The mechanism of sound production in organ pipes and cavity resonators

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A new unified method of treating the sounding mechanism of organ pipes and cavity resonators replaces the control volume method which until recently has been the most widely accepted theory. It is shown that shear layers contain a dividing surface whose motions control the production of sound in the resonant cavity. Both jet-drive and force drive contributions are found to be involved in the sounding mechanism. The present approach shows the drive system to be inherently linear over a range of shear layer widths, so that sinusoidal oscillation is readily achievable. For voicing situations calling for spectra rich in harmonics, nonlinear drive is attained by resorting to a smaller ratio of jet width to lip cutup which, at high enough Reynolds number, permits the growth of eddies which result in impulsive action at the lip.

Keywords: Sound production, Cavity resonator, Organ pipe, Separated shear layer, Jet instability, Flow tone

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1. INTRODUCTION

1.1 History of the Development of Organ Pipe Physics

The physics of organ pipes, a problem of classical mechanics left unsolved by the sudden onset of modern physics at the beginning of the twentieth century, had received some consideration by two scientific giants of the nineteenth century. H. Helmholtz, in his “On the Sensations of Tone,” expressed the view that pipe drive is due to fluctuating volume flow from the jet into the pipe. Rayleigh, in his “Theory of Sound” disagreed, believing that the oscillation is sustained, not by a volume flow but by a force or dipole source caused by the motions of the jet in the mouth of the pipe. Rayleigh also pointed to the fact that jet motion is inherently unstable and discovered how to calculate appropriate propagation constants for jets and free shear layers of simple structure, an important component to the understanding of jet/edge phenomena. He did not attempt a complete theory of the organ pipe problem however, and no further progress was made by his contemporaries.

Then in 1937 Brown published results of an investigation of edgetone oscillation including photographic visualizations of a ribbon jet playing on a sharp edge, with eddies being cast off at the period of the emitted tone. He showed that the frequency of an edgetone oscillator is proportional to the speed of the jet. This led to speculation that organ pipes are driven by the same mechanism as edgetones, except that for organ pipes, the frequency is held nearly constant because of the coupling to the pipe resonator. This view was generally accepted for a number of years but did not lead to a method of calculating the jet blowing pressure and sound amplitude for organ pipes.

The first progress in developing a quantitative theory of pipe mechanism came in the form of three papers delivered at the 5th International Congress on Acoustics in 1965 by scientists from Germany: Bechert, Cremer and Ising. These were later followed up with a seminal paper by Cremer and Ising in 1968 and a related doctoral dissertation by Ising in 1969. The “German” method was an adaptation of Helmholtz’s concept. Pipe drive was
assumed to take place by flow of jet fluid, $Q_j$, into the parallel resonant mechanical circuit consisting of positive mouth reactance and negative pipe reactance branches as shown in Fig. 1. The pipe is then assumed to act as a linear amplifier, producing an acoustic flow $Q_a = -(Z_p/Z_s)Q_j$, where $Z_p$ is the acoustic impedance of the cavity looking into the mouth and where $Z_s$ is the combined series impedance of the resonant system. The acoustic flow in turn, acting on the jet fluid in the plane of the mouth, produces more jet-drive flow $Q_j$, completing the feedback loop. Using root locus methods they were able to account at least qualitatively for many of the observed properties of organ pipe oscillation.

To get improved agreement with experiment they found it necessary to assume that entrained jet fluid added to the drive, resulting in an oscillation amplitude proportional to the square of the jet speed.

A divergent line of reasoning was followed by Coltman,8) in a paper which appeared also in 1968. Following Rayleigh, Coltman reasoned that the jet interacted with the pipe by generating a fluctuating force across the mouth associated with its own decay, a “force-drive” theory as opposed to “jet-drive.” This meant that the reactive elements of the pipe were driven in series rather than in parallel (see Fig. 2) and in addition implied that the amplitude of the oscillation should be proportional to the square of the jet speed without the need of entrained fluid, as postulated for the jet-drive approach. However in later measurements of the timing of the arriving jet pulse at the lip of the pipe, Coltman9) found that the phase relations seemed to support the jet-drive theory.

In 1973, Elder10) used a control volume approach to show that both jet-drive and force-drive were consistent and suggested that there should be a third, nonlinear, contribution to the oscillation due to intermodulation distortion caused by the nonlinearity of the momentum conservation process. This was followed in 1976 by a paper due to Fletcher11) which showed that Elder’s nonlinear term was unnecessary, leaving only the contributions due to force-drive and jet-drive. Fletcher and his associates have developed a comprehensive feedback theory12,13) which goes further than any to date in explaining the physics of organ pipes. Nevertheless there remain some areas where the model appears to be deficient, such as in the prediction of threshold blowing pressures, and the amplitude of underblown regimes.

In 1980 an important new approach was introduced by Yoshikawa and Saneyoshi,14) who suggested the possibility of direct feedback from the action of the jet as a vibrating diaphragm. This idea is the starting point for a new concept developed by Elder15) whose theory represents the latest effort in the field of organ pipe physics and which will be described in this paper.

1.2 Recent Studies of Cavity Resonators

A closely related phenomenon is the cavity resonator, or “Helmholtz” resonator problem. Wind blowing past a slot or cavity along the side of a moving vehicle may cause sound and vibration with frequency proportional to the speed. If a resonant volume or plate is present to couple with the oscillation, a strong fixed-frequency tone may be produced at certain speeds. This sometimes occurs in automobiles with a single window open, manifesting itself as a very low frequency (infrasonic) rumble or feeling of pressure sensation to the occupant of the automobile. The effect has been studied extensively in
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Fig. 3 Cavity resonator compared with organ pipe. a. Cavity mouth showing inflection point locus (dividing streamsurface) with displacement, \( f \). b. Organ pipe mouth showing jet emerging from flue of width \( f \). Dividing surfaces are along the inflection point loci of the shear layers on either side of jet.

recent years on account of its application to military vehicles.\(^{16}\) Besides its being an undesirable sound, or "noise," the cavity resonator problem differs from organ pipe dynamics in the fact that only a single free shear layer, or "halfjet," is present at the mouth of the cavity in contrast to the organ pipe jet, which has two shear layers (see Fig. 3).

1.3 A Unified Approach to Organ Pipes and Cavity Resonators

We will consider first the case of the cavity resonator, showing how the oscillation of the single free shear layer creates forces and drive flow in the mouth of the cavity. Then the same approach will be applied to the organ pipe jet, where two shear layers move in coupled motion in such as manner as to cooperate in the creation of acoustic flow in and out of the pipe.

2. THE SHEAR LAYER AS A VIBRATING MEMBRANE

2.1 Shear Layer Instability

The simplest case of shear layer instability is that of the layer with step-change in velocity profile—the Helmholtz "vortex sheet." Imagine a uniform mass of fluid moving over an inviscid stationary fluid. Rayleigh\(^{17}\) showed that the slightest disturbance will cause the interface between the layers to break into a wave which will begin to grow exponentially. If the disturbance has an angular frequency, \( \omega \), the surface of discontinuity will have a transverse displacement:

\[ \xi(x, t) = Ae^{i\omega t} e^{-(x-ct)/2C} \]  

where \( A \) is the amplitude of the disturbance at \( x \), \( t=0 \), and where \( U_c \), the wave speed, is half the speed of the upper fluid. Such waves, however, are an expression of vorticity and will eventually roll up into a row of discrete eddies if there is space enough, as demonstrated by Rosenhead\(^{18}\) who carried out Rayleigh's solution to the second order. This is shown in Fig. 4.

For the equivalent case of a complete jet consisting of a layer of fluid of width, \( \delta \), moving at speed

Fig. 5 Sinuous and varicose jet waves. a. Sinuous. b. Varicose.
$U$, surrounded above and below by masses of stationary fluid, instability also is present. However for this case, the two vortex sheets making up the edges of the jet will tend to move in synchronism with each other, forming either a "sinuous" pair of waves or a "varicose" pair, in the words of Rayleigh,\(^{19}\) as shown in Fig. 5. In this case the wave speed depends on the wavelength. Rayleigh’s solution for the sinuous case leads to a wave speed:

$$U_c/U = \pi \delta / \lambda \quad (2a)$$

for thin jets and

$$U_c/U = 1/2 \quad (2b)$$

in the limit for very thick jets, which is of course to be expected since for a very thick jet we should be aware of only one shear layer at a time. The jet waves also begin with exponential growth and tend to roll up into a row of eddies, first to one side and then the other, as may be seen in Brown’s photographs\(^{13}\) of edgetone jets.

2.2 The Inflection Point Streamline: A Flow Divider

Inviscid vortex sheets are a useful idealization, but in the real world the edge of a jet or separated shear layer always consists of a finite boundary layer of thickness, $\delta_b$. In Fig. 6 is a comparison of the velocity profiles for the ideal halfjet and “full” jet and their real-world viscous counterparts. At the upstream, or trailing, edge of a slot the existing wall boundary layer at that point must be continuously converted to a separated shear layer. The streamline that forms at the edge itself must therefore begin as a stagnation point in order to match the zero-velocity boundary layer condition at the wall. As the flow along this streamline accelerates up to the local stream speed, $U$, the streamline will become a dividing line for flow in the outside fluid, which is moving, and flow of the fluid inside the cavity, which initially is not. As the flow progresses along the mouth of a cavity or pipe, viscous (or turbulent) shear will produce a profile of positive curvature below this point due to acceleration of fluid while above the point the curvature will be negative due to deceleration of fluid. This means that the streamline that begins at the edge is actually the locus of points corresponding to the inflection point in the velocity profile. The inflection point locus is therefore the dividing streamsurface which separates outside from inside fluid as waves form along the shear layer.

2.3 Stability Wave Functions for Realistic Shear Layers

For realistic halfjets and jets as shown in Fig. 6b and Fig. 6d, the wave speed and growth rate must be computed by numerical integration based on some assumed velocity profile shape. One must also consider spatially growing waves rather than temporally growing waves assumed by Rayleigh. Some useful tables have been prepared by Michalke,\(^{20}\) based on a hyperbolic tangent curve shape, which give complex wave propagation constants as a function of the nondimensional frequency. The parameters evaluated by Michalke for spatially growing disturbances along shear layers of infinite extent are:

- $C_r =$ nondimensional phase velocity
- $\beta =$ nondimensional frequency
- $\alpha_r =$ nondimensional real propagation constant
- $\alpha_i =$ nondimensional imaginary propagation constant.

To make use of these for obtaining practical estimates for a real shear layer we need to know the angular frequency $\omega$, the free stream velocity, $U_w$, and the slope of the velocity profile at the inflection point, $(\partial U/\partial z)_o$. The following auxiliary formulas

Fig. 6  Velocity profiles: a. Ideal halfjet. b. Real shear layer. c. Ideal (top-hat) jet. d. Real jet.
are useful:

\[ U_c = \frac{C}{4}, U_{\infty} = \text{wave convection speed} \]
\[ d = \frac{U_{\infty}}{(2U_{\infty} \partial \alpha \partial z)_{\infty}} = \text{length parameter} \]
\[ \beta = \omega d / \left( \frac{2\partial U}{\partial z}_{\infty} \right) = \text{non-dimensional frequency} \]
\[ k = 2\pi / \lambda = \alpha / d = \text{real propagation constant} \]
\[ \alpha = - \alpha / d = \text{imaginary propagation constant}. \]

The wave function for a separated shear layer of infinite extent will then take the form:

\[ \xi(x, t) = A e^{\alpha x} e^{i(\omega t - kx)} \] (3)

3. WAVE FUNCTIONS FOR SHEAR WAVES FORMED IN THE MOUTH OF A CAVITY

3.1 Kutta Conditions at Leading and Trailing Edges

Except near boundaries fluids behave as though they were nearly inviscid, and over distances small compared to the acoustic wavelength a free shear layer is approximately incompressible. Therefore for some purposes we may deal with the shear layer by potential flow methods. This was the basis of Rayleigh's analysis of the instability problem. However, near either the upstream or downstream edges, the potential flow solution leads to infinite velocity. In a real fluid discontinuities in the velocity cannot occur, nature always finding a way to avoid the catastrophe. For the case of cavity resonators and organ pipe jets there seem to be two ways in which this is achieved: either by (1) the casting off of an eddy, or (2) by forming a stability wave/acoustic wave pair at the edge.

Both eddies and stability waves are, in fact, manifestations of the vorticity that is resident in the shear layer. Eddies may thought of as the “particle” nature of vorticity while stability waves exhibit its “wave” nature. In order for eddies to show up it is necessary that there be enough room in the mouth region for them to spin. The initial size of the eddy is of the order of the thickness of the upstream boundary layer. Therefore, when the boundary layer thickness, \( \delta_b \), is small compared to the mouth dimension, \( H \), eddies are an option for assuring velocity control at the upstream edge. More typically, \( \delta_b \) is of the order of \( H \) and the vorticity shows up as a stability wave.

3.2 Cavity Resonator Edge Conditions

In order to insure that the velocity vanish at \( x = 0 \) (i.e. at the upstream edge), forces in the shear layer create an acoustic wave across the mouth, either from the resonance of the cavity or from a dipole source at the downstream edge. The sum of the acoustic particle velocity and the stability wave then add up to zero at \( x = 0 \). For the case of the cavity resonator, Elder\(^{21,22}\) has shown that the form of the total wave disturbance near \( x = 0 \) is:

\[ \xi(x, t) = - (u_{\delta} / \omega) e^{i(\omega t - kx)} \] (4)

which is the sum of the acoustic particle displacement and the negative of the stability wave. Here “\( k \)” is twice the value which would be obtained from the formulas in Section 2.3 and the exponential gain term is missing, while \( u_{\delta} \) is the acoustic particle velocity amplitude. As explained in Ref. 23), the doubling of the propagation constant occurs on account of the fact that the shear layer is being accelerated by the sound field, causing the wave speed to be halved relative to that in the “laboratory” system. Note the absence of an imaginary component to the propagation constant. Exponential gain disappears very quickly in the growth of the wave because of the stretching of the shear layer that is produced by its transverse motion, reducing the strength of the fluctuating vorticity sources. Stability waves become neutrally stable once their lateral displacement approaches the shear layer width.

At the downstream edge, eddy creation is more common, there being room beyond the edge for the eddy to develop. However, for the case of the laminar cavity resonator, it has been shown\(^{25}\) that flow reattachment may occur at \( x = H \), allowing the acoustic wave/stability wave solution to take place in the same manner as at \( x = 0 \). On the other hand, when the downstream boundary condition is met by casting off an eddy, the acoustic wave is not generated at this edge and the total wave function in the vicinity of \( x = H \) becomes the stability wave alone:

\[ \xi(x, t) = (u_{\delta} / \omega) e^{i(\omega t - kx)} \] (5)

3.3 Organ Pipe Edge Conditions

For the organ pipe jet, there are two upstream edges, one for each shear layer. The condition \( \xi = 0 \) cannot be met simultaneously by both shear layers at \( x = 0 \) unless an additional boundary condition is imposed, namely that \( \partial \xi / \partial x = 0 \) at \( x = 0 \). This is sometimes called a “full” Kutta condition. Creamer and Ising\(^{15}\) invoked this condition in their description of organ pipe jet motion, showing that it leads to a second pair of stability waves.
along the jet, moving in the upstream direction. For the top-hat profile considered by Cremer and Ising, this leads to a jet centerline wave of the form:

$$\xi(x, t) = -(u_{\infty}/a)e^{i\omega t}[1 - C_1e^{-i\omega t} + C_2e^{i\omega t}]$$  \hspace{1cm} (6)

where

$$k_1^* = k_1 + j\alpha_1$$
$$k_2^* = k_2 + j\alpha_2$$

and where \(C_1\) and \(C_2\) are given in terms of \(k_1, k_2, \alpha_1, \alpha_2\).

For jet with a more realistic profile, numerical results are available to estimate the complex propagation constant\(^{13}\) needed to describe the wave motion.

4. THE DIVIDING SURFACE AS A VIBRATING MEMBRANE

4.1 Relation between \(v\) and \(\xi\)

Since the streamsurface through the locus of inflection points divides the shear layer into inside and outside fluid,\(^{24}\) its wave motions are in some ways analogous to the motions of the reed in a reed wind instrument. The transverse velocity of the dividing streamsurface contains contributions from the acoustic wave across the mouth and from the component of the main flow velocity of the shear layer or jet as it is deflected either into or out of the mouth. To determine the velocity in relation to the wave displacement \(\xi\), we use the method of Rayleigh\(^{25}\) and Lamb\(^{26}\):

Let the surface in question be represented by the equation

$$S = z - \xi(x, t) = 0$$  \hspace{1cm} (7)

We then require that

$$DS/ Dt = \partial S/ \partial t + U\partial S/ \partial x + v\partial S/ \partial z = 0$$  \hspace{1cm} (8)

where \(U\) is the local stream speed and \(v\) is the "membrane" velocity, \(\partial z/ \partial t\). Carrying out the differentiation,

$$-\partial \xi/ \partial t - U\partial \xi/ \partial x + v = 0$$  \hspace{1cm} (9)

This may be written

$$v = \partial \xi/ \partial t + U\partial \xi/ \partial x$$  \hspace{1cm} (10)

By substituting the appropriate value of \(\xi\) we may obtain \(v\) as a function of \(x\) over the mouth.

4.2 The Flow Conservation Rule

If the cavity or pipe is to be driven by a parallel-resonant circuit representation, as shown in Fig. 1, we must require that the jet-drive flow, \(Q_j\), must be introduced by means of a "branch" circuit into the loop containing the reactive elements, so that the total acoustic discharge into the cavity, \(Q_p\), is given by:

$$Q_p = Q_m + Q_j$$  \hspace{1cm} (11)

To establish the legitimacy of this relation, which is a kind of flow-conservation rule, we compute \(Q_p\) by integrating the dividing surface velocity over the entire surface of the mouth:

$$Q_p = \int_{-b/2}^{b/2} \int_0^H v\, dx\, dy$$  \hspace{1cm} (12)

where two-dimensional symmetry is assumed and the integration is over the streamwise dimension \(H\) and lateral dimension \(b\) of the mouth.

To simplify the discussion, let us assume an ideal halfjet with velocity profile given by Fig. 6a. Substituting \(v\) from Eq. (10) and neglecting the small flow deficit due to the stagnation point at the edge, we obtain:

$$Q_p = -jbH\, u_{\infty} e^{i\omega t} + bU[\xi(H, t) - \xi(0, t)]$$  \hspace{1cm} (13)

But we have indicated that \(\xi(0, t) = 0\) so this may be written:

$$Q_p = -jbH u_{\infty} e^{i\omega t} + bU \xi(H, t)$$  \hspace{1cm} (14)

The first term on the right is the integral of the acoustic particle velocity, or the acoustic discharge through the mouth, \(Q_m\), while the second term is of the form of jet-drive, \(Q_j\), which has been used in previous papers, for the case in which \(\xi\) has a finite value at the downstream edge. We have shown, therefore, that the distinction between \(Q_p\) and \(Q_m\), and with it the parallel-resonant drive situation, occurs whenever \(\xi(H, t)\) does not vanish.

Jet-drive is present for organ pipe oscillation, but not necessarily for cavity resonators. Where it holds, we must assume that at least a portion of the acoustic response of the system, \((Q_m)_{par}\), will be via parallel-resonant drive:

$$(Q_m)_{par} = -\left(\frac{Z_v}{Z_b}\right)Q_j$$  \hspace{1cm} (15)

If there is assumed to be no fluid motion below the dividing surface, the effect of the motion of the shear layer is to cause an amount of fluid \(Q_s\) to enter the cavity each cycle, but not to remove any. There is therefore both a dc and an ac component of \(Q_s\).

For a real shear layer, there is some motion of the fluid below the dividing surface, so that fluid will be also ejected from the cavity each cycle, but in
general this will be of a smaller amount. The ejected fluid, \( Q_j \), does not produce additional acoustic gain (such as indicated by Eq. (15)), since it is applied outside the cavity.

4.3 Forces on the Dividing Surface

The lateral displacement of the vorticity concentrated in the portion of the shear layer between the upstream and downstream edges produces fluctuating forces on the surrounding fluid which result in a pressure difference \( \Delta p \) across the mouth in series with the resonant elements. This produces an additional acoustic flow in the cavity given by:

\[
(Q_M)_{tot} = \frac{\Delta p}{Z_s} \tag{16}
\]

so that the total contribution to acoustic discharge in the cavity may be written:

\[
Q_p = -\left( \frac{Z_f}{Z_s} \right) Q_j + \frac{\Delta p}{Z_s} + Q_j \tag{17}
\]

Combining the terms in \( Q_p \) and making use of fact that \( Z_s = Z_M + Z_p \), we may write this in the form:

\[
Q_p = \frac{\Delta p + (Z_s - Z_f)Q_j}{Z_s} = \frac{\Delta p + Z_f Q_j}{Z_s} \tag{18}
\]

This shows clearly that the acoustic oscillation consists in general of two parts: a jet-drive contribution that is proportional to \( Q_j \), and a force-drive contribution that is not.

To compute the force term from which \( \Delta p \) is found, we must take the material derivative of the dividing surface velocity which is proportional to the force on the fluid per unit volume fluid due to shear layer motion:

\[
f = \rho \frac{Dv}{Dt} = \rho \frac{D^2 \xi}{Dt^2} \tag{19}
\]

or

\[
\rho \frac{D^2 \xi}{Dt^2} = \rho \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} \rho U^2 \right) \frac{\partial \xi}{\partial x} + \rho U^2 \frac{\partial^2 \xi}{\partial x^2} + 2 \rho U \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t} \tag{20}
\]

Given a function \( \xi(x, y, z, t) \) we may determine the total force by integrating this expression over the \( x \) and \( y \) dimensions of the mouth and vertically, over the portion of the shear layer containing the sources of the fluctuating vorticity. The geometry of the cavity mouth is shown in Fig. 7. Formally we may represent the total force on the fluid in the region of the mouth as:

\[
F = \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho \frac{\partial \xi}{\partial t} dx dy dz + \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho \frac{\partial}{\partial x} \left( \frac{1}{2} \rho U^2 \right) \frac{\partial \xi}{\partial x} dx dy dz + \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho U^2 \frac{\partial^2 \xi}{\partial x^2} dx dy dz + \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} 2 \rho U \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t} dx dy dz \tag{21}
\]

Now if the fluid upstream and downstream of the cavity is assumed to be free of forces in the \( z \)-direction, we may set \( F = 0 \), giving:

\[
-\int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho \frac{\partial \xi}{\partial t} dx dy dz = \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho \frac{\partial}{\partial x} \left( \frac{1}{2} \rho U^2 \right) \frac{\partial \xi}{\partial x} dx dy dz + \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho U^2 \frac{\partial^2 \xi}{\partial x^2} dx dy dz + \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} 2 \rho U \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t} dx dy dz \tag{22}
\]

Noting that the integrands of all the terms on the right side of this equation involve at least one space derivative of \( \xi \), and therefore operate on the stability wave alone, it is clear that the term on the left must contain any reaction forces associated with the purely acoustic field. If we further separate the integrals on the left into two parts, the first containing no \( x \)-dependent terms and the latter a function of \( x \), we may then identify the former term as due to the acoustic pressure difference, \( Z_s Q_M \), that drives the standing wave in the cavity:

\[
Z_s Q_M S_M = -\int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho \frac{\partial \xi}{\partial t} dx dy dz + \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho \frac{\partial}{\partial x} \left( \frac{1}{2} \rho U^2 \right) \frac{\partial \xi}{\partial x} dx dy dz + \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} \rho U^2 \frac{\partial^2 \xi}{\partial x^2} dx dy dz + \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_{0}^{H} 2 \rho U \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial t} dx dy dz \tag{23}
\]
where \( S_M \) is the area of the mouth \((bH)\). This gives, for the “force-drive” acoustic pressure difference:

\[
\Delta p = Z\frac{Q_S}{bH} - \frac{1}{bH} \int_{-b/2}^{b/2} \int_0^{H} \text{terms in } x \, dx \, dy \, dz
\]

\[
+ \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_0^H \frac{\partial}{\partial x} \left( \frac{1}{2} \rho U^2 \right) \frac{\partial \xi}{\partial x} \, dx \, dy \, dz
\]

\[
+ \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_0^H \rho U \frac{\partial \xi}{\partial x} \, dx \, dy \, dz
\]

\[
+ \int_{-\infty}^{\infty} \int_{-b/2}^{b/2} \int_0^H 2\rho U \frac{\partial^2 \xi}{\partial x \partial t} \, dx \, dy \, dz
\]

(24)

For later reference, we shall identify the terms on the right in Eq. (24) by the labels A, B, C, and D respectively. It should be remarked that term-B appears in Eq. (24) only because we must take into account the fact that the dividing streamline begins (and sometimes ends) at a stagnation point, and so the fluid speed \( U \), which is constant over most of the path, must be considered a variable over a small “boundary layer” region where acceleration and deceleration take place. This is an important source of coupling to the surrounding fluid.

5. APPLICATIONS OF THE THEORY

5.1 Cavity Resonator in Laminar Boundary Layer

We will treat this case first because it represents the simplest application of the theory. In Fig. 8 is shown a sequence of dividing streamline configurations obtained by computer-assisted hotwire experiment \(^{(23)}\) on a cavity acted on by a laminar boundary layer. Such experiments have shown that when a cavity mounted in a flat wall under conditions of laminar flow is driven at resonance, reattachment occurs at the downstream edge. Consequently the “acoustic” boundary condition is applied at both upstream and downstream edges and the \( \xi \) function specified in formula (4) applies uniformly along the mouth.

Before we can substitute this function into Eq. (24), however, we must specify the \( z \)-dependence of \( \xi \). For Rayleigh’s ideal halfjet, \( \xi \) is proportional to \( e^{kz} \), where the plus sign applies below the inflection point and the minus sign above.\(^{(27)}\) As a first approximation, let us therefore take the integration over \( z \) to be equivalent to multiplying the integrand by a constant of order \( 2/k \). Now we may make use of the \( \xi \) function to integrate \( \Delta p \). The result is:
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\[ Z_0 Q_\omega = j[2 \rho U^2 u_n e^{i\omega t} k^2 H/(1 - e^{-j k H}) \]
\[ - j[\rho U^2 u_n e^{i\omega t}/\omega H](e^{-j k H} + 1) \]
\[ - j[2 \rho U^2 u_n e^{i\omega t}/\omega H](e^{-j k H} - 1) \]
\[ + 4 j[\rho U u_n e^{i\omega t}/k H](e^{-j k H} - 1) \]  \hspace{1cm} (25)

where we have left the pressure terms in the order A, B, C, and D.

In a feedback oscillator the total phase shift around the “loop” must be 360° to produce self-sustained operation. Now for all types of jet/edge resonator systems, feedback is applied from the downstream edge or lip to the upstream edge with a 180° shift. Since the feedback is applied via some form of acoustic radiation, and so travels from the downstream to upstream edge effectively instantaneously (the acoustic wavelength being large compared to H), travel time of information along the jet or shear layer must adjust itself to one-half period of the oscillation. This means that the disturbance along the jet or shear layer must be approximately one-half wavelength long between \( x=0 \) and \( x=H \), a fact that has been experimentally verified by Coltman for organ pipes and edgetones with laminar jets, and by Fletcher for turbulent jets.

This can be expressed in terms of the real propagation constant:

\[ k_L H = \pi \]  \hspace{1cm} (26)

where \( k_L \) is the value of the propagation constant in the “laboratory” coordinate system, as distinguished from \( \kappa \) which is measured in the accelerated system of the shear layer itself. To illustrate the difference, we rewrite Eq. (4) in the form:

\[ \xi = -2j(u_n / \omega) \sin (k x/2) e^{i \omega t - k x/2} \]  \hspace{1cm} (27)

from which it is evident that the effective value of the propagation constant is \( k_L = k/2 \). This effect may be observed in the actual data for dividing surface waves along the mouth of a laminar cavity resonator in Fig. 8, obtained from Ref. 23), where it is plain that, in the system moving up and down with the shear layer, the mouth dimension \( H \) is one full wavelength long. Another way of looking at it is to trace the timing of the zero-crossing locus in Fig. 8 and compare it with the progress of the wave maximum, which moves exactly half as fast. The result implies that:

\[ k H = 2\pi \]  \hspace{1cm} (28)

Substituting this in Eq. (25) we discover that, at resonance, all but term-B vanishes, obtaining for the laminar cavity resonator the simple drive-pressure:

\[ \Delta p = -j 2 \rho U^2 u_n e^{i\omega t}/\omega H \]  \hspace{1cm} (29)

Here at last is the correct explanation of the “\( U^2 \)” dependence of the acoustic amplitude!

To find the amplitude and frequency of the oscillation at a given speed we make use of Eq. (18), in the form:

\[ \frac{\Delta p + Z_0 Q_\omega}{Q_\omega} \leftrightarrow Z_0 \]  \hspace{1cm} (30)

which may be read as follows: for simultaneous values of wind speed, \( U \), oscillation amplitude, \( u_n \), and sounding frequency, \( \omega \), that give the same complex number on either side of the equation, we have a solution point. This is a form of “root locus” method and the solution may be performed either by computer or by graphical analysis. Without actually going through this process, which is rather tedious, we can gain some useful insights by examining the known form of the \( Z_0 \) impedance function:

\[ Z_0 = R_L + (S_F/S_W)|u_n|/c + j(\rho c/S_W)(k_0 \Delta L - \cot k_0 L) \]  \hspace{1cm} (31)

where \( L \) is the depth of the cavity (here assumed to be a long tube), \( k_0 \) is the acoustic propagation constant, \( c \) is the speed of sound, \( \Delta L \) is the cavity end correction, and \( R_L \) is the linear part of the system resistance, containing losses due to radiation, viscous damping and thermal damping. The nonlinear part of the resistance is due to formation of periodically emitted vortices at the mouth orifice at large sound amplitude, while the imaginary terms represent the mouth and cavity reactances that form the resonant system.

Amplitude of the oscillation is ultimately limited by the nonlinear term in \( Z_0 \). So long as the phase condition for positive feedback is satisfied, the oscillation will tend to build up until system losses represented by the real part of \( Z_0 \) are just matched by a real, amplitude-dependent term on the left side of Eq. (30). The threshold for equality of the two sides of Eq. (30) can occur only when the amplitude is large enough for the nonlinear term in \( Z_0 \) to be of size comparable to the linear term. Thus by equating these two terms we can get an idea of the actual amplitude at which the oscillation will reach dynamic equilibrium.

For a pipe closed at the opposite end, the linear portion of the impedance at the lowest axial mode can be estimated from the “\( Q \)” or quality factor of the system at low amplitude from the formula:
\[ R_L = \pi \rho c / 4 S_p Q \]  
where \( S_p \) is the cross sectional area of the pipe. Equating this to the nonlinear term we obtain:

\[ |u_H| / c = \pi (S_h / S_p) / 4 Q \]

Since \( \xi(H) = 0 \) for laminar cavity at resonance, there is no contribution from \( Q_3 \) and the left side of Eq. (30) is

\[ 2 \rho U^2 |oH|^2 \]

which is real and so may be equated to approximately twice the nonlinear resistive impedance, giving:

\[ U^2 = (\pi/4)(1/Q)oH^2c/S_p \]

Thus the speed of the wind needed to excite a cavity of slot dimensions \( H \) and \( b \), quality factor \( Q \) and resonant frequency \( f \) is on the order of:

\[ U = \pi H \left[ f c / (Q S_p) \right]^{1/2} \]

This may be compared with some actual data, taken from Ref. 23), for which the resonant frequency, \( f \), is 300 Hz, and low-amplitude "Q" of the cavity is 26.3, and other parameters are given by:

\[ H = 1.96 \text{ cm} , \quad b = 2.4 \text{ cm} , \quad S_p = 9.316 \text{ cm}^2 , \quad U = 1,300 \text{ cm/s} \]

Substituting in Eq. (34), we obtain a computed value of \( U = 1,382 \text{ cm/s} \) which is in the neighborhood of the measured value, 1,300 cm/s.

5.2 Effect of Turbulent Boundary Layer on Cavity Resonance

In Fig. 9 is shown a sequence of dividing streamlines for data obtained on a cavity resonator mounted in turbulent boundary layer flow,\textsuperscript{21} Synchronous computer averaging was needed to reduce the background noise caused by turbulence. We assume that, since the turbulence is small scale, the average flow behaves in a quasi-laminar fashion so that it is still reasonable to talk about a dividing streamline for the flow. Experimentally it is found that the \( \xi \) function fits the experimental data best near \( x = 0 \), while, in the vicinity of \( x = H \), it is necessary to use a \( \xi_{II} \) function to get a match. This seems to indicate that there are no acoustic sources in the jet in the vicinity of the downstream edge, and we may assume that \( k \) and \( k_L \) are essentially indistinguishable here. The average value of \( k \) over the path from \( x = 0 \) to \( x = H \) is therefore given approximately by:

\[ \langle k \rangle = (2k_L + k_L) / 2 = (3/2)k_L \]  

Therefore the value of \( k_H \) to use in Eq. (25) should be

\[ \langle k \rangle H = (3/2)\pi \]

(The integration for turbulent boundary case is the same as for laminar, despite the fact that the \( \xi_{II} \) function applies over only part of the path. This is because we need integrate either function only over the integrals that contain functions of \( x \).) Substitution of \( k_H \) into this equation indicates that all four force terms contribute to the drive term resulting in:

\[ \Delta p = -j [2\rho U^2 |u_H| / oH] e^{j \omega t} \{ 2(U_c / U) \}
\]

\[ - (U_c / U)^2 - 0.5 \}

Interestingly, in this case, the force contains an expression (shown in curly brackets) that depends on the relative wave speed in the accelerated system, \( U_c / U \). The expression involving the wave speed has real values only for the range of \( U_c / U \) equal to or greater than 0.2929.

The theory therefore predicts that cavity resonance will not occur in a turbulent boundary layer unless the wave speed exceeds a critical threshold. This can be translated into a critical Strouhal number:

\[ kH = 2n f H / U_c = (3/2)\pi \]

from which we may derive the Strouhal number for the lowest shear-wave mode:

\[ \text{Strouhal Number} = (fH / U)_c = (3/4)(U_c / U)_{xx} \]

\[ = 0.2197 \]
This prediction is strikingly confirmed in the data of DeMetz and Farabee\(^3\) who observed a distinct cut-off effect for resonant cavities with lower Strouhal number!

From Fig. 9 it is clear that \(\xi\) does not vanish at \(x=H\) for this case, so there should be a contribution to \(Q_j\). However since the peak value of \(\xi\) arrives 90° out-of-phase with the sound pressure in the pipe, \(Q_j\) does not contribute to the power stroke for turbulent cavity resonance.

5.3 Organ Pipes

As shown in Fig. 3, the organ pipe jet may be thought of as a pair of shear layers of opposite vorticity, separated by the jet width, \(\delta\). Although on account of their proximity, the two shear layers exhibit coupled motion, with propagation constants characteristic of the system as a whole, force-drive and jet-drive may still be thought of as coming from each layer separately. The lower layer produces forces across the mouth plane according to Eq. (24), and delivers a jet-drive flow

\[ Q_j = bU\xi(H, t) \]  

(39)
to the pipe each cycle. On account of the instability growth of the jet displacement along its path, \(\xi(H, t)\) does not vanish for the organ pipe and we expect a contribution from jet-drive so long as there is a component of \(Q_j\) in phase with the sound pressure in the pipe. In an ideal top-hat jet, there is no moving fluid below the dividing surface and so no return flow is removed from the pipe by the lower layer. The upper layer, however, removes an amount \(-Q_j\) from the jet, that is, the same amount out-of-phase, and introduces no fluid from the outside. Provided the maximum displacement at the lip, \(\xi(H)\), is at least \(\delta/2\), all the jet fluid contributes to \(Q_j\).

The drive flow \(Q_j\) introduced inside the pipe is amplified by the acoustic resonator to give an acoustic standing wave through the mouth of the pipe, \(Q_s\), with total acoustic discharge into the pipe \(Q_r\) as given by Eq. (11). The flow \(-Q_j\) produces no amplification, but does combine with the acoustic flow in the mouth, \(-Q_m\), to produce the total radiated discharge, \(-Q_r\). Therefore, the total radiation from either end of an open pipe is the same, as shown experimentally by Coltman.\(^{32}\)

The forces from the upper layer act in phase with those from the lower layer when the jet motion is sinuous, doubling the total drive force. To calculate the force drive terms, one must determine the value for \(k_1H\). It seems likely that

\[ k_1H = k_1 \pi \]  

(40)

Over the early portion of its path, the jet centerline travels very fast on account of the action of the counter wave at \(x=0\), so that there is little phase shift. Over the rest of the path, the acoustic particle displacement is smaller than the displacement due to the stability wave, which has large gain in a jet, so that \(k_1\) and \(k_\delta\) are approximately the same over the entire path of the wave.

We will not attempt to integrate Eq. (24) for the fulljet. On account of the standing wave conditions near \(x=0\) and the fact that \(k_1^*\) is a complex number, a simple result such as Eq. (25) is not feasible. Some things can be said about the solution however. Generally speaking, jet width is small compared to the cut-up, \(H\). For a typical diapason pipe, for example, the ratio of \(\delta/H\) might be 1/30, while the ratio of \(k_\delta/H\) is even smaller. Therefore organ pipe jets voiced for bright tone must fall into the limiting category for which eddies are expected to travel along the jet. Currently halfjet resonators with \(\delta_0/H\) in this range are being studied at the U.S. Naval Academy (results are still unpublished), and it has been found that below a critical value of \(\delta_0/H\), the spectrum of the radiated sound suddenly becomes very rich, indicating transition to a new type of drive. It appears that for shear layers thin enough to permit the buildup of spinning vortices in the mouth, a large inertia is added to the motion of the layer, producing impulsive drive at the lip, with a resulting spectrum not unlike that of a diapason pipe. The matter is still under study, but already shows much promise for increasing the understanding of organ pipe physics, particularly with regard to the manipulations involved in voicing.

5.4 Organ Pipe with Wide Jet.

Until now we have tacitly assumed that jet widths are narrow enough that upper and lower shear layers are able to move in a coupled, sinuous mode and that both shear layers are able to react with the cavity. Clearly, as the width of the jet becomes greater, there must come a point at which one or both of these assumptions is not valid. Assuming that the width of the shear layer lies within the range

\[ \frac{\delta}{H} = \frac{\delta_0}{H} = 0.03 \]  

(41)
for instability for the halfjet, what are the limits on $\delta$ that distinguish halfjet from fulljet resonator and is there a continuous operational mode for the in-between state? Clearly there are two ratios that need to be specified in order to distinguish single shear layer from double shear layer situations: $\delta/\lambda$, where $\lambda$ is the wavelength of the stability wave along the shear layer, determines the complex propagation constant for the top-hat profile, while $\delta/H$ determines whether one or two shear layers interact with the lip. For the former, Rayleigh showed that the top-hat eigenvalues change smoothly into step-profile eigenvalues for $\delta/\lambda \gg 1$, so we are assured that there is a proper value of wave propagation constant, $k + i\omega$, for all widths of the jet. Likewise it seems obvious that for $\delta/H \gg 1$, it must be impossible for the full jet to interact with the lip by crossing into the pipe, regardless of whether the sinuous mode is present or not.

It would be interesting to investigate experimentally as to whether there is an in-between oscillatory state in which interaction at the lip is dominated by lower shear layer dynamics while a weak upper shear layer oscillation shows some coupling of the entire jet with the acoustic field in the pipe. For this case, we should expect the propagation constant to be somewhere between that for halfjet and that for jet, while at resonance the value of $kH$ might be in the neighborhood of $1.25\pi$, with the possibility of a critical value of $U_c/U$ according to the way in which type A and D forces are involved.

6. CONCLUSIONS

A new unified method of treating the sounding mechanism of organ pipes and cavity resonators has been developed to replace the control-volume method which until recently has been the most widely accepted theory. It has been shown that shear layers contain a dividing surface which acts as a vibrating membrane whose motions control the production of sound in the resonant cavity. This vorticity-containing shear layer, which is the principle agent in the dynamics of jet motion, can be studied apart from the whole jet. In this way insights into organ pipe physics can be found through the study of the simpler but related phenomenon of cavity resonators.

Both jet-drive and force drive contributions are found to be involved in the sounding mechanism. The historical controversy as to whether pipes are driven by force-drive or jet-drive is thus resolved in favor of both. Jet-drive is possible whenever the jet arrives at the lip with finite displacement and with a component in phase with the sound pressure in the pipe.

The present approach shows the drive system to be inherently linear over a range of shear layer widths, so that sinusoidal oscillation is readily achievable. For voicing situations calling for spectra rich in harmonics, nonlinear drive is attained by resorting to a smaller ratio of jet width to lip cutup which, at high enough Reynolds number, permits the growth of eddies which result in impulsive action at the lip.

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REFERENCES

See, for example, the tutorial material in the text by N. H. Fletcher and T. D. Rossing, *The Physics of Musical Instruments* (Springer-Verlag, New York, 1991), Chap. 16.


See Ref. 17), p. 402.


We must necessarily restrict our attention to the case for which the vorticity has not rolled up into an eddy.

See Ref. 17), p. 391.


Ideally, in a fully developed velocity profile, the average inflection point of a free shear layer lies along the plane z=0. In the simple model, velocity perturbations move about this point while the average profile remains stationary, even though the actual velocity profile moves up and down with the wave as it descends into the cavity along the path.

See examples of similar analysis in Refs. 6), 21), and 23).

See Refs. 6) or 21).

