Optically visualized sound field reconstruction using Kirchhoff–Helmholtz equation

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1. Introduction

The optical measurement of a sound field, originally developed in the area of ultrasonic and underwater acoustics [1,2], has received much attention since its extension to an audible sound field [3–7]. However, poor signal-to-noise ratio (SNR) due to the smallness of the quantity measured by optical methods is preventing its practical use in the audible range. In this paper, a method for reconstructing a measured sound field using the Kirchhoff–Helmholtz boundary integral equation is presented.

2. Physical model of optically visualized sound field

A sound field visualized by optical measurement techniques can be considered as a two-dimensional sound field generated by integration along optical paths [8]. In this paper, the homogeneous Helmholtz equation,

$$(\Delta + k^2)u = 0, \quad (1)$$

where u is the optically projected sound quantity, k is the wave number, and $\Delta (= \sum_n \partial^2 / \partial x_n^2)$ is the Laplace operator, is assumed inside the observed region $\Omega \subset \mathbb{R}^2$ so that only the information related to sound is extracted from noisy observation. Note that the solutions for the Helmholtz equation can be converted to the solution of the wave equation through the Fourier transform.

The well-known boundary integral form of the Helmholtz equation is the so-called Kirchhoff–Helmholtz equation:

$$u(y) = \int_{\partial\Omega} \left[\Phi(x, y) \frac{\partial u(x)}{\partial \nu_x} - \frac{\partial \Phi(x, y)}{\partial \nu_x} u(x) \right] dS(x), \quad (2)$$

where $\partial\Omega$ is the boundary of Ω , $x \in \partial\Omega$, $y \in \text{int}(\Omega)$, ν_x is the outward unit normal at x , $dS(x)$ is the surface element at x ,

$$\Phi(x, y) = \frac{i}{4} \left(\frac{k}{2\pi \|x - y\|_2} \right)^{\frac{d-2}{2}} H_{\frac{d-2}{2}}^{(1)}(k \|x - y\|_2) \quad (3)$$

is the fundamental solution of the Helmholtz equation, $\|\cdot\|_p$ is the l_p norm, $i = \sqrt{-1}$, d is the dimension of the space $\mathbb{R}^d$, and $H_{\mu}^{(1)}$ is the Hankel function of the first kind of order μ . Let us rewrite Eq. (2) shortened as

$$u(y) = (Ku)(y) \quad (4)$$

for convenience. Then, the square error of the estimation at several observation points y_m is

$$\sum_m |u(y_m) - (Ku)(y_m)|^2. \quad (5)$$

Direct minimization of this error results in a highly ill-posed problem. Thus, it is necessary to regularize the problem in order to obtain a physically meaningful solution. In this paper, the homogeneous Helmholtz equation is utilized as the prior,

$$\begin{aligned} &\underset{u(x)}{\text{minimize}} \sum_m |u(y_m) - (Ku)(y_m)|^2 \\ &\text{subject to } (\Delta + k^2)u(x) = 0, \end{aligned} \quad (6)$$

because the boundary is merely an arbitrarily chosen subspace of the free field; in other words, the boundary is virtual and does not affect any physical phenomenon.

3. Discretization by spectral method

Let the position on the boundary be parametrized as $x = x(\theta) \in \partial\Omega$, then the boundary condition can be represented as the Fourier series

$$u(x) = \sum_n \alpha_n e^{in\theta}, \quad \frac{\partial u(x)}{\partial \nu_x} = \sum_n \beta_n e^{in\theta}. \quad (7)$$

Substituting these equations into Eq. (2) generates

$$\begin{aligned} u(y) = \int_{-\pi}^{\pi} &\left[\Phi(x, y) \sum_n \beta_n e^{in\theta} \right. \\ &\left. - \frac{\partial \Phi(x, y)}{\partial \nu_x} \sum_n \alpha_n e^{in\theta} \right] \left| \frac{dx}{d\theta} \right| d\theta, \end{aligned} \quad (8)$$

where $dS(x) = |dx/d\theta|d\theta$. By assuming the summations to be finite, this equation becomes

$$\begin{aligned} u(y) = \sum_n \beta_n \int_{-\pi}^{\pi} &\Phi(x, y) \left| \frac{dx}{d\theta} \right| e^{in\theta} d\theta \\ &- \sum_n \alpha_n \int_{-\pi}^{\pi} \frac{\partial \Phi(x, y)}{\partial \nu_x} \left| \frac{dx}{d\theta} \right| e^{in\theta} d\theta. \end{aligned} \quad (9)$$

These Fourier integrals can be calculated only from x and y . Therefore, the estimation problem is reduced to the estimation of the Fourier coefficients α_n and β_n .

In this paper, we chose the l_4 -norm ball shape for the boundary, which is smooth and more suitable for data measured along the rectangular coordinate than the circle. Then, the point $x \in \partial\Omega$ can be represented as

$$x = [x_1 \ x_2]^T = r(\theta) [\cos \theta \ \sin \theta]^T, \quad (10)$$

where

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$$r(\theta) = a(\cos^4 \theta + \sin^4 \theta)^{-\frac{1}{4}}, \quad (11)$$

and a is the radius. In this case,

$$\left| \frac{dx}{d\theta} \right| = \frac{2\sqrt{5+3\cos 4\theta}}{(3+\cos 4\theta)^{\frac{5}{4}}} a. \quad (12)$$

3.1. Discretization of constraint term

The two-dimensional Laplace operator is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \quad (13)$$

in the polar coordinate system (r, θ) . Since the derivative w.r.t. θ can be calculated analytically from Eq. (7), only r derivatives are approximated.

Let x' be a point on an inner boundary,

$$x' = [x'_1 \ x'_2]^T = r'(\theta)[\cos \theta \ \sin \theta]^T, \quad (14)$$

where

$$r'(\theta) = (a - \delta)(\cos^4 \theta + \sin^4 \theta)^{-\frac{1}{4}} \quad (15)$$

and $\delta > 0$. Let the Fourier series representation of the inner boundary condition be written as

$$u(x') = \sum_n \phi_n e^{in\theta}, \quad \frac{\partial u(x')}{\partial v_{x'}} = \sum_n \psi_n e^{in\theta}. \quad (16)$$

Then, at $\theta = \xi\pi/2$ ($\xi \in \mathbb{Z}$), because $\partial\Omega$ is not a circle,

$$\frac{\partial u(x)}{\partial r} \approx \frac{1}{\delta} \left(\sum_n \alpha_n e^{in\theta} - \sum_n \phi_n e^{in\theta} \right), \quad (17)$$

$$\frac{\partial^2 u(x)}{\partial r^2} \approx \frac{1}{\delta} \left(\sum_n \beta_n e^{in\theta} - \sum_n \psi_n e^{in\theta} \right), \quad (18)$$

which gives the following approximation:

$$\begin{aligned} (\Delta + k^2)u(x) &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} + k^2 \right) u(x) \\ &\approx \frac{1}{\delta} \left(\sum_n \beta_n e^{in\theta} - \sum_n \psi_n e^{in\theta} \right) - \frac{1}{a^2} \sum_n \alpha_n n^2 e^{in\theta} \\ &\quad + \frac{1}{a} \left(\sum_n \alpha_n e^{in\theta} - \sum_n \phi_n e^{in\theta} \right) + k^2 \sum_n \alpha_n e^{in\theta}. \end{aligned} \quad (19)$$

Moreover, since this approximation should satisfy the identity

$$\frac{\partial u(x)}{\partial r} = \frac{\partial u(x)}{\partial v_x} \quad (20)$$

at $\theta = \xi\pi/2$ ($\xi \in \mathbb{Z}$), the following equation is added to the constraint:

$$\frac{1}{\delta} \left(\sum_n \alpha_n e^{in\theta} - \sum_n \phi_n e^{in\theta} \right) - \sum_n \beta_n e^{in\theta} \approx 0. \quad (21)$$

3.2. Norm ball constraint

Let

$$\beta = [\beta_{-N_c} \dots \beta_0 \dots \beta_{N_c} \ \alpha_{-N_c} \dots \alpha_0 \dots \alpha_{N_c}]^T, \quad (22)$$

$$A_\beta^{(m,n)} = \int_{-\pi}^{\pi} \Phi(x, y_m) e^{in\theta} \left| \frac{dx}{d\theta} \right| d\theta, \quad (23)$$

$$A_\alpha^{(m,n)} = - \int_{-\pi}^{\pi} \frac{\partial \Phi(x, y_m)}{\partial v_x} e^{in\theta} \left| \frac{dx}{d\theta} \right| d\theta, \quad (24)$$

where N_c is a cutoff number of Eq. (7) and $A^{(m,n)}$ denotes the (m, n) -th element of matrix A . Similarly,

$$\varphi = [\psi_{-N_c} \dots \psi_0 \dots \psi_{N_c} \ \phi_{-N_c} \dots \phi_0 \dots \phi_{N_c}]^T, \quad (25)$$

A_ψ and A_ϕ are defined. Then, the discretized problem of Eq. (6) can be formulated as

$$\begin{aligned} &\underset{\beta, \varphi}{\text{minimize}} \quad \|\mathbf{u} - A\beta\|_2^2 + \|\mathbf{u} - A'\varphi\|_2^2 \\ &\text{subject to} \quad \left\| H \begin{bmatrix} \beta \\ \varphi \end{bmatrix} \right\|_\infty \leq \varepsilon_1, \quad \left\| D \begin{bmatrix} \beta \\ \varphi \end{bmatrix} \right\|_\infty \leq \varepsilon_2, \end{aligned} \quad (26)$$

where $\mathbf{u} = [u(y_1) \dots u(y_M)]^T$, $A = [A_\beta \ A_\alpha]$, $A' = [A_\psi \ A_\phi]$, and H and D are the matrices corresponding to Eqs. (19) and (21), respectively. The constraints are expressed by the ∞ -norm balls to take the approximation error into account.

4. Initial value generation via sparsity

In the modeling using the boundary integral equation, it is implicitly assumed that the boundary condition $u(x)$ has higher amplitude than that at the inner point $u(y)$, because sound sources are always located outside Ω . This fact indicates that a minimum norm solution is not a favorable choice when A and A' are ill-conditioned, which is a typical situation for these kinds of estimation problems. Thus, generation of a good initial value is needed.

In this paper, a simple model reflecting the fact that sound sources are located outside the region is utilized. Let $u^{(k)}(y_m)$ be the influence of a point source located at $z_k \notin \bar{\Omega}$. Then, the observation $\mathbf{u}$ should be approximated by a linear combination of $\mathbf{u}^{(k)} = [u^{(k)}(y_1) \dots u^{(k)}(y_M)]^T$, which leads to the following LASSO problem:

$$\underset{\mathbf{c}}{\text{minimize}} \quad \|\mathbf{B}\mathbf{c} - \mathbf{u}\|_2^2 + \lambda \|\mathbf{c}\|_1, \quad (27)$$

where $\mathbf{B} = [\mathbf{u}^{(1)} \dots \mathbf{u}^{(M)}]$ and $\lambda > 0$. After solving the above problem, the Fourier transform of $\mathbf{B}\mathbf{c}$ should give a reasonable initial guess for the original problem.

5. Solving problems with ADMM

These optimization problems are convex; thus, any convex optimization algorithm allowing the sum of differentiable and composite nondifferentiable objective functions can be used to solve them globally. In this paper, we chose the alternating direction method of multipliers (ADMM) [9], which is an iterative algorithm,

$$\begin{aligned} \mathbf{x}_{n+1} &= \underset{\mathbf{x}}{\text{argmin}} \left[f(\mathbf{x}) + \frac{\rho}{2} \|\mathbf{L}\mathbf{x} + T\mathbf{z}_n - \mathbf{c} + \mathbf{y}_n\|_2^2 \right] \\ \mathbf{z}_{n+1} &= \underset{\mathbf{z}}{\text{argmin}} \left[g(\mathbf{z}) + \frac{\rho}{2} \|\mathbf{L}\mathbf{x}_{n+1} + T\mathbf{z} - \mathbf{c} + \mathbf{y}_n\|_2^2 \right] \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \mathbf{L}\mathbf{x}_{n+1} + T\mathbf{z}_{n+1} - \mathbf{c}, \end{aligned} \quad (28)$$

for solving a convex optimization problem with the following form:

$$\begin{aligned} &\text{minimize} \quad f(\mathbf{x}) + g(\mathbf{z}) \\ &\text{subject to} \quad \mathbf{L}\mathbf{x} + T\mathbf{z} = \mathbf{c}, \end{aligned} \quad (29)$$

where $f: \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g: \mathbb{R}^M \rightarrow \mathbb{R} \cup \{+\infty\}$ are the proper lower semicontinuous convex functions, $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{z} \in \mathbb{R}^M$, $L \in \mathbb{R}^{S \times N}$, $T \in \mathbb{R}^{S \times M}$, and $\mathbf{c} \in \mathbb{R}^S$. Note that f and g can be nondifferentiable functions; in that case, a proximal operator,

$$\text{prox}_f(\mathbf{x}) = \underset{\mathbf{y}}{\text{argmin}} \left[f(\mathbf{y}) + \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|_2^2 \right], \quad (30)$$

plays an important role in calculating $\mathbf{x}$ and/or $\mathbf{z}$ -update in Eq. (28).

5.1. Solver for Eq. (27)

In order to adapt to the above algorithm, the original problems in complex number systems are mapped to the corresponding real systems [10]. This transformation converts the l_p norm into the $l_{2,p}$ mixed norm:

$$\|\mathbf{x}\|_{2,p} = \left(\sum_{g \in \mathcal{G}} \|\mathbf{x}_g\|_2^p \right)^{\frac{1}{p}}, \quad (31)$$

where $\mathcal{G}$, in this case, is the pair of real and imaginary elements of a vector. The proximal operator for the $l_{2,1}$ mixed norm can be evaluated easily as [11]

$$(\text{prox}_{\lambda \|\cdot\|_{2,1}}(\mathbf{x}))_g = \left(1 - \frac{\lambda}{\|\mathbf{x}_g\|_2} \right)_+ \mathbf{x}_g, \quad (32)$$

where $(\cdot)_+ = \max\{0, \cdot\}$; therefore, the following iterative procedure starting from any $\mathbf{x}_0$, $\mathbf{y}_0$, and $\mathbf{z}_0$ solves Eq. (27), the LASSO problem:

$$\begin{aligned} \mathbf{x}_{n+1} &= (B^T B + \rho I)^{-1} (B^T \mathbf{u} + \rho(\mathbf{z}_n - \mathbf{y}_n)) \\ \mathbf{z}_{n+1} &= \text{prox}_{\frac{\lambda}{\rho} \|\cdot\|_{2,1}}(\mathbf{x}_{n+1} + \mathbf{y}_n) \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \mathbf{x}_{n+1} + \mathbf{z}_{n+1}. \end{aligned} \quad (33)$$

5.2. Problem reformulation of Eq. (26)

The constraint terms in Eq. (26) do not fit the problem, Eq. (29), that can be solved by ADMM. Moreover, it is empirically known that a solution to the above LASSO problem gives a very good initial guess; a final solution should stay near the initial value. Therefore, using the indicator function

$$\iota_C(\mathbf{x}) = \begin{cases} 0 & (\mathbf{x} \in C) \\ \infty & (\mathbf{x} \notin C), \end{cases} \quad (34)$$

Eq. (26) is reformulated as the nonconstrained form with an additional constraint for the initial value $[\beta_0^T \varphi_0^T]^T$:

$$\begin{aligned} \underset{\mathbf{u}, \mathbf{u}'}{\text{minimize}} \quad & \|\mathbf{u} - A\beta\|_2^2 + \|\mathbf{u} - A'\varphi\|_2^2 + \iota_{C_{\varepsilon_1}} \left(H \begin{bmatrix} \beta \\ \varphi \end{bmatrix} \right) \\ & + \iota_{C_{\varepsilon_2}} \left(D \begin{bmatrix} \beta \\ \varphi \end{bmatrix} \right) + \iota_{C_{\varepsilon_3}} \left(W \left\{ \begin{bmatrix} \beta \\ \varphi \end{bmatrix} - \begin{bmatrix} \beta_0 \\ \varphi_0 \end{bmatrix} \right\} \right), \end{aligned} \quad (35)$$

where

$$C_{\varepsilon} = \{\mathbf{x} \mid \|\mathbf{x}\|_{2,\infty} \leq \varepsilon\} \quad (36)$$

is the $l_{2,\infty}$ -norm ball, and W is a diagonal weighting matrix.

5.3. Solver for Eq. (35)

Let f be the l_2 norm terms, g be the indicator functions, $\mathbf{x}_0 = [\beta_0^T \varphi_0^T]^T$, $\mathbf{y} = [\mathbf{y}^{(1)T} \mathbf{y}^{(2)T} \mathbf{y}^{(3)T}]^T$, $\mathbf{z} = [\mathbf{z}^{(1)T} \mathbf{z}^{(2)T} \mathbf{z}^{(3)T}]^T$, $L = [H^T D^T W^T]^T$, $T = -I$, and $\mathbf{c} = 0$.

Table 1 Parameters for the numerical experiment.

Sampling points	$16 \times 16 (= 256)$ points
Sampling point interval	0.1 m
Sound speed	340 m/s
Sound source position	$(-1, -1)$
Distance b/w boundaries	$\delta = 2^{-28} \approx 3.7$ nm
Regularization parameters	$\lambda = 2^{-4} = 0.0625$, $\rho = 1$
SNR of synthetic data	0, 10, 20, 30, 40, 50 dB
Spatial Nyquist frequency	1,700 Hz

Then Eq. (35) can be interpreted as Eq. (29), and thus the following ADMM algorithm is derived:

$$\begin{aligned} \mathbf{x}_{n+1} &= (A^T A + A'^T A' + \rho L^T L)^{-1} \\ &\quad ((A + A')^T \mathbf{u} + \rho L^T (\mathbf{z}_n - \mathbf{y}_n)) \\ \mathbf{z}_{n+1}^{(1)} &= P_{C_{\varepsilon_1}} (H \mathbf{x}_{n+1} + \mathbf{y}_n^{(1)}) \\ \mathbf{z}_{n+1}^{(2)} &= P_{C_{\varepsilon_2}} (D \mathbf{x}_{n+1} + \mathbf{y}_n^{(2)}) \\ \mathbf{z}_{n+1}^{(3)} &= W \mathbf{x}_0 + P_{C_{\varepsilon_3}} (W \mathbf{x}_{n+1} + \mathbf{y}_n^{(3)} - W \mathbf{x}_0) \\ \mathbf{y}_{n+1}^{(1)} &= \mathbf{y}_n^{(1)} + H \mathbf{x}_{n+1} + \mathbf{z}_{n+1}^{(1)} \\ \mathbf{y}_{n+1}^{(2)} &= \mathbf{y}_n^{(2)} + D \mathbf{x}_{n+1} + \mathbf{z}_{n+1}^{(2)} \\ \mathbf{y}_{n+1}^{(3)} &= \mathbf{y}_n^{(3)} + W \mathbf{x}_{n+1} + \mathbf{z}_{n+1}^{(3)}, \end{aligned} \quad (37)$$

where

$$P_{C_{\varepsilon}}(\mathbf{x}) = \mathbf{x} - \text{prox}_{\varepsilon \|\cdot\|_{2,1}}(\mathbf{x}) \quad (38)$$

is the projection operator to C_{ε} directly obtained from the Moreau decomposition [11].

6. Numerical evaluation

To confirm the effectiveness of the proposed method, a numerical experiment was performed. The simulation settings are shown in Table 1 and Fig. 1. Unlisted parameters were determined, using the initial value, as $\varepsilon_1 = \max\{|H\mathbf{x}_0|\}$, $\varepsilon_2 = \max\{|D\mathbf{x}_0|\}$, $\varepsilon_3 = 1.2$, and $W = \text{diag}(1/|\mathbf{x}_0|)$. Then, each matrix (i.e., H , D , and W) was normalized by its matrix norm. Epsilons were also scaled by the same values. Initial values for the LASSO problem were chosen as zero vectors, and the maximum iteration number for each problem was set to 1,000. After estimating the Fourier coefficients of the boundary conditions by solving Eqs. (27) and (35), the sound field was reconstructed from Eq. (9).

SNR of the reconstructed sound fields was calculated as

$$\text{SNR}_{\text{result}} = 10 \log_{10} \frac{\sum |u|^2}{\sum |u - \hat{u}|^2}, \quad (39)$$

where u is the original sound field at a point inside Ω , and $\hat{u}$ is the reconstructed sound field at the same point. The experiments were performed for several observed data whose SNR,

$$\text{SNR}_{\text{data}} = 10 \log_{10} \frac{\sum |u|^2}{\sum |w|^2}, \quad (40)$$

where w denotes the Gaussian noise, was arbitrarily chosen by adjusting the level of the noise, in this case, $\{0, 10, 20, 30, 40, 50\}$ dB.

Fig. 1 Simulation setup. l_4 -norm ball-shaped boundary was set around the sampling points. Positions of point sources for forming bases in Eq. (27) are chosen from the region outside the boundary.

Figure 2 shows the SNR of the reconstruction results ($\text{SNR}_{\text{result}}$) versus the frequency of the field. Each line denotes the SNR of the noisy observation (SNR_{data}), which was calculated from $16 \times 16 = 256$ sampling points, whereas the vertical axis ($\text{SNR}_{\text{result}}$) was calculated from $301 \times 301 = 90,601$ reconstructed points. This result confirms that the proposed method can be used to estimate the original field from sparsely observed data when the spatial Nyquist frequency of the observation is greater than the frequency of the sound source.

7. Conclusion

In this paper, a reconstruction method for an optically visualized sound field using the Kirchhoff–Helmholtz integral equation was proposed. ADMM was employed to solve the initial value estimation via sparsity and the norm ball constrained least squares problem. The experiment using synthetic data confirmed the effectiveness of the method. Future works include reconstruction of the three-dimensional sound pressure distribution from optically measured data.

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Fig. 2 $\text{SNR}_{\text{result}}$ of each reconstruction result versus frequency of the field. Each line is for different SNR_{data} . Synthetic data were composed of the true sound field and additive Gaussian noise whose level was chosen so that SNR of the data became $\{0, 10, 20, 30, 40, 50\}$ dB. These noisy observations were sampled at $16 \times 16 (= 256)$ points, whereas the SNR for the vertical axis was calculated for $301 \times 301 (= 90,601)$ reconstructed points. The vertical line at 1,700 Hz denotes the spatial Nyquist frequency.

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