Applicability of an explicit time-domain finite-element method on room acoustics simulation

Takeshi Okuzono^{1,*}, Toru Otsuru² and Kimihiro Sakagami¹

¹*Environmental Acoustic Laboratory, Department of Architecture, Graduate School of Engineering, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe, 657-8501 Japan*

²*Department of Architecture and Mechatronics, Architecture Course, Faculty of Engineering, Oita University, 700 Dannoharu, Oita, 870-1192 Japan*

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1. Introduction

Recently the time-domain finite-element method (TD-FEM) is becoming an efficient wave-based numerical method for wideband room acoustics simulation up to several kilohertz frequencies, with the drastic advancement of computer technology [1,2]. However, a cost-efficient simulation up to high frequencies using the method is still considered to be a difficult task, due to the high computational cost. A dispersion error, which is an inherent discretization error coming from spatial and time discretizations of a computational domain, is one of the reasons for the high computational cost. Because of the error, a spatial discretization requirement known as a rule of thumb is imposed in a mesh generation process to yield reliable results. Also, a time discretization error must be considered in a time-domain analysis. These requirements engender the solution of large-scale problems with many degrees of freedom (DOF) and many time steps for a simulation at high frequencies.

Many dispersion reduction methods have been developed to increase the computational efficiency [1–6]. The authors have also proposed some dispersion-reduced TD-FEM based on an implicit method to solve the large-scale problems efficiently [1,2]. Also, there exists an explicit TD-FEM with a simple dispersion reduction technique called modified integration rules (MIR) [5], in which fourth-order accuracy with respect to the dispersion error can be obtained for the idealized case using square or cubic finite elements (FEs). This explicit method is very attractive for realizing an efficient room acoustics simulation because it does not need a solution of a linear system of equations at each time step. However, as presented in the literature [5], the explicit method does not consider a dissipation term for treating an absorption at boundaries, which is important for room acoustics simulation.

When a sound field inside a room with an finite impedance boundary is analyzed using this explicit method, a time derivative of sound pressure related to the dissipation term has to be approximated by a less-accurate backward difference even though time derivatives of other physical quantities are approximated by a second-order accurate

central difference. This might produces low accurate results at higher frequencies, unless a significantly small time interval is used. Therefore, whether the explicit TD-FEM with the dissipation term is efficient or not on room acoustics simulation at high frequencies still remains unclear.

To study the applicability of the explicit method on room acoustics simulation, this paper examines the accuracy and efficiency of the explicit TD-FEM with the dissipation term on room acoustics simulation at high frequencies through a numerical comparison with the implicit TD-FEM [1] for large-scale analysis. After describing the theory of the explicit TD-FEM with the dissipation term as well as the implicit TD-FEM, the accuracy and efficiency of the explicit TD-FEM are demonstrated through sound field analysis inside a cubic cavity with an infinite impedance boundary, in which the analytical solution by the method of variable separation is used for the estimation of the accuracy. Then, the effect of using the backward difference approximation in the dissipation term of the explicit TD-FEM on the resulting accuracy is tested for sound field analyses inside a cubic cavity with a finite impedance boundary.

2. Time-domain finite-element method

2.1. Implicit method

Following the standard FE procedure based on the principle of minimum potential energy, the semi-discretized matrix equation for a sound field can be obtained as

$$M\ddot{\mathbf{p}} + c_0^2 \mathbf{K}\mathbf{p} + c_0 \mathbf{C}\dot{\mathbf{p}} = \mathbf{f}, \quad (1)$$

where $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{C}$, respectively, denote the global mass matrix, the global stiffness matrix, and the global dissipation matrix. Further, $\mathbf{p}$, $\mathbf{f}$, c_0 , respectively, denote the sound pressure vector, the external force vector, and the speed of sound. '·' and '·' respectively signify first-order and second-order derivatives with respect to time. The implicit method solves the above second-order ordinary differential equation (ODE) by using a direct time integration method. In this paper, an efficient formulation [1] for large-scale analysis with many degrees of freedom is used to solve the Eq. (1), in which 8-node hexahedral elements and Fox–Goodwin method are respectively used for spatial and time discretizations, with MIR. A Krylov subspace iterative method is also used to

*e-mail: okuzono@port.kobe-u.ac.jp

solve the large-scale linear system of equations at each time step efficiently. The stability condition for three dimensional analysis in the case using only rectangular FEs is given as [1]

$$\Delta t \leq \frac{1}{c_0 \sqrt{\frac{1}{d_x^2} + \frac{1}{d_y^2} + \frac{1}{d_z^2}}}, \quad (2)$$

where d_x , d_y , and d_z respectively represent element length in x , y , and z directions.

2.2. Explicit method

In contrast to the implicit method, the explicit method solves a first-order ODE. By introducing a diagonal mass matrix $\mathbf{D}$ lumped from $\mathbf{M}$ and a vector $\mathbf{v} = \dot{\mathbf{p}}$, the second-order ODE of Eq. (1) can be transformed into [4]

$$\mathbf{D}\dot{\mathbf{p}} = \mathbf{M}\mathbf{v}, \quad (3)$$

$$\mathbf{D}\dot{\mathbf{v}} = \mathbf{f} - c_0^2 \mathbf{K}\mathbf{p} - c_0 \mathbf{C}\dot{\mathbf{p}}. \quad (4)$$

Discretizations of $\dot{\mathbf{p}}$ in Eq. (3) and $\dot{\mathbf{v}}$ in Eq. (4) using the second-order accurate central difference and $\dot{\mathbf{p}}$ in Eq. (4) using the first-order accurate backward difference lead to the following explicit scheme as

$$\mathbf{p}^n = \mathbf{p}^{n-1} + \Delta t \mathbf{D}^{-1} \mathbf{M} \mathbf{v}^{n-\frac{1}{2}}, \quad (5)$$

$$\mathbf{v}^{n+\frac{1}{2}} = \mathbf{v}^{n-\frac{1}{2}} + \Delta t \mathbf{D}^{-1} \times \left[\mathbf{f}^n - c_0^2 \mathbf{K} \mathbf{p}^n - \frac{c_0}{\Delta t} \mathbf{C} (\mathbf{p}^n - \mathbf{p}^{n-1}) \right], \quad (6)$$

where n represents the time step. Note that the backward difference is necessary for $\dot{\mathbf{p}}$ in the dissipation term because it needs to be discretized at $t = n$. For further numerical efficiency, $\mathbf{M}$ and $\mathbf{K}$ are stored by a sparse matrix storage format and a lumped $\mathbf{C}$ is used. Here, these techniques are naturally used in the implicit method. Therefore, two sparse matrix-vector products, $\mathbf{M} \mathbf{v}^{n-\frac{1}{2}}$ and $\mathbf{K} \mathbf{p}^n$, are the main numerical operation of this scheme.

For this explicit scheme, cubic shaped eight-node hexahedral FEs are used. The element matrix of $\mathbf{D}$ is lumped using the standard Gauss-Legendre rule. Meanwhile, element matrices of $\mathbf{K}$ and $\mathbf{M}$ are constructed using MIR where the respective modified integration points are given as [5]

$$\alpha_k = \sqrt{\frac{2}{3}}, \quad \alpha_m = \sqrt{\frac{1}{3} \left[4 - \left(\frac{c_0 \Delta t}{d} \right)^2 \right]}. \quad (7)$$

Here, α_k and α_m are the numerical integration points of element matrices in $\mathbf{K}$ and $\mathbf{M}$, and d is the element length of cubic elements. With the modified integration points, a fourth-order accuracy with respect to the dispersion error is achieved for the idealized case using cubic elements in a free space [5]. Note that the programming is easy because the components of element matrices can be calculated explicitly using only the parameter of element length, as shown in Ref. [7].

3. Numerical experiments

3.1. Cubic cavity with infinite impedance boundaries

The sound field analysis inside a cubic cavity with rigid boundaries, which is a problem, “B0-1T, Task A,” in the benchmark platform on computational methods for architec-

Fig. 1 A benchmark problem, B0-1T.

tural/environmental acoustics in Ref. [8], was selected to compare the accuracy and efficiency between the implicit method and the explicit method because the analytical method in Ref. [9] is available. It is the problem of computing transient response at three receiving points R2, R3 and R4 inside the cubic cavity (Fig. 1), in which c_0 and air density ρ_0 were respectively assumed to be 343.7 m/s and 1.205 kg/m³. For the quantitative estimation of the accuracy, the relative error in sound pressures between the analytical method and the numerical method was defined as

$$\|e\|_2 = \frac{1}{N_{\text{step}}} \sqrt{\frac{\sum_{i=1}^{N_{\text{step}}} [p_{\text{ana.}}(x, t_i) - p_{\text{FEM}}(x, t_i)]^2}{\sum_{i=1}^{N_{\text{step}}} p_{\text{ana.}}(x, t_i)^2}}, \quad (8)$$

where $p_{\text{ana.}}(x, t_i)$ and $p_{\text{FEM}}(x, t_i)$ are the sound pressure at a receiving point x obtained from the analytical method and the FE analysis, respectively. N_{step} and $\|\cdot\|$ represent the number of time step and 2-norm, respectively.

The sound source used here was a modulated Gaussian pulse. Its waveform and frequency characteristics are depicted in Fig. 2. Here, the upper limit of the frequency was assumed as 8 kHz where the gain is -3 dB. An FE mesh with the spatial resolution $\lambda/d = 6.8$ was constructed using only cubic elements, where λ represents the wavelength at the upper limit of the frequency. The DOF of the FE mesh is 4,173,281. The sound pressure was calculated up to 0.01 s with $\Delta t = 1/96,000$ s, which was determined by the stability condition of Eq. (2). Note that Eq. (2) is not exact stability condition for the explicit method, but it is considered as the safe side because the stability condition of the explicit method in two-dimensional analysis is slightly looser than that of the implicit method [5]. The diagonal scaled CG method with the convergence tolerance of 10^{-6} was used to solve the linear system of equations in the implicit method.

Figure 3(a) and 3(b) respectively present the comparison of sound pressures at R2 obtained using the analytical method and the implicit/explicit methods. Overall, the sound pressures obtained by both implicit and explicit methods agree well with the analytical solution, but larger numerical oscillation can be observed in the sound pressure obtained using the explicit method. Table 1 lists the mean relative error over receiving points for both schemes where the relative error of the explicit method is approximately two times larger than that of the implicit method. This is because the

Fig. 2 Waveform and frequency characteristics of the sound source.

Fig. 3 Comparison of sound pressures at R2 with the FE mesh of $\lambda/d = 6.8$: (a) implicit TD-FEM vs. analytical method and (b) explicit TD-FEM vs. analytical method. The both lines are overlapped well except for the numerical oscillation in the TD-FEM.

Table 1 Mean relative errors $\|e\|_2$ for implicit and explicit methods with the FE mesh of $\lambda/d = 6.8$.

Method	$\ e\ _2, \%$
Implicit	0.015
Explicit	0.029

magnitude of the spatial discretization error in the explicit method is larger than that in the implicit method, which can be confirmed by the comparison of the relationship between maximum dispersion error and spatial resolution of FE mesh calculated using the two-dimensional dispersion relation in both methods [5], with the presented numerical setting. The maximum dispersion errors of both methods calculated by

Fig. 4 Comparison of sound pressures at R2 obtained using the explicit TD-FEM with the FE mesh of $\lambda/d = 8.0$ and the analytical method. The both lines are overlapped well except for the numerical oscillation in the explicit method.

the dispersion relation are respectively 0.13% for the implicit method and 0.24% for the explicit method. For the explicit method, the dispersion relation also indicates the use of the FE mesh with approximately $\lambda/d = 8.0$ is necessary to obtain the same accurate results as the implicit method. Considering this, the comparison of sound pressures obtained using the analytical method and the explicit method with the FE mesh of $\lambda/d = 8.0$ is shown in Fig. 4. Here, the DOF of the FE mesh was 6,539,203 and Δt was set to $1/120,000$ s. The agreement of sound pressures between them was improved. The mean relative error is 0.013%, which is the same level of magnitude as the implicit method. Further, in the comparison of computational cost of both methods to achieve the same accurate result, the computational time of the explicit method was only 1/2.4 of the implicit method, whereas the explicit method required 1.4 times larger memory than the implicit method.

3.2. Cubic cavity with finite impedance boundaries

Sound fields inside the cubic cavity (Fig. 1) with finite impedance boundaries were computed to reveal the effect of the use of the backward difference in the dissipation term of the explicit method on the resulting accuracy in analysis at high frequencies. Computed sound pressures were compared with the reference solution obtained using the implicit method. In the computations, the time interval of the explicit method was again determined by the stability condition of the implicit method and was set to near the critical value. Four kinds of absorbing conditions were assumed for estimating the effect of degrees of absorption, in which different degrees of absorption was given on the floor. The accuracy was also estimated using the relative error defined by Eq. (8), but the reference solution was used instead of the analytical solution.

Regarding the four kinds of absorption conditions, frequency independent normalized acoustic impedance ratio z_n , which corresponds to statistical absorption coefficient α_s of 0.2, 0.4, 0.6 and 0.8, was given for the floor, where the z_n was calculated using Paris' formula with the assumption that z_n is a real number. z_n corresponding to $\alpha_s = 0.01$ was given for the remaining boundaries. The FE meshes with $\lambda/d = 6.8$ and 8.0 were respectively used for the implicit method and the

Fig. 5 Comparison of sound pressures at R3 in the case with $\alpha_s = 0.8$: implicit TD-FEM vs. explicit TD-FEM. The both lines are overlapped well.

Table 2 Mean relative errors $\|e\|_2$ for various absorption conditions.

Absorbing condition	$\ e\ _2 \times 10^{-3}, \%$
Rigid	2.57
$\alpha_s = 0.2$	2.12
$\alpha_s = 0.4$	2.14
$\alpha_s = 0.6$	2.69
$\alpha_s = 0.8$	3.90

explicit method. The sound pressure was calculated up to 0.01 s with Δt of 1/120,000 s. Other settings are the same as previous section.

As an example, Fig. 5 presents the comparison of sound pressure at R3 obtained using the implicit method and the explicit method for the case with $\alpha_s = 0.8$. Good agreement can be found in sound pressures calculated by both methods. Table 2 lists the mean relative error over receiving points for all cases. Here, a reference value of the mean relative error is also presented as “rigid,” which was calculated from the condition that all boundaries are rigid. This table indicates the magnitude of mean relative errors increase for higher absorption conditions, but the magnitude is almost the same as reference value in the presented numerical condition. From these results, it can be indicated that the use of the first order backward difference in the dissipation term is acceptable from the perspective of applicability. Furthermore, the results also indicate that the use of Δt determined by the stability condition of Eq. (2) provides a sufficiently small value for approximating $\dot{p}$ in Eq. (4).

4. Conclusions

The accuracy and efficiency of the explicit TD-FEM with the dissipation term approximated by the first order accurate

backward difference on room acoustics simulation at high frequencies was tested in the comparison with the efficient implicit TD-FEM for large-scale analysis. Numerical results showed that the explicit method is computationally more efficient than the implicit method from the perspective of computational time in the case using only cubic FEs. It was also revealed that the backward difference approximation of $\dot{p}$ in the dissipation term has no significant effect on the resulting accuracy with the time interval determined by the stability condition of Eq. (2), in the presented numerical conditions. Furthermore, this explicit TD-FEM has the advantage in the easiness of programming. Thus, it can be said that this method is very useful for room acoustics simulation, in which the room shape can be approximated by cubic FEs. The applicability of the explicit method for more generalized cases using rectangular FEs and distorted FEs will be presented in the future, as well as derivations of dispersion relation and stability condition for three-dimensional analysis.

References

- [1] T. Okuzono, T. Otsuru, R. Tomiku and N. Okamoto, “Application of modified integration rule to time-domain finite-element acoustic simulation of rooms,” *J. Acoust. Soc. Am.*, **132**, 804–813 (2012).
- [2] T. Okuzono, T. Otsuru, R. Tomiku and N. Okamoto, “A finite-element method using dispersion reduced spline elements for room acoustics simulation,” *Appl. Acoust.*, **79**, 1–8 (2014).
- [3] T. Otsuru and R. Tomiku, “Basic characteristics and accuracy of acoustic element using spline function in finite element sound field analysis,” *Acoust. Sci. & Tech.*, **21**, 87–95 (2000).
- [4] S. Krenk, “Dispersion-corrected explicit integration of the wave equation,” *Comput. Methods Appl. Mech. Eng.*, **191**, 975–987 (2001).
- [5] B. Yue and M. N. Guddati, “Dispersion-reducing finite elements for transient acoustics,” *J. Acoust. Soc. Am.*, **118**, 2132–2141 (2005).
- [6] L. L. Thompson, “A review of finite-element methods for time-harmonic acoustics,” *J. Acoust. Soc. Am.*, **119**, 1315–1330 (2006).
- [7] T. Okuzono, T. Otsuru, R. Tomiku, N. Okamoto, K. Maruyama and M. Nizam, “Sound field analysis of rooms by finite element method—accuracy and computational efficiency of a finite element using modified integration rule—,” *Tech. Rep. Res. Comm. Archit. Acoust.*, AA2010-40, 8 pages (2010) (in Japanese).
- [8] T. Otsuru, T. Sakuma and S. Sakamoto, “Constructing a database of computational methods for environmental acoustics,” *Acoust. Sci. & Tech.*, **26**, 221–224 (2005).
- [9] S. Sakamoto, “Phase-error analysis of high-order finite difference time domain scheme and its influence on calculation results of impulse response in closed sound field,” *Acoust. Sci. & Tech.*, **28**, 295–309 (2007).