are usually referred to as fingers, are barely observed. Especially, the third harmonic has almost no side-lobes within the framework of the present receiver system, whose noise floor level, including the hydrophone, is approximately 190 dB.
Excellent agreement between the theory and experiment provides that the present theoretical model is effective for evaluating focused sound fields by an elliptic surface lens with a widely opening aperture.

The temperature of the bath water was changed from 20°C to 15°C, and on-axis pressure profile at that temperature was measured in the same way as in Fig. 3. The result is shown in Fig. 8 with the data at 20°C. It should be noted that the pressure profile at 15°C moves by 2 or 3 mm on the whole compared with the profile at 20°C. When the temperature falls, sound speed in water gets slow. In contrast, the speed in the lens gets fast. Hence, the refractive index at 15°C is smaller than 0.544, so that the pressure profile might move toward the lens by such few millimeters.

4. CONCLUSIONS

Focusing of a plane progressive ultrasonic wave by a plano-concave lens has been investigated theoretically and experimentally. To reduce spherical aberration for a widely opening aperture lens, an elliptic surface lens has been introduced in place of a spherical surface lens. The configuration of the elliptic lens is based on geometrical optics. The primitive approach to linear sound field analysis in the preceding report has been extended to the case where harmonic generation in finite-amplitude sound beams becomes significant due to the inherent nonlinearity of the fluid. For describing nonlinear propagation of the beams, we used the spheroidal beam equation (SBE) associated with an appropriate boundary conditions. To make sure of the effectiveness of the present theory, experiments were carried out in water using an elliptic surface lens of eccentricity 0.544 which is attached tightly to a 1.7-MHz planar transducer with a circular aperture of 75-mm in diameter. It has been verified that the SBE model is useful in predicting quantitatively the sound pressure amplitudes of the higher harmonics, at least of the first third harmonics. Not only the decrease of the directional cosine but also the existence of sound absorption in the lens would always cause the amplitude of the particle velocity on the lens/water interface to taper off as the radial distance from the axis is increased, providing the suppression of the side-lobe level, particularly for the third harmonic components.

REFERENCES