Introduction of generalized curvilinear coordinates to spectral nodal Galerkin methods for irregular-shaped two-dimensional sound field analysis

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1. Introduction

When sound fields cannot be analyzed theoretically because of their complex shapes or boundary conditions, we have to use numerical analysis methods such as finite difference methods (FDMs) or finite element methods (FEMs). Spectral methods have been proposed besides the methods and known to be able to offer greater accuracy than the other methods [1]. In the spectral methods, a solution is approximated with global interpolation functions spreading over an analysis domain and their orders increase as the number of nodes is increased.

In our previous paper [2], we introduced generalized curvilinear coordinates to the spectral collocation method for irregular-shaped sound field analysis, where ordinary spectral methods have some difficulties. We showed that our proposed spectral collocation method could offer more accurate solutions with fewer degrees of freedom than the FEM with triangular linear elements. In addition, we showed that the use of iterative methods can make the amount of required memory smaller for the proposed method than for the FEM, while the use of direct methods cannot do so.

The purposes of this paper are introduction of generalized curvilinear coordinates to another type of spectral method, called the spectral nodal Galerkin method [3], and validation of the proposed method for analysis of an irregular-shaped sound field. Since the coefficient matrix of the matrix equation through the proposed method becomes symmetric, the matrix equation is expected to be solved more efficiently when using iterative methods than the spectral collocation method whose coefficient matrix is nonsymmetric. After giving the formulation of the proposed method, we discuss the computation efficiency of the proposed method and that of the spectral collocation method. This paper is a derivative report of our previous paper [2]. Therefore, in this paper, to ensure consistency, a problem used in the previous paper [2] is again solved in the same way as that paper. Through this procedure, we can evaluate the proposed method in this paper, from the same standpoint in the previous paper.

2. Proposed method

2.1. Helmholtz equation in generalized curvilinear coordinates

The Helmholtz equation is written as

$$\nabla^2 \phi + k^2 \phi = -q, \quad (1)$$

where ϕ is the velocity potential, k is the wave number, and q is the volume velocity of a sound source distribution. In order to analyze an irregular-shaped domain with a spectral method, generalized curvilinear coordinates (ξ, η) as shown in Fig. 1 is introduced. Use of a map $(x, y) = (X(\xi, \eta), Y(\xi, \eta))$ between the physical space coordinates (x, y) and computational space coordinates (ξ, η) transforms the Laplacian into the following form [3]

$$\nabla^2 \phi = \frac{1}{J} \left(\frac{\partial \Phi^1}{\partial \xi} + \frac{\partial \Phi^2}{\partial \eta} \right), \quad (2)$$

where

$$\Phi^1 = \frac{1}{J} \left\{ (X_\eta^2 + Y_\eta^2) \frac{\partial \phi}{\partial \xi} - (X_\xi X_\eta + Y_\xi Y_\eta) \frac{\partial \phi}{\partial \eta} \right\}, \quad (3)$$

$$\Phi^2 = \frac{1}{J} \left\{ -(X_\xi X_\eta + Y_\xi Y_\eta) \frac{\partial \phi}{\partial \xi} + (X_\xi^2 + Y_\xi^2) \frac{\partial \phi}{\partial \eta} \right\}, \quad (4)$$

$$J = X_\xi Y_\eta - X_\eta Y_\xi. \quad (5)$$

The superscripts of Φ^1 and Φ^2 represent the contravariant components of gradient $\Phi = \nabla \phi$. The subscripts represent the derivatives with respect to the variables: $X_\xi = \partial X / \partial \xi$, $X_\eta = \partial X / \partial \eta$, $Y_\xi = \partial Y / \partial \xi$, and $Y_\eta = \partial Y / \partial \eta$.

The map $(X(\xi, \eta), Y(\xi, \eta))$ can be obtained by using transfinite interpolation [3] among four defined boundaries $\Gamma_1(\xi)$, $\Gamma_2(\eta)$, $\Gamma_3(\xi)$ and $\Gamma_4(\eta)$. Defining the normal vectors $\hat{n}^1$ and $\hat{n}^2$ at the boundaries as shown in Fig. 1, we can express the particle velocities v_n^1 and v_n^2 as [3]

$$v_n^1 = -\frac{\partial \phi}{\partial \hat{n}^1} = -\hat{n}^1 \cdot \nabla \phi = -\frac{|J|}{J} \frac{\Phi^1}{\sqrt{X_\eta^2 + Y_\eta^2}}, \quad (6)$$

$$v_n^2 = -\frac{\partial \phi}{\partial \hat{n}^2} = -\hat{n}^2 \cdot \nabla \phi = -\frac{|J|}{J} \frac{\Phi^2}{\sqrt{X_\xi^2 + Y_\xi^2}}. \quad (7)$$

2.2. Formulation of spectral nodal Galerkin method

Equation (1) in the generalized curvilinear coordinates is

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Fig. 1 A map between the physical space coordinates (x, y) and computational space coordinates (ξ, η) .

solved with the spectral nodal Galerkin method, which is one of weighted residual methods. In the method, Lagrange basis functions $\ell_i(\xi)$ and $\ell_j(\eta)$ are adopted as both interpolation functions and weighting functions.

Discretizing the ξ -coordinate and η -coordinate into $(L + 1)$ and $(M + 1)$ Legendre-Gauss-Lobatto (LGL) nodes, respectively, let $\ell_i(\xi)\ell_j(\eta)$ be the interpolation function concerned with a node (ξ_i, η_j) . Multiplying both sides of Eq. (1) by the weighting function $\ell_i(\xi)\ell_j(\eta)$ and integrating the left-hand side by parts, we obtain

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 (\ell'_i \ell_j \Phi^1 + \ell_i \ell'_j \Phi^2) d\xi d\eta \\ & - \int_{-1}^1 [\ell_i \ell_j \Phi^1]_{\xi=-1}^1 d\eta - \int_{-1}^1 [\ell_i \ell_j \Phi^2]_{\eta=-1}^1 d\xi \\ & - k^2 \int_{-1}^1 \int_{-1}^1 \ell_i \ell_j q J d\xi d\eta = \int_{-1}^1 \int_{-1}^1 \ell_i \ell_j q J d\xi d\eta. \end{aligned} \quad (8)$$

The integrals of both sides can be calculated approximately with LGL quadrature.

First, by using LGL quadrature with weights $w_l w_m$ at nodes (ξ_l, η_m) , the first integral of the left-hand side is given by

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 (\ell'_i \ell_j \Phi^1 + \ell_i \ell'_j \Phi^2) d\xi d\eta \\ & \approx \sum_{l=0}^L \sum_{m=0}^M \{ \ell'_i(\xi_l) \ell_j(\eta_m) \Phi_{l,m}^1 + \ell_i(\xi_l) \ell'_j(\eta_m) \Phi_{l,m}^2 \} w_l w_m \\ & = \sum_{l=0}^L \ell'_i(\xi_l) w_l w_j \Phi_{l,j}^1 + \sum_{m=0}^M \ell'_j(\eta_m) w_i w_m \Phi_{i,m}^2 \\ & = \sum_{l=0}^L D_{li}^{(\xi)} w_l w_j \Phi_{l,j}^1 + \sum_{m=0}^M D_{mj}^{(\eta)} w_i w_m \Phi_{i,m}^2, \end{aligned} \quad (9)$$

where $\Phi_{l,m}^1 = \Phi^1(\xi_l, \eta_m)$ and $\Phi_{l,m}^2 = \Phi^2(\xi_l, \eta_m)$. We used properties of Lagrange basis functions $\ell_i(\xi_l) = \delta_{il}$ and $\ell'_j(\xi_i) = D_{ij}^{(\xi)}$ where $D_{ij}^{(\xi)}$ represents an entry of a differentiation matrix $\mathbf{D}_\xi$. Next, in the same way, the fourth integral of the left-hand side and the integral of the right-hand side of Eq. (8) are written as

$$\begin{aligned} & -k^2 \int_{-1}^1 \int_{-1}^1 \ell_i \ell_j q J d\xi d\eta \\ & \approx -k^2 \sum_{l=0}^L \sum_{m=0}^M \ell_i(\xi_l) \ell_j(\eta_m) J(\xi_l, \eta_m) w_l w_m q(\xi_l, \eta_m) \end{aligned}$$

$$= -k^2 J(\xi_i, \eta_j) w_i w_j q(\xi_i, \eta_j), \quad (10)$$

$$\int_{-1}^1 \int_{-1}^1 \ell_i \ell_j q J d\xi d\eta \approx J(\xi_i, \eta_j) w_i w_j q(\xi_i, \eta_j). \quad (11)$$

Finally, the second integral of the left-hand side of Eq. (8), which imposes boundary conditions, is written as

$$\begin{aligned} & - \int_{-1}^1 [\ell_i \ell_j \Phi^1]_{\xi=-1}^1 d\eta \\ & = - \int_{-1}^1 \{ \ell_i(1) \ell_j(\eta) \Phi^1(1, \eta) - \ell_i(-1) \ell_j(\eta) \Phi^1(-1, \eta) \} d\eta \\ & = - \sum_{m=0}^M \{ \ell_i(1) \ell_j(\eta_m) \Phi_{L,m}^1 - \ell_i(-1) \ell_j(\eta_m) \Phi_{0,m}^1 \} w_m \\ & = - \{ \ell_i(1) \Phi_{L,j}^1 - \ell_i(-1) \Phi_{0,j}^1 \} w_j. \end{aligned} \quad (12)$$

The third integral is also given by

$$\begin{aligned} & - \int_{-1}^1 [\ell_i \ell_j \Phi^2]_{\eta=-1}^1 d\xi \\ & = - \{ \ell_j(1) \Phi_{i,M}^2 - \ell_j(-1) \Phi_{i,0}^2 \} w_i. \end{aligned} \quad (13)$$

Given specific acoustic impedance z_n on the boundaries, from Eqs. (6) and (7), we obtain

$$\Phi_{L,j}^1 = - \frac{j\omega\rho}{z_n} \left(\frac{J}{|J|} \sqrt{X_\eta^2 + Y_\eta^2} \right)_{L,j} \phi_{L,j}, \quad (14)$$

$$\Phi_{0,j}^1 = \frac{j\omega\rho}{z_n} \left(\frac{J}{|J|} \sqrt{X_\eta^2 + Y_\eta^2} \right)_{0,j} \phi_{0,j}, \quad (15)$$

$$\Phi_{i,M}^2 = - \frac{j\omega\rho}{z_n} \left(\frac{J}{|J|} \sqrt{X_\xi^2 + Y_\xi^2} \right)_{M,i} \phi_{i,M}, \quad (16)$$

$$\Phi_{i,0}^2 = \frac{j\omega\rho}{z_n} \left(\frac{J}{|J|} \sqrt{X_\xi^2 + Y_\xi^2} \right)_{0,i} \phi_{i,0}, \quad (17)$$

where ω is angular frequency and ρ is the density of air.

Substituting Eqs. (9)–(13) into Eq. (8) for $i = 0, 1, \dots, L$, $j = 0, 1, \dots, M$ gives the following matrix equation

$$\begin{aligned} & [\mathbf{I}_\eta \otimes \mathbf{D}_\xi]^T \mathbf{W} \Phi^1 + [\mathbf{D}_\eta \otimes \mathbf{I}_\xi]^T \mathbf{W} \Phi^2 \\ & + j\omega \mathbf{C} \Phi - k^2 \mathbf{M} \Phi = \mathbf{M} \mathbf{q}, \end{aligned} \quad (18)$$

where

$$\mathbf{W} = \text{diag}(w_i w_j), \quad (19)$$

$$\mathbf{M} = \text{diag}(J(\xi_i, \eta_j) w_i w_j), \quad (20)$$

$\mathbf{D}_\xi$ and $\mathbf{D}_\eta$ are differentiation matrices with respect to ξ and η , and $\mathbf{I}_\xi$ and $\mathbf{I}_\eta$ are identity matrices. $\mathbf{C}$ is a diagonal damping matrix and its entries are given by

$$j\omega C_{j(M+1)+i, j(M+1)+i} = \Phi_{i,j}^1 w_j / \phi_{i,j}, \quad (21)$$

$$i = 0, L, j = 1, \dots, M-1,$$

$$j\omega C_{j(M+1)+i, j(M+1)+i} = \Phi_{i,j}^2 w_i / \phi_{i,j}, \quad (22)$$

$$i = 1, \dots, L-1, j = 0, M,$$

$$j\omega C_{j(M+1)+i, j(M+1)+i} = (\Phi_{i,j}^1 w_j + \Phi_{i,j}^2 w_i) / \phi_{i,j}, \quad (23)$$

$$i = 0, L, j = 0, M.$$

Fig. 2 An analyzed pentagonal room.

From Eqs. (3) and (4), Φ^1 and Φ^2 are expressed as

$$\Phi^1 = \left[\text{diag} \left(\frac{X_\eta^2 + Y_\eta^2}{J} \right) I_\eta \otimes D_\xi - \text{diag} \left(\frac{X_\xi X_\eta + Y_\xi Y_\eta}{J} \right) D_\eta \otimes I_\xi \right] \phi, \quad (24)$$

$$\Phi^2 = \left[-\text{diag} \left(\frac{X_\xi X_\eta + Y_\xi Y_\eta}{J} \right) I_\eta \otimes D_\xi + \text{diag} \left(\frac{X_\xi^2 + Y_\xi^2}{J} \right) D_\eta \otimes I_\xi \right] \phi. \quad (25)$$

The resultant matrix equation (18) requires a large amount of memory in the case of using direct methods because one needs to store the entire coefficient matrix, which is dense, of the left-hand side of Eq. (18). However, it is composed of sums and products of small matrix components such as D_ξ and diagonal matrix components such as W . Therefore, the use of iterative methods, which only needs to store the matrix components separately, can reduce required memory [2]. In addition, one can easily confirm that the entire coefficient matrix of the left-hand side of Eq. (18) is symmetric. As will be shown in the next chapter, this feature can raise the computational efficiency of iterative methods, compared with the spectral collocation method whose coefficient matrix is nonsymmetric.

3. Trial calculations

3.1. Numerical conditions

We computed a frequency response function and impulse response of the pentagonal room as shown in Fig. 2 using the proposed method in this paper and collocation method given in our previous paper [2], and we evaluated their computational costs under the same numerical conditions and evaluation strategy as the paper. Specific acoustic impedance on all the boundaries was set to 15,000 Pa·s/m. The frequency response function between the source and receiving point was calculated at intervals of 2,000/2,048 Hz over the frequency range of 100 to 1,000 Hz, then the impulse response was obtained by computing the inverse DFT of the frequency response function after our previous paper [2].

The FEM with 9,333 triangular linear elements and 4,782 nodes also computed the impulse response. We considered it a

Fig. 3 Cross-correlation coefficients of impulse responses as functions of the number of unknowns.

Fig. 4 Cross-correlation coefficients of impulse responses as functions of required memory.

reference solution and evaluated the cross-correlation coefficient between the reference solution and one calculated by each spectral method. Although the spectral collocation method proposed in the previous paper [2] might be considered a reference instead of the FEM, a numerical solution computed by the spectral method showed an unfavorable convergence property. Therefore, we adopted a numerical solution computed by the FEM as a reference solution for safety.

3.2. Convergence property of cross-correlation coefficient

Figure 3 shows cross-correlation coefficients of impulse responses computed by the three methods as functions of the number of unknowns and Fig. 4 shows the cross-correlation coefficients as functions of memory required to store matrices and vectors in each method. One cannot see a marked difference between the convergence properties of the spectral nodal Galerkin method (NGM) and the spectral collocation method (SCM); both of the numerical solutions converge with a much smaller number of unknowns and smaller amount of memory than the FEM. Note that, when using iterative methods, the amount of required memory of NGM is a little bit larger than that of SCM because NGM needs to store the quadrature weights, which SCM does not need. However, since the increase of required memory is too small, this aspect of NGM is trivial.

Table 1 The number of matrix-vector products operated in each method.

		125 Hz	250 Hz	500 Hz	1,000 Hz
NGM	COCG	348	374	486	764
	CGS	654	830	1,320	3,508
	BiCGStab	484	812	2,720	8,148
SCM	CGS	992	1,172	1,706	5,250
	BiCGStab	3,962	4,992	12,656	28,642
FEM	COCG	330	340	450	664

3.3. Comparison of iterative methods

We investigated the numbers of matrix-vector products required for iterative solutions to converge. Since the coefficient matrix of the proposed method is an indefinite symmetric matrix as well as that of an FEM, types of iterative methods were selected referring to literature [4]. The CGS and BiCGStab methods were applied to both the proposed method and collocation method. The COCG method [5], which is an iterative method for symmetric matrix equations and is often used as a solver of FEMs in acoustic problems, was also applied to the proposed method. Although the GPBiCG method was also applied to both the proposed method and collocation method, it diverged in all the cases.

The diagonal preconditioner, which is the simplest of all the preconditioners, was adopted in all the iterative methods. The initial solutions were set to $\mathbf{0}$ in all the methods and the initial shadow residuals in the CGS and BiCGStab methods were set to random vectors.

Table 1 shows the number of matrix-vector products at which relative residual norm of each iterative method converges below 10^{-12} . From Fig. 3, we can recognize that the cross-correlation coefficients of the spectral methods reach 0.99 with 441 degrees of freedom. All the computations were executed under this same condition on the number of nodes. In Table 1, two spectral methods can be compared directly because the two methods require an equivalent number of products per matrix-vector product. We can see that the numbers of matrix-vector products of the CGS and BiCGStab methods in the proposed method are smaller than those in the collocation method. The numbers of matrix-vector products of the BiCGStab method was significantly decreased in particular. Furthermore, the COCG method can reduce the number of matrix-vector products because only one matrix-vector product every iteration is operated in the COCG method whereas two matrix-vector products every iteration are needed in the CGS and BiCGStab methods [5].

The numbers of matrix-vector products of the COCG method in the FEM using triangular linear elements with 2,331 degrees of freedom, which also achieves the cross-correlation coefficient of 0.99 to the reference solution, are also shown in Table 1. The number of products per matrix-

vector product in the FEM is different from those of the two spectral methods. Those results of the FEM are shown just for reference. Compared with this, the proposed method (NGM) requires a little bit large number of matrix-vector products despite a fewer degrees of freedom. The result is attributed to large condition numbers of matrices in spectral methods [6]. If we used preconditioners other than the diagonal one, we could improve the convergence property.

4. Conclusion

Generalized curvilinear coordinates were introduced to the spectral nodal Galerkin method for irregular-shaped sound field analysis and its numerical properties were investigated. Although there was no significant difference in the accuracy, required degree of freedom and memory between the proposed method and the collocation method, the numbers of matrix-vector products required for iterative solutions to converge could be reduced in the proposed method. This is because the COCG method can be applied to the proposed method involving symmetric coefficient matrices. However, the convergence property of the iterative solutions is not preferable enough due to ill-conditioned matrices of spectral methods compared with the FEM. In future work, it is necessary to study other effective preconditioners such as the finite element preconditioner [6] in order to improve the convergence property. Furthermore, although the proposed method was evaluated on a small-scale problem in the present paper, we would like to apply the proposed method to large-scale or complicated problems and evaluate it in more detail.

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