Finite element modeling of acoustic resonators with thermal and viscous boundary layers

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1. Introduction

Predicting the absorption performance of resonators is of great interest for noise control engineers. The thermal and viscous boundary layers (TVBL) around a resonator's neck play important roles in the absorption, particularly when there is no porous material near the neck. Several papers have been dedicated to the numerical prediction of resonator's absorption performance. The recently proposed approaches have been based on computational fluid dynamics [1] or the viscothermal acoustic elements [2]. Accordingly, relatively high computation and implementation costs are required. This paper concerns another simple approach that is based on the equivalent boundary condition of TVBL in the conventional Helmholtz equation [3,4]. Despite its effectiveness, the details of the implementation have not been presented, especially where giving the TVBL boundary condition. Furthermore, to the authors' best knowledge, three-dimensional analysis has not been performed yet. This paper presents the finite element implementation of the TVBL boundary condition for three-dimensional frequency-domain problems. This implementation is applied to analyze the absorption characteristics of slit-type and Helmholtz resonators, and the appropriate calculation setting is clarified.

2. Analysis theory

2.1. TVBL boundary condition

The time convention is $e^{j\omega t}$ throughout this paper. The boundary condition taking the effect of thermal and viscous boundary layers into account is given in [5] as

$$\frac{\partial p}{\partial n} = -j\omega\rho_0 \left[\frac{1}{Z_n} + (1+j) \left\{ \frac{R_t}{\rho_0^2 c_0^2} - \frac{R_v \nabla_{\tan}^2}{\rho_0^2 \omega^2} \right\} \right] p, \quad (1)$$

where p , ρ_0 , c_0 , and Z_n are the sound pressure, air density, sound speed, and normal-incidence surface impedance, respectively. $\nabla_{\tan}^2$ is the Laplacian operator in the tangential plane of the boundary and $\partial/\partial n$ is the normal derivative to the plane. R_t and R_v are the thermal and viscous resistances, which are related to the thicknesses of the thermal and viscous boundary layers, δ' and δ , as follows:

$$R_t = \frac{1}{2} \rho_0 \omega (\gamma - 1) \delta', \quad \delta' = \sqrt{\frac{2\mu}{\rho_0 \omega N_{pr}}},$$

$$R_v = \frac{1}{2} \rho_0 \omega \delta, \quad \delta = \sqrt{\frac{2\mu}{\rho_0 \omega}},$$

where γ , μ , and N_{pr} are the heat capacity ratio, viscous coefficient, and Prandtl number of the air, respectively. Note that several signs in the above are modified from those in [5] owing to the time convention $e^{j\omega t}$. To simplify Eq. (1), it is rewritten as

$$\frac{\partial p}{\partial n} = -jk\beta p - (j\omega)^{3/2} c_T p - (j\omega)^{-1/2} c_V \nabla_{\tan}^2 p, \quad (2)$$

where β is the normalized admittance and the frequency-independent coefficients, c_T and c_V , are defined as

$$c_T = \frac{\gamma - 1}{c_0^2} \sqrt{\frac{\mu}{\rho_0 N_{pr}}} \quad \text{and} \quad c_V = \sqrt{\frac{\mu}{\rho_0}}.$$

Obviously, the TVBL boundary condition involves a fractional derivative in the time domain expression, which makes it difficult to use in time domain analysis. On the other hand, this paper focuses on the treatment of the boundary condition in frequency domain analysis.

2.2. Weak formulation of the Helmholtz equation

The weak form of the Helmholtz equation, $\nabla^2 p + k^2 p = 0$, is given as

$$\int_{\Omega} \nabla \delta p \cdot \nabla p dV - k^2 \int_{\Omega} \delta p p dV - \int_{\Gamma} \delta p \frac{\partial p}{\partial n} dS = 0, \quad (3)$$

where δp is the admissible variation and ∇ is the nabla operator. Substituting Eq. (2) into Eq. (3), the following equation is obtained:

$$\int_{\Omega} \nabla \delta p \cdot \nabla p dV - k^2 \int_{\Omega} \delta p p dV + jk\beta \int_{\Gamma} \delta p p dS$$

$$+ (j\omega)^{3/2} c_T \int_{\Gamma} \delta p p dS + (j\omega)^{-1/2} c_V \int_{\Gamma} \delta p \nabla_{\tan}^2 p dS = 0.$$

The main difficulty is to evaluate the fifth boundary integral term. Although the interpolation function must be differentiable twice or more to evaluate the term directly, this requirement can be reduced by applying Green's first identity in two-dimensions as follows:

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$$\int_{\Gamma} \delta p \nabla_{\tan}^2 p dS = - \int_{\Gamma} \nabla_{\tan} \delta p \cdot \nabla_{\tan} p dS + \int_{\partial \Gamma} \delta p \frac{\partial p}{\partial n_i} dl, \quad (4)$$

where $\nabla_{\tan}$ is the gradient operator in the tangential plane and $\partial/\partial n_i$ is the normal derivative to the contour of the boundary surface. Accordingly, the energy dissipation along the contour should be evaluated in the latter term, which makes the global system matrix asymmetric. The numerical implementation of Eq. (4) is provided in Appendix.

3. Results and discussion

3.1. Slit-type resonators

In [3], the TVBL boundary condition was implemented in the two-dimensional boundary element method (BEM), where the term of $\nabla_{\tan}^2 p$ was directly evaluated using the finite differences of sound pressures between neighboring elements. This implementation was applied to estimate the normal-incidence absorption coefficients of slit-type resonators, and the results were confirmed to be in good agreement with measured values. In this section, to clarify the appropriate setting of the implementation, slit-type resonators are examined by the three-dimensional finite element method (FEM), and the results are compared with BEM results and the measured values in [3].

In the FEM analysis, a 100-mm-square duct with a slit-type resonator is modeled, where all surfaces are assumed to be rigid ($\beta = 0$). Figure 1 shows the horizontal geometry of the model and the notation of the boundaries. These conditions correspond to those of the measurement in [3]. As a broad setting, the TVBL boundary condition is given to the end surface of the duct (Γ^D), the front surface of the cavity (Γ^C), and the side, upper, and bottom surfaces of the slit neck (Γ^N). On the contours of Γ^D and Γ^C , $\partial p/\partial n_i$ is zero except along the edges of the neck. Thus, contour integrals are evaluated along the edges ($\partial \Gamma^D$, $\partial \Gamma^C$, and $\partial \Gamma^N$). Second-order prism and hexahedral elements are used, and a fine mesh is applied as illustrated later in Fig. 4(a). The normal-incidence absorption coefficients are calculated by the transfer function method at 1/24-octave band center frequencies [6].

Figure 2 shows the calculated and measured results for four slit-type resonators with different combinations of the

Fig. 1 Horizontal geometry of the duct with a slit-type resonator.

width and depth of the neck. The FEM results are given for three different settings, and the BEM and measured results are cited from [3]. The FEM results with the above broad TVBL setting are represented by red lines. The calculated values for the 4-mm-width slit correspond relatively well with the BEM and measured ones, whereas those for the 2-mm-width slit are slightly larger than the others. Nevertheless, this TVBL setting for the FEM seems reasonably reliable.

Next, a narrow TVBL setting only in the neck (Γ^N) is examined. As can be seen from the results represented by blue lines, the calculated values are considerably greater than those with the broad setting. This overestimation of the absorption coefficient may be caused by intensification of the edge potential around the openings of the neck due to the lack of a tangential drag force on Γ^D and Γ^C . Therefore, it is suggested that the TVBL boundary condition should be given to two surfaces along an edge.

If the asymmetric contour integral term for edges, I_c , can be neglected, the calculation becomes more efficient. However, the absorption coefficients calculated without I_c , represented with green lines, are lower than the red lines. This demonstrates that the energy dissipation at edges around a resonator's neck must be considered to predict the absorption coefficient accurately.

Fig. 2 Normal-incidence absorption coefficients of four slit-type resonators calculated by the FEM with three different settings. The BEM and measured values for PVC are cited from [3].

Fig. 3 Normal-incidence absorption coefficient calculated with two different mesh resolutions.

Fig. 4 Meshes for the examination of effects of mesh resolution around neck.

In the above calculation, fine meshes were used because the acoustic field around the neck is expected to change sharply. Figure 3 shows the results calculated with the fine and coarse meshes illustrated in Fig. 4. As a fixed condition, two hexahedral elements are generated in the direction of the slit width. The maximum lengths around the neck are 2 and 10 mm for fine and coarse meshes, respectively. It can be seen that the results for the coarse mesh are slightly greater than those for the fine mesh, but the difference is not significant when predicting the absorption coefficient.

3.2. Helmholtz resonators

As a typical example of three-dimensional resonators, Helmholtz resonators are analyzed under a similar setting to that in the previous section, where the cross sections of the cavity and duct are identical and square. In the FEM calculation, the TVBL boundary condition is given to the front and back surfaces of the perforated baffle and the cylindrical surface of the neck. The normalized admittance β is assumed to be zero for all surfaces. The FEM results are compared with the existing theoretical models of Atalla and Sgard [7] and Maa [8]. The resistance and inertial correction terms are approximately derived for periodic resonators in the Atalla–Sgard model and for a single resonator in the Maa model. Note that the resistance correction of the Maa model is based on [7].

Figure 5 shows the calculated results for two resonators with different sizes. It can be seen that all the theoretical and FEM results are relatively correlated but slightly different in terms of the peak value and frequency. The sound field in the duct is equivalent to the normal-incidence field over the

Fig. 5 Normal-incidence absorption coefficients calculated for two Helmholtz resonators with different sizes, Types (a) and (b).

periodic resonators. However, the FEM results are closer to the Atalla–Sgard model for Type (a) but closer to the Maa model for Type (b). This inconsistency implies that the interactive motion of the neck and cavity considerably affects the resistance and inductance corrections for the neck's impedance. From this viewpoint, the FEM may be advantageous because the effects of the interaction are automatically incorporated.

4. Conclusion

This paper presented and verified the finite element modeling of acoustic resonators with thermal and viscous boundary layers. It was confirmed that the TVBL boundary condition should be given not only to the side surfaces in the resonator's neck but also to the front and back surfaces of the perforated baffles. Furthermore, the energy dissipation at the neck's edges was not negligible. Thus, an asymmetric matrix equation should be solved. Numerical methods are advantageous for predicting the absorption performance of resonators not only because of their capability of handling resonators with complex shapes, but also because the resistance and reactance corrections are automatically incorporated, reflecting the acoustic field around the neck. In future work, the presented results should be verified by comparison with those of more detailed methods such as those in [1,2].

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Appendix: Implementation of Eq. (4)

The surface and contour integral terms in the right-hand side of Eq. (4) are referred to as I_s and I_c , respectively. The numerical evaluation procedures for I_s and I_c are provided here by taking an example of the triangle element shown in Fig. 6. Since the inner product of a pair of vectors is not dependent on the coordinates, I_s and I_c can be calculated in the local Cartesian coordinate system for each element.

Firstly, the local coordinates (x' , y') of each node in an element are calculated (see Fig. 6(a)). Let us define the position vector of the i th node in the global coordinates as $\mathbf{r}_i$ and the direction vector from the i th to j th nodes as $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$. Then the unit vectors ($\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$) of the local Cartesian coordinate system are obtained as $\mathbf{e}_{x'} = \mathbf{r}_{12}/|\mathbf{r}_{12}|$, $\mathbf{e}_{z'} = \mathbf{r}_{12} \times \mathbf{r}_{13}/|\mathbf{r}_{12} \times \mathbf{r}_{13}|$, $\mathbf{e}_{y'} = \mathbf{e}_{z'} \times \mathbf{e}_{x'}$. Subsequently, the coordinates of the i th node, (x'_i, y'_i), are obtained as $x'_i = \mathbf{r}_i \cdot \mathbf{e}_{x'}$ and $y'_i = \mathbf{r}_i \cdot \mathbf{e}_{y'}$. In the following, the numerical evaluations for I_s and I_c are performed in the x' - y' plane. For all surface elements, I_s

Fig. 6 Local axes for the tangential Laplacian calculation.

can be calculated in the same way as the stiffness matrix in two dimensions. On the other hand, I_c should be calculated for surface elements adjacent to $\partial\Gamma$ as follows:

$$I_c = \{\delta \mathbf{p}\} \left(\int_{\partial\Gamma} [\mathbf{N}]^T \begin{bmatrix} n_{x'} \\ n_{y'} \end{bmatrix}^T \begin{bmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{bmatrix} [\mathbf{N}] dl \right) \{\mathbf{p}\}$$

$$= \{\delta \mathbf{p}\} \left(\int_{\xi(\eta)} [\mathbf{N}]^T \begin{bmatrix} n_{x'} \\ n_{y'} \end{bmatrix}^T [\mathbf{J}]^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} [\mathbf{N}] |J| d\xi(d\eta) \right) \{\mathbf{p}\},$$

where $\{\mathbf{p}\}$ and $\{\delta \mathbf{p}\}$ are the vector of nodal sound pressure and its admissible variation, respectively. $[\mathbf{N}]$ is the interpolation function matrix, and $n_{x'}$ and $n_{y'}$ are the components of the normal vector to the contour of a surface element. $[\mathbf{J}]$ is the Jacobian matrix for the coordinate transform in the numerical integration, which is defined as

$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x'}{\partial \xi} & \frac{\partial y'}{\partial \xi} \\ \frac{\partial x'}{\partial \eta} & \frac{\partial y'}{\partial \eta} \end{bmatrix},$$

and $|J|$ is the determinant of the Jacobian matrix. In the triangle element, the line integral can be performed on I-II ($\eta = 0, 0 \leq \xi \leq 1$), II-III ($\eta = 1 - \xi, 0 \leq \xi \leq 1$) and III-I ($\xi = 0, 0 \leq \eta \leq 1$), respectively.