Evaluation of a semi-analytical method for calculating multiple acoustic scattering fields

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1. Introduction

Various numerical methods such as the boundary element method (BEM) and the finite element method (FEM) have been presented to solve acoustic scattering fields. These numerical methods have the advantage that arbitrary shape and boundary conditions can be treated. However, the discretization of either boundary surfaces or full space of acoustic fields must be solved, which leads to increased computational complexity.

Other approaches to solve acoustic scattering problems are semi-analytical methods using spherical wave functions [1–6]. These semi-analytical methods are valid in the case of calculating acoustic scattering fields or radiation from a single scatterer constructed of a single closed surface, i.e., a finite cylinder or spheroid. However, in the case of complex shape and boundary conditions, these semi-analytical methods are not applicable. In addition to this, these methods cannot be applied to multiple acoustic problems from many scatterers.

In this letter, we extend a semi-analytical method to solve multiple acoustic scattering problems from many scatterers by introducing the multipole reexpansion technique [7]. There are two purposes to formulating this semi-analytical method. First, we will be able to solve multiple acoustic scattering problems from an arbitrary number of scatterers. Second, if the scatterer has complex shape or boundary conditions, we can use the decomposition of the original shape into objects of simpler geometry. Consequently, we will be able to treat complex shapes or boundary conditions semi-analytically. We assume that each scatterer of simple geometry has the following features.

- Each scatterer is constructed of a single closed surface.
- Its shape and boundary conditions are expressed simply by using spherical coordinates.

We compare the results of the computations with the proposed semi-analytical method and BEM.

2. Proposed semi-analytical method

We consider the problem of acoustic scattering by N scatterers situated in a three-dimensional space as shown in

Fig. 1. The coordinates of the centers of the scatterers are denoted as $\mathbf{r}'_p = (r'_p, \theta'_p, \phi'_p)$, $p = 1, 2, \dots, N$. We assume that the source is a monopole source at $\mathbf{r}_s$. The acoustic field is governed by the Helmholtz equation,

$$\nabla^2 \phi + k^2 \phi = 0. \quad (1)$$

Here, k is the wave number. Because of the linearity of the sound field, the velocity potential $\phi(\mathbf{r})$ at an arbitrary point $\mathbf{r}$ is presented in the form

$$\phi(\mathbf{r}) = \phi_{\text{in}}(\mathbf{r}) + \phi_s(\mathbf{r}), \quad (2)$$

where $\phi_{\text{in}}(\mathbf{r})$ is the incident field, and $\phi_s(\mathbf{r})$ is the scattered field. Owing to the linearity of the acoustic field, the scattered field can be divided into the influence of each scatterer,

$$\phi_s(\mathbf{r}) = \sum_{p=1}^N \phi_p(\mathbf{r}). \quad (3)$$

Here, we assume that the velocity potential $\phi_p(\mathbf{r})$ includes the interaction between the scatterers. We also assume that $\phi_p(\mathbf{r})$ is expanded into a series of singular solutions of the Helmholtz equation around the center of the p th scatterer $\mathbf{r}'_p$,

$$\begin{aligned} \phi_p(\mathbf{r}) &= \sum_{n=0}^{\infty} \sum_{m=-n}^n A_n^{(p)m} S_n^m(\mathbf{r}_p), \\ S_n^m(\mathbf{r}_p) &= h_n^{(1)}(kr_p) Y_n^m(\theta_p, \phi_p), \\ \mathbf{r}_p &= \mathbf{r} - \mathbf{r}'_p, \quad p = 1, 2, \dots, N, \end{aligned} \quad (4)$$

where $h_n^{(1)}(kr_p)$ is a spherical Hankel function of the first kind, $Y_n^m(\theta_p, \phi_p)$ is spherical harmonics, and $A_n^{(p)m}$ is an unknown expansion coefficient to be calculated.

2.1. Multipole reexpansion

To solve the multiple scattering problem, we use the multipole reexpansion technique developed by Gumerov and Duraiswami [7].

Now, we consider the q th scatterer. Near the center of the q th scatterer, all the velocity potentials $\phi_p(\mathbf{r})$, $p \neq q$, are regular. Accordingly, the singular solutions $S_n^m(\mathbf{r}_p)$, $p \neq q$, can be reexpanded into a series of regular elementary solutions $R_l^j(\mathbf{r}_q)$ of the Helmholtz equation near the q th scatterer, using the following expression under the condition $|\mathbf{r}_q| \leq |\mathbf{r}'_{pq}| = |\mathbf{r}'_q - \mathbf{r}'_p|$,

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Fig. 1 Multiple scattering problem.

$$\begin{aligned}
 S_n^m(\mathbf{r}_p) &= \sum_{l=0}^{\infty} \sum_{s=-l}^l (S|R)_{ln}^{sm}(\mathbf{r}'_{pq}) R_l^s(\mathbf{r}_q), \\
 R_l^s(\mathbf{r}_q) &= j_l(kr_q) Y_l^s(\theta_q, \varphi_q), \\
 p, q &= 1, 2, \dots, N, \quad p \neq q,
 \end{aligned} \tag{5}$$

where $(S|R)_{ln}^{sm}(\mathbf{r}'_{pq})$ is the translation reexpansion coefficient, and $j_l(kr_q)$ is a spherical Bessel function of the first kind. The incident field can also be represented using a similar series,

$$\begin{aligned}
 \phi_{\text{in}}(\mathbf{r}) &= \sum_{l=0}^{\infty} \sum_{s=-l}^l C_l^s(\mathbf{r}'_q) R_l^s(\mathbf{r}_q), \\
 C_l^{(in)s}(\mathbf{r}'_q) &= jk Q S_l^{-s}(\mathbf{r}_s - \mathbf{r}'_q).
 \end{aligned} \tag{6}$$

Here, $C_l^{(in)s}(\mathbf{r}'_q)$ is the expansion coefficient of the incident field, Q is the volume velocity, and $j = \sqrt{-1}$. Therefore, the velocity potential $\phi(\mathbf{r})$ near the q th scatterer can be given by

$$\begin{aligned}
 \phi(\mathbf{r}) &= \sum_{l=0}^{\infty} \sum_{s=-l}^l \{B_l^{(q)s} j_l(kr_q) + A_l^{(q)s} h_l^{(1)}(kr_q)\} Y_l^s(\theta_q, \varphi_q), \\
 B_l^{(q)s} &= C_l^{(in)s}(\mathbf{r}'_q) + \sum_{\substack{p=1 \\ p \neq q}}^N \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^n (S|R)_{ln}^{sm}(\mathbf{r}'_{pq}) A_n^{(p)m} \right\}.
 \end{aligned} \tag{7}$$

2.2. Construction of linear system using the Kirchhoff-Huygens formula

To obtain a linear system for the unknown expansion coefficients $A_n^{(q)m}$, we use the Kirchhoff-Huygens formula, which is an integral form of the Helmholtz equation. Using the Kirchhoff-Huygens formula, the velocity potential at an arbitrary point $\mathbf{r}$ is expressed as

$$\begin{aligned}
 \phi(\mathbf{r}) &= QG(\mathbf{r}_s, \mathbf{r}) \\
 &+ \iint_{\Gamma} \left(\phi(\mathbf{r}_0) \frac{\partial G(\mathbf{r}_0, \mathbf{r})}{\partial \mathbf{n}_0} - \frac{\partial \phi(\mathbf{r}_0)}{\partial \mathbf{n}_0} G(\mathbf{r}_0, \mathbf{r}) \right) dS_0,
 \end{aligned} \tag{8}$$

where Γ is the surface of all scatterers, and G is Green's function. We use the following equation as Green's function:

$$G(\mathbf{r}_0, \mathbf{r}) = \frac{1}{4\pi|\mathbf{r}_0 - \mathbf{r}|} e^{jk|\mathbf{r}_0 - \mathbf{r}|}. \tag{9}$$

The Green's function $G(\mathbf{r}_0, \mathbf{r})$ can also be expanded into a series of regular elementary solutions around the center of the q th scatterer $\mathbf{r}'_q$,

$$\begin{aligned}
 G(\mathbf{r}_0, \mathbf{r}) &= jk \sum_{l=0}^{\infty} \sum_{s=-l}^l R_l^{-s}(\mathbf{r}_0 - \mathbf{r}'_q) S_l^s(\mathbf{r}_q), \\
 |\mathbf{r}_0 - \mathbf{r}'_q| &< |\mathbf{r}_q|.
 \end{aligned} \tag{10}$$

We assume that the source is a monopole source, thus Eqs. (2), (3), and (8) give

$$\begin{aligned}
 \phi_s(\mathbf{r}) &= \iint_{\Gamma} \left(\phi(\mathbf{r}_0) \frac{\partial G(\mathbf{r}_0, \mathbf{r})}{\partial \mathbf{n}_0} - \frac{\partial \phi(\mathbf{r}_0)}{\partial \mathbf{n}_0} G(\mathbf{r}_0, \mathbf{r}) \right) dS_0, \\
 \sum_{q=1}^N \phi_q(\mathbf{r}) &= \sum_{q=1}^N \iint_{\Gamma_q} \left(\phi(\mathbf{r}_0) \frac{\partial G(\mathbf{r}_0, \mathbf{r})}{\partial \mathbf{n}_0} - \frac{\partial \phi(\mathbf{r}_0)}{\partial \mathbf{n}_0} G(\mathbf{r}_0, \mathbf{r}) \right) dS_0,
 \end{aligned} \tag{11}$$

where Γ_q is the surface of the q th scatterer. From Eq. (11), we obtain

$$\begin{aligned}
 \phi_q(\mathbf{r}) &= \iint_{\Gamma_q} \left(\phi(\mathbf{r}_0) \frac{\partial G(\mathbf{r}_0, \mathbf{r})}{\partial \mathbf{n}_0} - \frac{\partial \phi(\mathbf{r}_0)}{\partial \mathbf{n}_0} G(\mathbf{r}_0, \mathbf{r}) \right) dS_0, \\
 q &= 1, 2, \dots, N.
 \end{aligned} \tag{12}$$

Substituting Eq. (7) and Eq. (10) into Eq. (12) and using the orthogonality of $Y_l^j(\theta_q, \varphi_q)$, we obtain

$$\begin{aligned}
 A_i^{(q)j} &= \sum_{l=0}^{\infty} \sum_{s=-l}^l A_l^{(q)s} W_{il}^{(q)js} + \sum_{l=0}^{\infty} \sum_{s=-l}^l B_l^{(q)s} \tilde{W}_{il}^{(q)js}, \\
 i &= 1, 2, \dots, \quad j = -i, \dots, i, \quad q = 1, 2, \dots, N, \\
 W_{il}^{(q)js} &= jk \iint_{\Gamma_q} S_l^s(\mathbf{r}_0 - \mathbf{r}'_q) \\
 &\quad \times \left\{ \frac{\partial}{\partial \mathbf{n}_0} R_i^{-j}(\mathbf{r}_0 - \mathbf{r}'_q) - \frac{j\omega\rho}{z} R_i^{-j}(\mathbf{r}_0 - \mathbf{r}'_q) \right\} dS_0, \\
 \tilde{W}_{il}^{(q)js} &= jk \iint_{\Gamma_q} R_l^s(\mathbf{r}_0 - \mathbf{r}'_q) \\
 &\quad \times \left\{ \frac{\partial}{\partial \mathbf{n}_0} R_i^{-j}(\mathbf{r}_0 - \mathbf{r}'_q) - \frac{j\omega\rho}{z} R_i^{-j}(\mathbf{r}_0 - \mathbf{r}'_q) \right\} dS_0,
 \end{aligned} \tag{13}$$

where z is the specific acoustic impedance. Thus, we obtain a linear system to determine the unknown expansion coefficient:

$$\begin{aligned}
 A_i^{(q)j} &- \sum_{l=0}^{\infty} \sum_{s=-l}^l \left\{ A_l^{(q)s} W_{il}^{(q)js} \right. \\
 &\quad \left. + \sum_{\substack{p=1 \\ p \neq q}}^N \sum_{n=0}^{\infty} \sum_{m=-n}^n (S|R)_{ln}^{sm}(\mathbf{r}'_{pq}) A_n^{(p)m} \tilde{W}_{il}^{(q)js} \right\} \\
 &= \sum_{l=0}^{\infty} \sum_{s=-l}^l C_l^{(in)s}(\mathbf{r}'_q) \tilde{W}_{il}^{(q)js}, \\
 i &= 1, 2, \dots, \quad j = -i, \dots, i, \quad q = 1, 2, \dots, N.
 \end{aligned} \tag{14}$$

If we truncate the series in Eq. (14) at the order M , the number of unknowns becomes $N \times (M + 1)^2$.

3. Numerical examples

We apply the proposed semi-analytical method to a

problem involving two acoustically hard finite cylinders with the diameter $a = 1.15$ m and the height $b = 1.15$ m as shown in Fig. 2. We set the receiving points at $r = 1.9$ m, $\theta = 90^\circ$, $\varphi = 0-180^\circ$. The computational environment is as follows.

- CPU: Intel Core i7 M620 2.67 GHz
- Memory: 4 GB
- OS: Windows 7 (64 bit)
- Language: Python

First, we calculate the directional characteristics of the scattering wave. In this computation, the computational results of BEM using 3,072 triangular constant elements are used as reference values. The maximum mesh length is 0.1337 m, which is one-fifth the wavelength of a frequency of about 500 Hz. Figure 3 shows the computational results of the directional characteristics of the scattering wave. SWE refers to the computational result obtained by the proposed semi-analytical method and the numerical values inside the

Fig. 2 Geometry of two finite cylinders, a monopole source, and receiving points.

parentheses indicate the number of unknowns and truncation number M used in the computation. To determine truncation number M , we calculate average relative error Δ using the following equation:

$$\Delta = \frac{1}{181} \sum_{n=0}^{180} \left| \frac{p_{\text{ref}}(\mathbf{r}_n) - p(\mathbf{r}_n)}{p_{\text{ref}}(\mathbf{r}_n)} \right| \times 100[\%], \quad (15)$$

where p_{ref} is the sound pressure of reference values obtained by BEM using 3,072 triangular constant elements, and p is the sound pressure of the result of the proposed method. We select truncation number M such that average relative error Δ is less than 3%. The computational results of the proposed semi-analytical method are in good agreement with the computational results of BEM. Figure 3(a) for 63 Hz shows that the BEM needs at least 192 elements to obtain an accurate computational result. In contrast, the proposed semi-analytical method needs only 18 unknowns to obtain an accurate computational result. This characteristic is also observed in other frequencies. Thus, the proposed semi-analytical method has less dimensionality than BEM.

Next, we calculate the frequency response function of the scattering wave at the receiving point ($r = 1.9$ m, $\theta = 90^\circ$, $\varphi = 0^\circ$). In this computation, the computational results of BEM using 6,262 triangular constant elements are used as reference. The maximum mesh length is 0.0971 m, which is one-fifth the wavelength of a frequency of about 700 Hz. Figure 4 shows the computational result of the frequency response function of the scattering wave. It is again observed that the computational result of the proposed semi-analytical method is in good agreement with the computational result of BEM. However, the computational result of BEM has several

Fig. 3 Directional characteristics of scattering wave obtained by the proposed semi-analytical method and BEM. Bold font indicates number of unknowns required to obtain accurate results in each method.

Fig. 4 Frequency response functions of scattering wave obtained by the proposed semi-analytical method and BEM.

irregular points. These numerical errors are caused by the non-uniqueness failure of BEM. The computational result of the proposed semi-analytical method does not produce such irregular points. Thus, the computational result of the frequency response function of the scattering wave indicates that the proposed semi-analytical method does not suffer from the non-uniqueness failure of BEM. This is a consequence of the fact that the proposed semi-analytical method uses the Kirchhoff-Huygens formula instead of the surface Helmholtz integral equation.

Finally, we measure computational time for calculating directional characteristics. If we use the same number of unknowns in both the proposed method and BEM, the proposed method requires a much longer computational time than BEM. This is because the proposed method needs to calculate reexpansion coefficients and each coefficient matrix element needs to calculate the integral over the surface of the q th scatterer. However, the proposed method has less dimensionality than BEM to obtain an accurate result. Table 1 shows the computational times of the proposed semi-analytical method and BEM for the same accuracy at each frequency. As can be seen, if the proposed method is compared with BEM at the same accuracy, the proposed method requires shorter computational time than BEM in this multiple scattering problem.

4. Conclusion

We have formulated a semi-analytical method by using the multipole reexpansion technique and the Kirchhoff-Huygens formula for multiple acoustic scattering fields by multiple scatterers. It has been observed that the computational results of the proposed method are in good agreement

Table 1 Computational times of the proposed semi-analytical method and BEM when both methods produce the same accuracy.

	BEM		SWE	
	No. of unknowns	CPU time [s]	No. of unknowns	CPU time [s]
63 Hz	192	8.39E+00	18 (M = 2)	5.77E-01
125 Hz	192	8.39E+00	72 (M = 5)	3.48E+00
250 Hz	432	3.10E+01	162 (M = 8)	1.06E+01
500 Hz	768	8.60E+01	392 (M = 13)	4.21E+01

with the computational results of BEM, even when the proposed method has less dimensionality than BEM. It was also observed that the proposed method does not suffer from the non-uniqueness failure of BEM. However, it is yet unclear whether the computational complexity of the proposed method is reduced compared with that of BEM in any other situation.

In further studies, we should apply the proposed method to the case that the boundary surface is in the absorption boundary condition and the height-to-diameter ratio of the cylinder is varied. We should also estimate the computation complexity theoretically.

References

- [1] W. Williams, M. G. Parke, D. A. Moran and C. H. Sherman, "Acoustic radiation from a finite cylinder," *J. Acoust. Soc. Am.*, **36**, 2316–2322 (1964).
- [2] P. C. Waterman, "New formulation of acoustic scattering," *J. Acoust. Soc. Am.*, **45**, 1417–1429 (1969).
- [3] W. Tobocman, "Comparison of the T -matrix and Helmholtz integral equation methods for wave scattering calculations," *J. Acoust. Soc. Am.*, **77**, 369–374 (1985).
- [4] P. C. Waterman, "Comment on "Comparison of the T -matrix and Helmholtz integral equation methods for wave scattering calculations" [*J. Acoust. Soc. Am.*, **77**, 369–374 (1985)]," *J. Acoust. Soc. Am.*, **78**, 804 (1985).
- [5] W. Tobocman, "Addendum and Erratum: "Comparison of the T -matrix and Helmholtz integral equation methods for wave scattering calculations" [*J. Acoust. Soc. Am.*, **77**, 369–374 (1985)]," *J. Acoust. Soc. Am.*, **78**, 805–808 (1985).
- [6] A. Yokomichi and T. Samejima, "Possibility of avoidance of nonuniqueness on acoustic scattering analysis with quasi-theoretical methods," *Proc. Autumn Meet. Acoust. Soc. Jpn.*, pp. 739–740 (2006) (in Japanese).
- [7] N. A. Gumerov and R. Duraiswami, "Computation of scattering from N spheres using multipole reexpansion," *J. Acoust. Soc. Am.*, **112**, 2688–2701 (2002).