Stability conditions for impedance boundaries in the finite-difference time-domain method

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1. Introduction

The finite-difference time-domain (FDTD) method was developed by Yee [1] for electromagnetic problems, and Madariage initially extended it to elastodynamics for seismologic analyses [2]. Recently, the FDTD method has been used in various acoustical fields [3]. In architectural acoustics, this method has been applied to situations including room acoustics [4–6], traffic noise problems [7], and floor impact sound predictions [8,9].

In room acoustics, enclosures and scatterers such as walls, ceilings, floors, and furniture often have sound absorption properties. Some techniques have been proposed to consider such absorptions in FDTD calculations [10–16]. Among them, the easiest technique is to apply the real number of the surface impedance to the boundary, which herein is called an impedance boundary. However, the standard FDTD algorithm cannot consider a reflection delay due to the boundary because it cannot deal with complex numbers. Additionally, the frequency characteristics of absorption cannot be taken into account in a single FDTD calculation with a constant impedance. Nevertheless, an impedance boundary is often used, especially for preliminary tests due to its ease of application and low cost.

To the best of the author's knowledge, an impedance boundary works perfectly if it is applied to a one-dimensional sound field. However, the results for a two- or three-dimensional sound field with an impedance boundary may diverge even when the Courant-Friedrichs-Lewy (CFL) condition [17] is satisfied. For stable calculations with an impedance boundary, the stability conditions in this study are derived from the state transition equation regarding the boundary.

2. Formulation

Figure 1 shows an impedance boundary given a surface impedance z_b and an adjacent fluid cell in a three-dimensional sound field, the left surface of which is assumed to be the impedance boundary, where p , v_x , v_y , v_z , and κ_0 are the sound pressure, the x -, y -, z -directional particle velocities, and the bulk modulus, respectively. Except for the impedance boundary, all boundaries are assumed to work as perfectly absorbing ones. Thus, this model represents one cell extracted

from a semi-infinite region bounded by an impedance boundary. The perfectly absorbing boundaries are herein approximated by giving the surfaces the characteristic impedance of the fluid z_0 . It should be noted that the boundaries can work as perfectly absorbing ones for a normal incidence of plane wave to them but cannot work perfectly for an oblique incidence. In this case, each boundary condition is written as

$$-v_{x1} = \frac{p}{z_b}, \quad (1)$$

$$-v_{y1} = -v_{z1} = \frac{p}{z_0}, \quad (2)$$

$$v_{x2} = v_{y2} = v_{z2} = \frac{p}{z_0}. \quad (3)$$

The continuous equation in the cell is given by

$$\frac{\partial p}{\partial t} = -\kappa_0 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right). \quad (4)$$

Discretizing these equations by the central difference schemes with the staggered grid system gives the approximated equations, which are expressed as

$$-v_{x1}^{n+0.5} = \frac{p^n}{z_b}, \quad (5)$$

$$-v_{y1}^{n+0.5} = -v_{z1}^{n+0.5} = \frac{p^n}{z_0}, \quad (6)$$

$$v_{x2}^{n+0.5} = v_{y2}^{n+0.5} = v_{z2}^{n+0.5} = \frac{p^n}{z_0}, \quad (7)$$

$$\begin{aligned} \frac{p^{n+1} - p^n}{\Delta t} = & -\kappa_0 \left(\frac{v_{x2}^{n+0.5} - v_{x1}^{n+0.5}}{\Delta x} \right. \\ & + \frac{v_{y2}^{n+0.5} - v_{y1}^{n+0.5}}{\Delta y} \\ & \left. + \frac{v_{z2}^{n+0.5} - v_{z1}^{n+0.5}}{\Delta z} \right). \end{aligned} \quad (8)$$

From these equations, the state transition equation regarding the sound pressure in the cell is given as

$$p^{n+1} = \left\{ 1 - \kappa_0 \frac{\Delta t}{\Delta x} \left(\frac{1}{z_0} + \frac{1}{z_b} \right) \right\}$$

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Fig. 1 Impedance boundary with a three-dimensional adjacent fluid cell.

$$-\kappa_0 \frac{\Delta t}{\Delta y} \left(\frac{2}{z_0} \right) - \kappa_0 \frac{\Delta t}{\Delta z} \left(\frac{2}{z_0} \right) \Big\} p^n. \quad (9)$$

To realize a stable system expressed by the state transition equation, the absolute value of the contents of the curly brackets on the right side of Eq. (9) must be less than or equal to one. Considering the relationship $\kappa_0 = c_0 z_0$, where c_0 is the speed of sound, this condition yields

$$\Delta t \leq \frac{1}{c_0 \left\{ \frac{1}{\Delta x} \left(\frac{1}{2} + \frac{1}{2z_b/z_0} \right) + \frac{1}{\Delta y} + \frac{1}{\Delta z} \right\}}. \quad (10)$$

Similarly, the stability conditions for two- and one-dimensional sound fields are respectively obtained as

$$\Delta t \leq \frac{1}{c_0 \left\{ \frac{1}{\Delta x} \left(\frac{1}{2} + \frac{1}{2z_b/z_0} \right) + \frac{1}{\Delta y} \right\}}, \quad (11)$$

$$\Delta t \leq \frac{1}{c_0 \left\{ \frac{1}{\Delta x} \left(\frac{1}{2} + \frac{1}{2z_b/z_0} \right) \right\}}. \quad (12)$$

3. Numerical examples

The CFL conditions for three-, two-, and one-dimensional sound fields are written as [17]

$$\Delta t \leq \frac{1}{c_0 \sqrt{\left(\frac{1}{\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2 + \left(\frac{1}{\Delta z} \right)^2}}, \quad (13)$$

$$\Delta t \leq \frac{1}{c_0 \sqrt{\left(\frac{1}{\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2}}, \quad (14)$$

$$\Delta t \leq \frac{\Delta x}{c_0}. \quad (15)$$

The real surface impedance z_b is related with the absorption coefficient α as [18]

$$\frac{z_b}{z_0} = \frac{1 + \sqrt{1 - \alpha}}{1 - \sqrt{1 - \alpha}} \geq 1. \quad (16)$$

Fig. 2 Maximum time intervals normalized by the spatial interval derived from the stability conditions for a two-dimensional sound field.

Fig. 3 Maximum time intervals normalized by the spatial interval derived from the stability conditions for a three-dimensional sound field.

Considering the CFL condition for a one-dimensional sound field expressed by Eq. (15) and the relationship expressed by Eq. (16), the stability condition of an impedance boundary placed in a one-dimensional sound field, which is expressed by Eq. (12), is unconditionally satisfied. For two- and three-dimensional cases, the maximum time intervals, which satisfies Eqs. (10), (11), (13), and (14), are normalized by the spatial interval Δh ($= \Delta x = \Delta y = \Delta z$) and respectively plotted in Figs. 2 and 3 where the sound speed c_0 is set to 343.5 m/s. The maximum time intervals satisfying the stability conditions of the impedance boundary are always smaller than those of the CFL conditions. In addition, they tend to be even smaller as the absorption coefficient shifts higher.

A zero absorption coefficient indicates a rigid boundary and not an absorptive boundary. In this case, the author has yet to experience an unstable FDTD calculation while satisfying the CFL condition. Therefore, the maximum time intervals of an impedance boundary with a zero absorption coefficient should be equal to those of the CFL conditions. This unreasonability is due to the approximation of perfectly absorbing boundaries expressed by Eqs. (2) and (3).

4. Conclusion

Herein, the stability conditions for impedance boundaries, which are often applied to FDTD calculations, are derived from the state transition equation. An impedance boundary in a one-dimensional sound field is unconditionally stable, but this is not the case in a two- or three-dimensional sound field. A two- or three-dimensional sound field can be unstable even when the CFL condition is satisfied, especially when the absorption coefficient of the boundary is very high. In two- and three-dimensional cases, only one of the boundaries is herein assumed to be an impedance boundary. If multiple boundaries are assumed to be impedance boundaries, the stability conditions must be different from those presented here. However, considering that the higher absorption coefficient leads to the smaller maximum Δt , the stability conditions presented here are on the safe side. Further studies are necessary because the maximum time interval satisfying the stability condition of the impedance boundary does not asymptotically approach that of the CFL condition when the absorption coefficient shifts lower. Nevertheless, the stability conditions derived here should be useful for stable calculations as these conditions are on the safe side and easily calculated.

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