Effects of time window on the estimation of coherence function for the system with delay

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The effects of the shape of time window on the estimation of the coherence function for the system having group delay are investigated. It is well known that the time-windowing determines the spectral resolution and the sidelobes, and that the suitable time window should be selected from that point of view. Time-windowing has another effect on the estimation when the system has group delay. Since the cross power spectrum obtained from DFT's corresponds to the circular convolution of the input and output sequences in time domain, the windowed input and output include the uncorrelated components in the initial and final parts of the windows due to group delay. Consequently, the estimation of the coherence function is biased toward zero. The estimation is improved by using the Hanning window as far as the delay is sufficiently shorter than the window duration, but the rectangular window becomes superior to the Hanning window when the delay is longer than about 0.18 times of the window duration. Theoretical and experimental results indicate the effects of time window on the estimation of the coherence function.

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1. INTRODUCTION

FFT algorithm and special-purpose hardware have made it practical to compute the coherence function from the auto and cross power spectra. The coherence function is very useful for acoustic measurements, as is applied to the following fields: noise source identification, measurement of signal-to-noise power ratio, error analysis for transfer function measurement and a directive spectrum analyzer.¹⁻³⁾ Since acoustic signals generally contain frequency components over the wide bandwidth, the sampling rate in AD conversion should be set high for the digital signal processing. Thus the maximum duration of time window in DFT, which depends on the capacity of a processor, is often too short to accurately estimate the coherence, as in the case of the acoustic system with long delay or reverberation.⁴⁾ For the accurate estimation of the coherence function and the transfer function of the

system with significant delay, input or output sequence should be shifted by the delay prior to computation. Acoustic systems, however, often contain multiple transmission paths with different propagation time, and in such cases the estimation problem due to delay cannot be solved exclusively by the shift of time window.⁵⁾ If the relation between delay and the bias of the coherence estimate for a single path is obtained, it can suggest the bias for multiple paths. This paper discusses the effects of time window on the estimation of coherence between input and output signals of the system with delay, and presents the relations between delay and the biased coherence estimate.

Various window functions for computing the DFT of time series signals are proposed in order to diminish the information loss due to cutting a part of waveform. Since the shape and length of time window determine the spectral resolution and sidelobes, the proper time window should be selected

from that point of view.⁶⁾ Time-windowing has another effect on the estimation of transfer characteristics when the system has propagation delay.

In this paper the definition of the coherence function and the estimation method are described in the first part. The estimation error of the coherence function due to delay is then pointed out from the results of acoustic experiments. The effects of window function on the estimation error of coherence between input and output signals of the system with delay are theoretically presented. For the rectangular and $\cos^n$ windows, the relations between the ratio of delay to window duration and the coherence estimate are given from computer simulations.

2. DEFINITION AND INTERPRETATION OF COHERENCE FUNCTION

Considering the system shown in Fig. 1, the output power spectrum $Y(f)$ is expressed as

$$Y(f) = X(f)H(f) + N(f) \quad (1)$$

where $X(f)$, $H(f)$ and $N(f)$ are the input spectrum, the transfer function and the additive noise spectrum, respectively. In digital signal processing, Fourier transforms in the above equation change to Discrete Fourier transforms (DFT's) and Eq. (1) is discretely expressed as

$$Y_k = X_k H_k + N_k. \quad (2)$$

For simplicity, in the succeeding formulas, the dependence upon k which is required is omitted from the notation.

The estimation method of the coherence function γ^2 is described as follows. Both sides in Eq. (2) being multiplied by X^* , the complex conjugate of X , and the products being averaged, the additive noise terms is eliminated and the cross power spectrum is obtained as

$$W_{xy} = \overline{YX^*} = \overline{HXX^*} + \overline{NX^*} = HW_{xx} \quad (3)$$

where $\overline{\quad}$ indicates averaging operation and W_{xx} expresses the input power spectrum. From Eq. (3), H is given by

$$H = \frac{W_{xy}}{W_{xx}}. \quad (4)$$

The coherence function γ^2 between input and output signals is defined by

$$\gamma^2 = \frac{|W_{xy}|^2}{W_{xx}W_{yy}}. \quad (5)$$

Fig. 1 Block diagram of the system with additive noise.

Fig. 2 Flowchart for estimating coherence and transfer functions using DFT.

From Eqs. (2), (3) and (5), γ^2 can be rewritten as

$$\gamma^2 = \frac{|H|^2 W_{xx}}{|H|^2 W_{xx} + W_{nn}}. \quad (6)$$

The numerator in Eq. (6) is the output power spectrum due to the input $x(t)$. The denominator is the total power spectrum of output which contains the response to the input and additive noise $n(t)$. Therefore, γ^2 is a value lying between 0 and 1.0. If the system is perfectly linear and the output is only due to the input, γ^2 is 1.0.

The coherence function and transfer function are practically computed as shown in Fig. 2, where time windows of finite length are used in implementing the DFT's. If the acoustic system has propagation delay, the coherence estimate is biased toward zero because of the finite length window.

3. ESTIMATION OF COHERENCE FUNCTION FOR THE SYSTEM WITH DELAY

As the sequence of infinite length cannot be processed in the digital signal processing, the coherence function and the transfer function are computed from DFT's using the finite length windows of T_w seconds as shown in Fig. 3. If the system has group delay of d seconds and if input $x(t)$ and output $y(t)$ are cut out by the same time window, the windowed input in the final d seconds duration and the output in the initial d seconds duration are uncorrelated with the windowed output and input, respectively.

Figures 4, 5, and 6 show the experimental results of the acoustic transfer characteristics between the driving voltage of a loudspeaker and the sound pressure at the observing point. The sampling frequency is 3 kHz and the rectangular or Hanning window of 128 samples (43 ms) is used. In the case of no significant delay γ^2 becomes nearly equal to 1.0 for both windows as shown in Fig. 4. The co-

Fig. 3 Time-windowing for the estimation of transfer characteristics of the system with group delay.

Fig. 4 Experimental transfer characteristics of an acoustic system without significant delay (time window: 43 ms).

herence function γ^2 is slightly higher for the Hanning window than for the rectangular window over the frequency. Figures 5 and 6 depict the results for delays of 6.7 ms and 13.5 ms, respectively. In Fig. 5 γ^2 is 0.6–0.8 for both windows, but γ^2 is smaller at the upper and lower ends of frequency range for the rectangular window than for the Hanning window. In Fig. 6 γ^2 is smaller for the Hanning window than for the rectangular window. From these experimental results it is found that the bias of coherence estimates depends upon window functions.

The transfer characteristics in Fig. 7 are obtained by using the shifted time window as shown in Fig. 3, where the acoustic system is the same as in Fig. 5 and shifting length is 6.0 ms (18 samples). Because of the shifting, γ^2 in Fig. 7 increases to about 1.0 and the validity of the estimated transfer function increases. On the other hand, the compensation corresponding to the shifting length is necessary for obtaining true $\angle H_{xy}$. Generally, in the estimation

Fig. 5 Experimental transfer characteristics of an acoustic system with delay of 6.7 ms (time window: 43 ms).

Fig. 6 Experimental transfer characteristics of an acoustic system with delay of 13.5 ms (time window: 43 ms).

Fig. 7 Experimental transfer characteristics of the acoustic system with delay of 6.7 ms obtained by shifting the input sequence by 6.0 ms (time window: 43 ms).

of the transfer characteristics of the system with delay, the input sequence should be shifted by the propagation delay. In the acoustic system, however, there frequently exist the multiple propagation paths whose propagation delays are different. Therefore, the bias of the coherence estimate due to delay still exists even if the shifting is carried out. In the next section the relationships between delay and the coherence estimate are investigated. The bias of coherence estimates for multiple paths can be suggested in terms of the relationships for a single path.

4. EFFECTS OF TIME WINDOW

For the case where the system has flat frequency response and group delay of d seconds, the relation between input $x(t)$ and output $y(t)$ is schematically represented in Fig. 8. In order to estimate the transfer characteristics, $x(t)$ and $y(t)$ are cut out by the same time window $w(t)$ of T_w seconds duration as shown in Fig. 3. Then the auto and the cross power spectra are computed as shown in Fig. 2.

Fig. 8 Schematic representation of time-windowing for the system with group delay.

Fig. 9 Relation between input and output spectra whose uncorrelated parts due to the circular convolution are considered as additive noise.

$X(f)$, $Y(f)$: observed spectrum, $X'(f)$, $Y'(f)$: spectrum without additive noise, $M(f)$, $N(f)$: Spectrum of the uncorrelated part corresponding to additive noise.

The cross power spectrum is obtained by averaging the complex conjugate products of input and output DFT's. The product corresponds to the circular convolution of input and output sequences in time domain. Therefore, the windowed output of initial d seconds and the input of final d seconds are uncorrelated to the windowed input and output, respectively. When the uncorrelated components are regarded as the additive noise, the relation between input and output is depicted in Fig. 9. In this figure the observed input spectrum $X(f)$ is decomposed into the spectrum $X'(f)$ without additive noise and the additive noise spectrum $M(f)$ as

$$X(f) = X'(f) + M(f). \quad (7)$$

The same decomposition is made for the output spectrum $Y(f)$ as

$$Y(f) = Y'(f) + N(f). \quad (8)$$

DFT's of observed input and output are computed and the complex conjugate products are averaged. Thus, the auto and cross power spectra are obtained as

$$W_{xx} = W_{xx'} + W_{mm}, \quad (9)$$

$$W_{yy} = W_{yy'} + W_{nn}, \quad (10)$$

$$\begin{aligned} W_{xy} &= \overline{(X' + M)(Y' + N)^*} \\ &= \overline{X'Y'^*} + \overline{X'N^*} + \overline{MY'^*} + \overline{MN^*} \\ &= W_{xy'}. \end{aligned} \quad (11)$$

Then γ^2 is given by

$$\begin{aligned} \gamma^2 &= \frac{|W_{xy}|^2}{W_{xx}W_{yy}} \\ &= \frac{(W_{xx} - W_{mm})(W_{yy} - W_{nn})}{W_{xx}W_{yy}} \end{aligned}$$

$$\begin{aligned} & \cdot \frac{|W_{xy}|^2}{(W_{xx} - W_{mm})(W_{yy} - W_{nn})} \\ & = \gamma_c^2 \cdot \gamma_s^2 \end{aligned} \quad (12)$$

where

$$\begin{aligned} \gamma_c^2 &= \left(1 - \frac{W_{mm}}{W_{xx}}\right) \left(1 - \frac{W_{nn}}{W_{yy}}\right), \\ \gamma_s^2 &= \frac{|W_{xy}|^2}{(W_{xx} - W_{mm})(W_{yy} - W_{nn})}. \end{aligned}$$

The ratio R of the output additive noise power W_{nn} due to delay to the total windowed output power W_{yy} is given by

$$R = \frac{\int_0^d w^2(t) dt}{\int_0^{T_w} w^2(t) dt}. \quad (13)$$

The rectangular window $w_r(t)$ and the Hanning window $w_h(t)$ are defined by

$$w_r(t) = 1.0 \quad ; \quad 0 \leq t < T_w \quad (14)$$

$$w_h(t) = \frac{1}{2} \left(1 - \cos 2\pi \frac{t}{T_w}\right); \quad 0 \leq t < T_w. \quad (15)$$

For the rectangular and Hanning windows, R_r and R_h are obtained as

$$R_r = \frac{d}{T_w}, \quad (16)$$

$$R_h = \frac{d}{T_w} - \frac{2}{3\pi} \left(\sin 2\pi \frac{d}{T_w} - \frac{1}{8} \sin 4\pi \frac{d}{T_w} \right). \quad (17)$$

The ratio R is the same in the output and input when the system has group delay. Consequently, the coherence function, γ_c^2 , becomes

$$\gamma_c^2 = (1 - R)^2 \quad (18)$$

because of the circular convolution of input and output sequences.

Theoretical curves shown in Fig. 10 represent the relations between delay and γ_c^2 in Eq. (18). Within the range of small d/T_w , γ_c^2 is higher for the Hanning window than for the rectangular window. In the case of $d/T_w > 0.5$, γ_c^2 is higher for the rectangular window than for the Hanning window. The experimental curves in Fig. 11 are obtained from the white noise simulation. In comparison of the theoretical and experimental results, both agree well with each other for the rectangular window. For the Hanning window, however, the experimental γ^2 is lower than the theoretical γ_c^2 owing to some other

Fig. 10 Theoretical γ_c^2 versus d/T_w . The bias of γ_c^2 is only due to the circular convolution.

Fig. 11 Experimental γ^2 and theoretical γ_c^2 versus d/T_w .

reasons besides the uncorrelated noise, W_{mm} and W_{nn} , in Eq. (12).

In Fig. 8 the hatched parts of input and output sequences are completely coherent, weighted by the different part of the Hanning window. Thus, the different window operation in DFT produces incoherent frequency components in the estimated spectra. Consequently, when the weighting function is used, the factor $|W_{xy}|^2/(W_{xx}W_{yy})$ in Eq. (12) becomes

$$\frac{|W_{xy}|^2}{W_{xx}W_{yy}} = \gamma_s^2 < 1.0. \quad (19)$$

The coherence estimate γ_s^2 in Fig. 12 is obtained

Fig. 12 Experimental γ_s^2 versus d/T_w for Hanning window. The bias of γ_s^2 is only due to the different operation to the coherent parts of input and output.

from the white noise simulation where the signal sequences are only given to the hatched portions of the Hanning window shown in Fig. 8 and zero sequences are appended to the remaining portions of the window. This experimental result is in accord with Eq. (19) and γ_s^2 is less than unity in the case of group delay. The result indicates that estimation error also depends upon the difference in the window spectrum between input and output. For the Hanning window the coherence estimate, γ^2 , becomes

$$\gamma^2 = \gamma_c^2 \times \gamma_s^2 \quad (20)$$

from Eqs. (12) and (19) when the system has group delay. The product of theoretical γ_c^2 and experimental γ_s^2 is given together with the direct estimate γ^2 in Fig. 13. As both results are in good agreement, it is verified that the biased estimate γ^2 is given by Eq. (20) for the Hanning window.

Figure 14 presents the relations between the experimental γ^2 and the ratio of delay to window duration for the $\cos^n$ windows. The $\cos^3$ window best heightens γ^2 for small d/T_w in Fig. 14. The Hanning window lowers γ^2 more than the rectangular window does when $d/T_w > 0.18$, but improves the estimation for small d/T_w . That is, the bias of coherence estimates decreases for small d/T_w and increases rapidly for large d/T_w , as the used window function smoothly lightens the weight at boundaries of the time window. The results in Figs. 4, 5 and 6 can be explained in terms of the relations between

Fig. 13 $\gamma_c^2 \times \gamma_s^2$ in Eq. (20) and direct estimate γ^2 versus d/T_w for Hanning window.

Fig. 14 Experimental γ^2 versus d/T_w for $\cos^n$ windows.

γ^2 and d/T_w for rectangular and Hanning windows in Fig. 14.

From these theoretical and experimental results, the window function should be selected from the point of view above discussed if the system has delay.

5. CONCLUSIONS

The effects of the shape of time window on the bias of coherence estimates between the input and output sequences for the system with delay were studied. The relationships between the coherence estimate and delay were presented theoretically and

experimentally. In coherence estimation as well as power spectrum estimation the Hanning window improves the estimation when $d/T_w < 0.18$. But the Hanning window lowers the coherence estimate more than the rectangular window does when $d/T_w > 0.18$, and of course, in such a range the coherence estimate is considerably biased even if any shape of time window is utilized.

The experimental and theoretical results indicate that the bias of coherence estimates generally decreases for short delay as the weight of the used window function is smoothly lightened at the boundaries. Therefore, the weighted time window is useful for the accurate estimation of the coherence function as far as the window duration is sufficiently longer than the propagation delay.

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