Improved stability conditions for impedance boundaries in the finite-difference time-domain method

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(Received 5 November 2018, Accepted for publication 13 November 2018)

Keywords: FDTD method, Surface impedance, Boundary condition, Stability condition, State transition equation
PACS number: 43.55.Ev, 43.55.Ka [doi:10.1250/ast.40.148]

1. Introduction

For finite-difference time-domain (FDTD) analyses, techniques that take the surface absorptivity in account have been proposed [1–7]. Among them, the easiest way to consider the surface absorptivity is to introduce the real number of the surface impedance to the boundary, which is hereafter called the impedance boundary. However, a high absorption impedance boundary can cause instability in the calculation.

The author previously tried to derive the stability condition by considering the impedance boundary and the adjacent cell [8]. The results showed that the impedance boundary does not affect the stability of a one-dimensional sound field, but the stabilities of two- and three-dimensional sound fields can deteriorate significantly. However, the results seem to be unreasonable. If the Courant-Friedrichs-Lewy (CFL) condition [9] is satisfied, a low absorption impedance boundary, which includes a rigid boundary, does not cause instability even in two- and three-dimensional sound fields. Therefore, the stability condition derived previously must be improved.

The previous discussion did not take the incident direction of the sound wave into account and all boundaries except for an impedance boundary were assumed to be perfectly absorptive with a one-dimensional consideration in each direction. Consequently, the stability conditions were underestimated. In this letter, the oblique incidence of a plane wave is assumed and the formulation is rebuilt based on the state transition equation regarding the impedance boundary.

2. Formulation

Figure 1 shows an impedance boundary given a surface impedance z_b and an adjacent fluid cell in a three-dimensional sound field, where p , v_x , v_y , v_z , Δx , Δy , Δz , κ_0 , c_0 , and z_0 are the sound pressure, the x -, y -, z -directional particle velocities, spatial intervals, the bulk modulus, the speed of sound, and the characteristic impedance of the fluid, respectively. Except for the impedance boundary, all boundaries are assumed to work as perfectly absorbing ones for the oblique incidence of the plane wave, and have impedances of z_x , z_y , and z_z at surfaces normal to x , y , and z directions, respectively. Thus, this model approximately represents a semi-infinite region bounded by an impedance boundary.

Each boundary condition is written as

$$-v_{x1} = \frac{p}{z_b}, \quad (1)$$

$$v_{x2} = \frac{p}{z_x}, \quad (2)$$

$$-v_{y1} = v_{y2} = \frac{p}{z_y}, \quad (3)$$

$$-v_{z1} = v_{z2} = \frac{p}{z_z}. \quad (4)$$

Now let v_0 be the particle velocity parallel to the direction of the plane wave and assume that

$$v_0^2 = v_{x2}^2 + v_{y2}^2 + v_{z2}^2. \quad (5)$$

Then considering $v_0 = p/z_0$, Eq. (5) can be rewritten with Eqs. (2)–(4) as

$$\frac{1}{z_0^2} = \frac{1}{z_x^2} + \frac{1}{z_y^2} + \frac{1}{z_z^2}. \quad (6)$$

The continuous equation in the cell is given by

$$\frac{\partial p}{\partial t} = -\kappa_0 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right). \quad (7)$$

Discretizing Eqs. (1)–(4), and (7) by the central difference schemes gives the approximated equations, which are expressed as

$$-v_{x1}^{n+0.5} = \frac{p^n}{z_b}, \quad (8)$$

$$v_{x2}^{n+0.5} = \frac{p^n}{z_x}, \quad (9)$$

$$-v_{y1}^{n+0.5} = v_{y2}^{n+0.5} = \frac{p^n}{z_y}, \quad (10)$$

$$-v_{z1}^{n+0.5} = v_{z2}^{n+0.5} = \frac{p^n}{z_z}, \quad (11)$$

$$\begin{aligned} \frac{p^{n+1} - p^n}{\Delta t} = & -\kappa_0 \left(\frac{v_{x2}^{n+0.5} - v_{x1}^{n+0.5}}{\Delta x} \right. \\ & + \frac{v_{y2}^{n+0.5} - v_{y1}^{n+0.5}}{\Delta y} \\ & \left. + \frac{v_{z2}^{n+0.5} - v_{z1}^{n+0.5}}{\Delta z} \right). \end{aligned} \quad (12)$$

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Fig. 1 Impedance boundary with a three-dimensional adjacent fluid cell.

From these equations, the state transition equation regarding the sound pressure in the cell is given as

$$p^{n+1} = \left[1 - \kappa_0 \Delta t \left\{ \left(\frac{1}{z_x} + \frac{1}{z_b} \right) \frac{1}{2\Delta x} + \frac{2}{z_y \Delta y} + \frac{2}{z_z \Delta z} \right\} \right] p^n. \quad (13)$$

To realize a stable system expressed by the state transition equation, the absolute value of the content in the brackets on the right side of Eq. (13) must be less than or equal to one. Therefore, the stability condition is derived as

$$\kappa_0 \Delta t \left\{ \left(\frac{1}{z_x} + \frac{1}{z_b} \right) \frac{1}{2\Delta x} + \frac{1}{z_y \Delta y} + \frac{1}{z_z \Delta z} \right\} \leq 1 \quad (14)$$

Now let

$$D = \frac{1}{2z_x \Delta x} + \frac{1}{z_y \Delta y} + \frac{1}{z_z \Delta z}, \quad (15)$$

and

$$X = \frac{1}{z_x}, \quad Y = \frac{1}{z_y}, \quad Z = \frac{1}{z_z}, \quad R = \frac{1}{z_0}. \quad (16)$$

Equations (6) and (15) can be rewritten as

$$X^2 + Y^2 + Z^2 = R^2, \quad (17)$$

$$\frac{1}{2\Delta x} X + \frac{1}{\Delta y} Y + \frac{1}{\Delta z} Z = D, \quad (18)$$

respectively, which are interpreted as a sphere and a plane in XYZ space. D has a maximum value when the plane contacts the sphere at a point, which can be expressed with the parameter T as $(T/(2\Delta x), T/\Delta y, T/\Delta z)$. Considering the point is on the sphere and Eq. (17) is satisfied, then

$$T = \frac{R}{\sqrt{\left(\frac{1}{2\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2 + \left(\frac{1}{\Delta z} \right)^2}}. \quad (19)$$

Thus,

$$D \leq R \sqrt{\left(\frac{1}{2\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2 + \left(\frac{1}{\Delta z} \right)^2}. \quad (20)$$

After all, the stability condition, Eq. (14), can be rewritten as

$$\kappa_0 \Delta t \left\{ \frac{1}{2\Delta x \cdot z_b} + \frac{1}{z_0} \sqrt{\left(\frac{1}{2\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2 + \left(\frac{1}{\Delta z} \right)^2} \right\} \leq 1. \quad (21)$$

Considering the relationship $\kappa_0 = c_0 z_0$, this condition yields

$$\Delta t \leq \frac{1}{c_0 \left\{ \frac{1}{2\Delta x \cdot z_b/z_0} + \sqrt{\left(\frac{1}{2\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2 + \left(\frac{1}{\Delta z} \right)^2} \right\}}. \quad (22)$$

Similarly, the stability conditions for two- and one-dimensional sound fields are respectively obtained as

$$\Delta t \leq \frac{1}{c_0 \left\{ \frac{1}{2\Delta x \cdot z_b/z_0} + \sqrt{\left(\frac{1}{2\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2} \right\}}, \quad (23)$$

$$\Delta t \leq \frac{1}{c_0 \left(\frac{1}{2\Delta x \cdot z_b/z_0} + \frac{1}{2\Delta x} \right)}. \quad (24)$$

3. Numerical examples

The CFL conditions for three-, two-, and one-dimensional sound fields are respectively written as [9]

$$\Delta t \leq \frac{1}{c_0 \sqrt{\left(\frac{1}{\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2 + \left(\frac{1}{\Delta z} \right)^2}}, \quad (25)$$

$$\Delta t \leq \frac{1}{c_0 \sqrt{\left(\frac{1}{\Delta x} \right)^2 + \left(\frac{1}{\Delta y} \right)^2}}, \quad (26)$$

$$\Delta t \leq \frac{\Delta x}{c_0}. \quad (27)$$

The real surface impedance z_b is related with the absorption coefficient α as [10]

$$\frac{z_b}{z_0} = \frac{1 + \sqrt{1 - \alpha}}{1 - \sqrt{1 - \alpha}} \geq 1. \quad (28)$$

Considering that the CFL condition for a one-dimensional sound field is expressed by Eq. (27) and the relationship is expressed by Eq. (28), the stability condition of an impedance boundary placed in a one-dimensional sound field, which is expressed by Eq. (24), is unconditionally satisfied. For two- and three-dimensional cases, the maximum time intervals, which satisfy Eqs. (22), (23), (25), and (26), are respectively plotted in Figs. 2 and 3 where the spatial intervals Δh ($= \Delta x = \Delta y = \Delta z$) are set to 10 mm and the sound speed c_0 is set to 343.5 m/s. The maximum time intervals satisfying the stability conditions of the impedance boundary tend to be smaller as the absorption coefficient shifts higher, and eventually become lower than those of the CFL conditions.

As for a zero absorption coefficient, which indicates a rigid boundary and not an absorptive boundary, although the previous discussion [8] gave lower maximum time intervals

Fig. 2 Maximum time intervals derived from the stability conditions for a two-dimensional sound field.

Fig. 3 Maximum time intervals derived from the stability conditions for a three-dimensional sound field.

than those of the CFL conditions in two- and three-dimensional cases, the maximum time intervals derived in this study

are higher than those of the CFL conditions in all dimensional cases.

4. Conclusion

Herein the stability conditions for impedance boundaries, which are often applied to FDTD calculations, are rebuilt under the oblique incidence of a plane wave. Similar to the previous discussion, the results show an impedance boundary in a one-dimensional sound field is unconditionally stable, but this is not the case in a two- or three-dimensional sound field. A two- or three-dimensional sound field can be unstable even when the CFL condition is satisfied if the absorption coefficient of the boundary is very high.

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