Approximating an isotropic sound field as a composition of plane waves

Tatsuhiko Tanaka* and Makoto Otani†

Graduate School of Engineering, Kyoto University, Kyoto daigaku-Katsura, Nishikyo-ku, Kyoto, 615-8540 Japan

(Received 17 March 2021, Accepted for publication 14 April 2021)

Keywords: Isotropy, Plane-wave composition, Plane-wave decomposition

1. Introduction

Many theories and techniques on room acoustics are based on the assumption of a completely diffuse sound field. Examples include reverberation theories [1] and measurements of absorption coefficients and transmission losses of acoustic materials. A diffuse sound field is generally composed of two physical features: the homogeneity of a sound field inside a room and the isotropy of the directional properties of a sound field observed at a given point in a room [2]. As for isotropy, in a recent study [3], a measure called the isotropy indicator was proposed to evaluate the degree of isotropy of a sound field. This measure employs plane-wave decomposition [4] based on spherical harmonics expansion.

To apply theories based on the diffuse sound field assumption to real measurements or numerical simulations, a diffuse sound field should be artificially modeled. However, it is impossible to implement a completely diffuse sound field physically in real environments. Even in numerical simulations, a completely diffuse sound field cannot be modeled because an infinite number of sound waves is required, which is beyond the ability of computers. Nevertheless, to apply theories based on the concept of a diffuse sound field to real and numerical environments, it is significantly meaningful to synthesize a nearly complete diffuse sound field. Focusing on isotropy, a completely isotropic sound field can be modeled by superimposing an infinite number of plane waves. However, neither we in this study nor computers are capable of dealing with an infinite number of plane waves.

Therefore, in real and numerical environments, a completely isotropic sound field should be approximated from a nearly complete isotropic sound field with a finite number of plane waves. In this study, we investigate the number of plane waves necessary to realize a nearly complete isotropic sound field. Specifically, the sufficient number of plane waves required to realize a nearly complete isotropic sound field is numerically discussed utilizing the plane-wave composition (PWC) and plane-wave decomposition (PWD) for the synthesis and analysis of a sound field, respectively. The degree of isotropy in a simulated sound field is evaluated using the isotropy indicator [3].

2. Synthesis and analysis of a sound field

Herein, we describe the procedures used to synthesize and analyze an incoming sound field to a certain position using

PWC and PWD, respectively. We introduce a procedure to model a nearly complete isotropic sound field into the process of PWC. Figure 1 illustrates the geometric relationship between the spherical and Cartesian coordinate systems in this study.

2.1. PWC synthesis of a sound field

Consider a sound field composed of S harmonic plane waves with time t [s], the wave number k [rad/m], the direction of arrival (DoA) of the s -th plane wave, which is denoted as (θ_s, ϕ_s) [rad] ($s = 1, 2, \dots, S$), and the amplitude at the origin of the coordinate system $a_s(k, t)$. Then, the sound pressure at (r, θ, ϕ, t) can be written as [4]

$$\begin{aligned} p(k, r, \theta, \phi, t) &= \sum_{n=0}^N \sum_{m=-n}^n 4\pi i^n j_n(kr) \\ &\quad \times \sum_{s=1}^S a_s(k, t) [Y_n^m(\theta_s, \phi_s)]^* Y_n^m(\theta, \phi) \quad (1) \\ &= \sum_{n=0}^N \sum_{m=-n}^n p_{nm}(k, r, t) Y_n^m(\theta, \phi), \end{aligned}$$

where

$$p_{nm}(k, r, t) = 4\pi i^n j_n(kr) \sum_{s=1}^S a_s(k, t) [Y_n^m(\theta_s, \phi_s)]^*, \quad (2)$$

$j_n(\cdot)$ is the n -th-order spherical Bessel function of the first kind. $Y_n^m(\cdot)$ is the spherical harmonics of order n and degree m . $[Y_n^m(\cdot)]^*$ is its complex conjugate. Here, N denotes the truncation order of the spherical harmonics and $p_{nm}(k, r, t)$ represents the coefficients of the spherical Fourier transform of $p(k, r, \theta, \phi, t)$. Figure 2 illustrates the DoAs (θ_s, ϕ_s) for $S = 49$, determined using a Fibonacci spiral [5].

2.2. Approximating an isotropic sound field

Galdo *et al.* [6] employed the plane-wave model to describe a diffuse sound field at an arbitrary point as the sum of plane waves with equal magnitudes, uniformly distributed directions, and uniformly distributed random phases. In their model, it is assumed that the plane waves arrive from random directions for each time window (i.e., the DoAs of the plane waves vary every several tens of ms). However, this assumption makes it difficult to physically realize a sound field. For example, a loudspeaker array with fixed loudspeaker positions cannot be realized by simply matching the DoAs of the plane waves and the directions of the loudspeakers. In addition, even if a sound field reproduction theory such as higher-order Ambisonics [7] can reproduce a plane wave arriving from an arbitrary direction that differs from the loudspeaker directions, the reproduction area, reproducible

*e-mail: tanaka.tatsuhiko.38u@st.kyoto-u.ac.jp

†e-mail: otani@archi.kyoto-u.ac.jp

[doi:10.1250/ast.42.286]

Fig. 1 Spherical and Cartesian coordinate systems.

Fig. 2 DoAs of 49 plane waves.

frequency, and spatial resolution are limited by the truncation order and the number of loudspeakers. Hence, we here approximate a completely isotropic sound field by a finite number of plane waves arriving from fixed directions distributed uniformly with random amplitudes and phases.

A sound pressure excited by a plane wave is denoted by its absolute amplitude $|a_s(k, t)|$ and phase $\psi_s(k, t)$ [rad] as

$$a_s(k, t) = |a_s(k, t)|e^{i\psi_s(k, t)}, \quad (3)$$

where t denotes a discrete time defined as

$$t = \tilde{n}\Delta t, \quad (4)$$

with $\tilde{n} = 0, 1, \dots, 1,000$ and $\Delta t = 5$ ms. The simulation generated two sets of $1,001 \times S$ uniform random numbers: one between $(0, 1/\sqrt{S})$ and the other between $(0, 2\pi)$ for $|a_s(k, t)|$ and $\psi_s(k, t)$, respectively. Figure 3 shows the real part of $p(k, r, \theta, \phi, t)$ derived from Eq. (1) on the x - y plane for $S = 49$, $N = 15$, $k = 10$, and $t = 0$.

To simulate a naturally observed temporal decay in a real environment, the plane wave amplitudes are attenuated as

$$a_{s,T_{60}}(k, t) = a_s(k, t)\sqrt{10^{-6t/T_{60}}}, \quad (5)$$

where T_{60} [s] denotes the reverberation time. Substituting $a_{s,T_{60}}$ for a_s in Eq. (1) yields a sound field composed of plane waves with randomized amplitudes and phases when the reverberation time is T_{60} .

Fig. 3 Real part of $p(k, r, \theta, \phi, t)$ plotted on the x - y plane for $S = 49$, $N = 15$, $k = 10$, and $t = 0$.

2.3. PWD analysis of a sound field

The above-mentioned formulation can synthesize a sound field with arbitrary values for a given number of plane waves S , truncation order N , and reverberation time T_{60} . PWD is applied to the synthesized sound field to analyze the degree of isotropy, which yields the plane-wave density. From the synthesized sound fields, both instantaneous- and cumulative-directional plane-wave densities can be derived.

2.3.1. Instantaneous-directional plane-wave density

The instantaneous-directional plane-wave density $a(\theta_l, \phi_l, t)$ is derived as

$$\begin{aligned} a(\theta_l, \phi_l, t) &= \sum_{n=0}^N \sum_{m=-n}^n \sum_{s=1}^S a_{s,T_{60}}(k, t) \\ &\quad \times [Y_n^m(\theta_s, \phi_s)]^* Y_n^m(\theta_l, \phi_l) \\ &= \sum_{n=0}^N \sum_{m=-n}^n a_{nm}(k, t) Y_n^m(\theta_l, \phi_l), \end{aligned} \quad (6)$$

$$a_{nm}(k, t) = \sum_{s=1}^S a_{s,T_{60}}(k, t) [Y_n^m(\theta_s, \phi_s)]^*, \quad (7)$$

where $a_{nm}(k, t)$ indicates the coefficients of the spherical Fourier transform of $a(\theta_l, \phi_l, t)$. Figure 4 depicts the normalized absolute of $a(\theta_l, \phi_l, t)$ obtained using Eq. (6) for $S = 49$, $N = 15$, $kr = 5$, and $t = 0$.

2.3.2. Cumulative-directional plane-wave density

The cumulative-directional plane-wave density $A(\theta_l, \phi_l, t)$ can be formulated as

$$\begin{aligned} A(\theta_l, \phi_l, t) &= \sum_{n=0}^N \sum_{m=-n}^n \sum_{s=1}^S \left(\int_t^{T_{60}} |a_{s,T_{60}}(k, \tau)| d\tau \right) \\ &\quad \times [Y_n^m(\theta_s, \phi_s)]^* Y_n^m(\theta_l, \phi_l) \\ &= \sum_{n=0}^N \sum_{m=-n}^n A_{nm}(k, t) Y_n^m(\theta_l, \phi_l), \end{aligned} \quad (8)$$

$$A_{nm}(k, t) = \sum_{s=1}^S \left(\int_t^{T_{60}} |a_{s,T_{60}}(k, \tau)| d\tau \right) [Y_n^m(\theta_s, \phi_s)]^*, \quad (9)$$

where $A_{nm}(k, t)$ indicates the coefficients of the spherical Fourier transform of $A(\theta_l, \phi_l, t)$.

Fig. 4 Normalized $|a(\theta_l, \phi_l, t)|$ for $N = 15$, $kr = 5$, and $t = 0$. White crosses indicate DoAs of the 49 plane waves.

We compute the instantaneous-directional plane-wave density to visualize the DoAs of the plane waves, as demonstrated in Fig. 4, but do not evaluate its isotropy. This is because the evaluation of the isotropy of the instantaneous-directional plane-wave density is equivalent to that of the random numbers at the interval of $(0, 1/\sqrt{S})$. Consequently, the degree of isotropy has a random value regardless of the number of plane waves S , which is beyond the scope of this study. In contrast, as the temporal integral interval increases, the temporal integration of the plane-wave density (i.e., a summation of the random numbers) for each direction should approach a similar value. Thus, we employ the cumulative-directional plane-wave density to evaluate the isotropy of a sound field because it should be uniform for a large number of plane waves.

2.4. Evaluation of isotropy

To evaluate the degree of isotropy of the synthesized sound field, we employ the isotropy indicator [3], which is denoted as

$$\iota(k, t) = \frac{|A_{00}(k, t)|}{\sum_{n=0}^{\infty} \sum_{m=-n}^n |A_{nm}(k, t)|}, \quad (10)$$

where $A_{nm}(k, t)$ can be computed from Eq. (9). Here, A_{00} is the omnidirectional component of the cumulative-directional plane-wave density. Equation (10) is used to calculate the ratio of the magnitude of the omnidirectional component to that of all the components for estimating the degree of isotropy. By replacing the infinite summation with the finite summation for a truncation order N , Eq. (10) can be rewritten as

$$\iota(k, t) = \frac{|A_{00}(k, t)|}{\sum_{n=0}^N \sum_{m=-n}^n |A_{nm}(k, t)|}. \quad (11)$$

As an example, Fig. 5 exhibits the resulting values of the isotropy indicator obtained from Eq. (11) for $S = 49$, $N = 5$, and $T_{60} = 5$.

3. Evaluation of isotropy for synthesized sound field

To clarify the relationship between the number of plane waves employed to model a sound field and the degree of

Fig. 5 Isotropy indicator for $S = 49$, $N = 5$, and $T_{60} = 5$.

isotropy, we performed numerical simulations using various values of S and N . For each value of the truncation order of N between 1 and 10, the sound fields were repeatedly generated, and the value of the isotropy indicator was calculated as the number of plane waves S increased until it satisfied the following two conditions:

$$\iota_S \geq 0.95, \quad (12)$$

and

$$\frac{|\iota_S - \iota_{S-1}| + |\iota_{S-1} - \iota_{S-2}| + |\iota_{S-2} - \iota_{S-3}|}{3} \leq 10^{-2}, \quad (13)$$

where ι_S is the isotropy indicator of a sound field composed of S plane waves for $t = 0$. In Eq. (12), ι_S is guaranteed to be sufficiently large, while in Eq. (13), the cases in which ι_S is accidentally large are excluded.

4. Results and discussion

Figure 6 shows the evolution of the isotropy indicator of synthesized sound fields as the number of plane waves increases. Table 1 depicts the minimum number of plane waves S satisfying Eqs. (12) and (13) for $N = 1, 2, \dots, 10$.

Fig. 6 Relationship between the isotropy indicator and the number of plane waves S for truncation order $N = 1, 2, \dots, 10$.

Table 1 Minimum number of plane waves S satisfying Eqs. (12) and (13) for truncation orders $N = 1, 2, \dots, 10$.

N	1	2	3	4	5	6	7	8	9	10
S	25	26	66	157	362	653	1,006	2,462	3,638	4,336

These results suggest a minimum number of plane waves required for realizing a nearly complete isotropic sound field for each truncation order. Additionally, a larger number of plane waves is required to achieve a larger value of the isotropic indicator. The sufficient number of plane waves increases with the truncation order (i.e., spatial resolution).

Because an arbitrary directional plane-wave density at a given point of a sound field, for truncation order N , is expressed with $(N+1)^2$ basis functions, the density can be synthesized with $(N+1)^2$ plane waves if their DoAs are equally distributed. Thus, we hypothesize that a sound field synthesized with $(N+1)^2$ equally distributed plane waves yields a similar value of the isotropy indicator $\iota_{(N+1)^2}$ regardless of the truncation order N . To test this hypothesis, normalized $\sum_{n=0}^N \sum_{m=-n}^n |A_{nm}(k, t)|$ of sound fields composed of $(N+1)^2$ plane waves for $N = 1, 2, \dots, 10$ were computed (Fig. 7). The isotropy of the sound fields is almost the same between ι of 0.7 and 0.8. However, the ratio of the magnitude of the omnidirectional component A_{00} tends to decrease as the truncation order increases. There are two possible reasons for this decrease: the incomplete modeling of the near-isotropic sound field or the isotropy indicator underestimating the degree of isotropy for higher truncation orders. Future work should focus on a more appropriate estimator of isotropy and/or improving the near-isotropic sound field model.

5. Summary

In this study, we introduced synthesis and analysis methods to approximate a completely isotropic sound field using PWC with random amplitudes and phases. The isotropy indicator evaluates the degree of isotropy of sound fields with various truncation orders and numbers of plane waves. The results reveal that a sufficient number of plane waves is required to create a nearly complete artificial isotropic sound field. The sufficient number of plane waves increases with the spatial resolution (i.e., truncation order).

Fig. 7 Normalized $\sum_{n=0}^N \sum_{m=-n}^n |A_{nm}(k, t)|$ for sound fields composed of $(N+1)^2$ plane waves where $k = 10$, $t = 0$, and $N = 1, 2, \dots, 10$. In each bar, $\sum_{m=-n}^n |A_{nm}(k, t)|$ for $n = 0, 1, \dots, N$ are shown from the bottom to the top.

Acknowledgement

This work was partly supported by Grants-in-Aid from MEXT, Japan (JP19H04153 and JP19H04145). We greatly appreciate the editor, the anonymous reviewer, and the proofreaders for their valuable comments.

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