The vibration of a struck string (Acoustical research on the piano, Part 1)

Isao Nakamura

*Teikyo University of Technology,
2289, Uruido, Ichihara, 290-01 Japan*

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Vibrations of piano strings struck with an elastic hammer are studied by using equivalent-circuit models based on the impedance analogy and the mobility analogy respectively. When the tension of a piano string alone is considered, neglecting its elasticity, the string and hammer system has an analogy with an electric transmission line, the hammer being represented by an inductance and a capacitance. An analogue simulator based on the models was designed and used for the study of the string and hammer system. Various waveforms, contact times, and the maximum contact force in each case were obtained by using the simulator. The reason for the appearance of even harmonics series in a high frequency region of piano keys, clearly shown in the simulator, is explained; and the method of their adjustment is suggested. A velocity waveform is determined by the resultant of two vertical tension waveforms in the simulator. The characteristics of these waveforms are shown compared with those from an actual piano. The contact forces, from which their vertical tensions can be reduced, are calculated from the equivalent circuits. Waveforms in all the cases become very similar to damped half sinusoidal waves. The energy transmission from a hammer to a string, including its efficiency at a low-frequency region, is discussed.

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1. INTRODUCTION

The piano, which has the widest frequency range next to the pipe organ, is popularly used for the basic instrument of music exercises today. A great many works²⁾ on the acoustics of the piano have been published since H. Helmholtz,¹⁾ and yet it is still difficult to describe the characteristic of the piano as a whole. This is due to the lack of study of the whole process of a piano's sound generation over its full tone range.

This research aims at finding the basic patterns of piano sound by studying the process of sound generation. Part I of this research is devoted to a study of string-hammer interaction in the piano.

Many works on struck strings of the piano were

published in the 1920 s and 30 s, including experiments using optical measurements and theories based on their results. The experimental results obtained by W. H. George³⁾ have been used to date. W. E. Kock⁴⁾ published a study of the piano using equivalent circuits in 1937, although no further development. Nakamura *et al.*⁵⁾ initiated an analogue simulation of the piano in 1964. Recently, L. Hiller and P. Ruiz,⁶⁾ Nakamura,⁷⁾ and some other workers^{8,9)} have published studies of the same subject using computer-simulation. However, none of these works covers the process of formation of vibration waveforms and their characteristics over the whole tone range of the piano. Note, the piano has typically 88 keys from A_0 (fundamental frequency, 27.5 Hz) to C_8 (4,186 Hz).

A method of representing the mechanism of struck strings by using an equivalent-circuit model to understand the basic characteristic of the phenomenon; the design of an analogue simulator using electrical networks and electronic switches based on the models; discussions on waveforms of strings; and transmission of energy from a hammer to a string are described in this paper. In other words, the tone colour and intensity of sound generated by struck strings over the full frequency range of the piano are discussed.

2. BASIC EQUATIONS AND EQUIVALENT CIRCUITS

Assuming that the displacement of each part of a flexible string is small compared with its full length, the vibration can be represented by

$$\sigma \frac{\partial^2 y}{\partial t^2} + r \frac{\partial y}{\partial t} = T \frac{\partial^2 y}{\partial x^2} \quad (1)$$

where σ is the uniform linear-density of the string, r is the frictional resistance per unit length in the air, and T is the tension. The y -component of the tension (vertical tension), f , is given by

$$f = -T \frac{\partial y}{\partial x}. \quad (2)$$

By using the vibration velocity of the string $v = \partial y / \partial t$, Eqs. (1) and (2) can be written as

$$-\frac{\partial f}{\partial x} = \sigma \frac{\partial v}{\partial t} + rv, \quad -\frac{\partial v}{\partial x} = \frac{1}{T} \frac{\partial f}{\partial t}. \quad (3)$$

The equation for an uniformly-distributed-constant electric transmission line shown in Fig. 1(a) is

$$-\frac{\partial e}{\partial x} = L \frac{\partial i}{\partial t} + Ri, \quad -\frac{\partial i}{\partial x} = C \frac{\partial e}{\partial t}. \quad (4)$$

When the impedance analogy (*i.e.* the force-voltage and velocity-current analogy) is applied to this line, σ should correspond to L , T to $1/C$, and r to R respectively. The equation for another uniformly-distributed-constant electric transmission line shown in Fig. 1(b) is

Fig. 1 Equivalent circuit elements for a string. (a) Impedance analogy. (b) Mobility analogy.

$$-\frac{\partial i}{\partial x} = C \frac{\partial e}{\partial t} + Ge, \quad -\frac{\partial e}{\partial x} = L \frac{\partial i}{\partial t}. \quad (5)$$

When the mobility analogy (*i.e.* the force-current and velocity-voltage analogy) is applied to this line, σ should correspond to C , T to $1/L$, and r to G respectively.

For the boundary conditions of the equations, the fixed end of a string is represented by an open end of the electric line in the impedance analogy; and that is represented by a short end of the electric line in the mobility analogy.

Let us now consider a string-and-hammer system of an upright piano as shown in Fig. 2(a), and its ideal model as shown in Fig. 2(b). The following equation holds:

$$M_H \frac{dv_H}{dt} = -K_f \int_0^t (v_H - v_0) dt = -f_H \quad (6)$$

where v_H and v_0 are the velocities of the hammer and string respectively, f_H is the force at the contact point of the hammer and string, M_H is the mass of the hammer, K_f is the stiffness of the hammer felt.

A uniformly-distributed-constant line with two discrete components inserted at point x_0 as shown in Fig. 3(a) is represented by,

$$L_H \frac{di_H}{dt} = -\frac{1}{C_f} \int_0^t (i_H - i_0) dt = -e_H \quad (7)$$

where L_H and C_f are a inductance and a capacitance respectively, i_H and i_0 are electric currents flowing through the inductance and the whole line respectively; and e_H is the terminal voltage of the inserted elements.

In the impedance analogy, therefore, M_H corresponds to L_H and K_f to $1/C_f$. The condition before the contact of the hammer is represented by

Fig. 2 String and hammer system. (a) Model of upright piano. (b) Ideal model.

Fig. 3 Equivalent circuits for a string and hammer system. (a) Impedance analogy. (b) Kock's analogy. (c) Mobility analogy.

the initial current I_H (which corresponds to V_H), the switch S being closed. The striking of the string with the hammer is represented by opening S , i_H corresponding to the velocity of the hammer v_H , and i_0 to the velocity of the string at the contact point. The state where the hammer is bounced by the string is represented by closing S when its terminal voltage becomes zero. Note that the contact force of the hammer f_H always compresses the string, and never pulls it.

W. E. Kock⁴⁾ has proposed the momentum-voltage and displacement-current analogy for a string with a hard hammer shown in Fig. 3(b). This figure must be incorrect, since the correct circuit should be the same as that of the impedance analogy. His theory described in his text is, however, still correct.

Since M_H corresponds to C_H , and K_f to $1/L_f$ in the mobility analogy,

$$C_H \frac{de_H}{dt} = -\frac{1}{L_f} \int_0^t (e_H - e_0) dt = -i_H \quad (8)$$

or

$$L_f \frac{di_H}{dt} = e_H - e_0, \quad C_H \frac{de_H}{dt} = -i_H \quad (9)$$

is obtained. Figure 3(c) shows the equivalent circuit for this expression. The initial voltage E_H representing the initial velocity of the hammer V_H

is given to the terminals of the capacitor C_H with S opened. When S is closed, the current flows into the line giving its voltage and current. After i_H has reduced to zero, S is opened again representing a state where the hammer is separated from the string. i_H represents the contact force of the hammer at the striking point, f_H .

The impedance analogy and the mobility analogy are in a dual relationship, and one of them can easily be obtained from the other one. For a complex system such as a system with a hammer and soundboard, the mobility analogy is easier, since this has the same form as a mechanical system. The elasticity of a string can be simulated by an equivalent circuit if necessary.¹⁰⁾ However, if the struck string is the main subject, this can be ignored since the time involved is very short.

3. DESIGN OF SIMULATOR

An analogue simulator based on the above-described equivalent circuit model was constructed. The simulator consists of a circuit which is equivalent to a transmission line representing a piano string, and a switch circuit with L_H and C_f representing the mass and the compliance of the hammer respectively. The impedance analogy was employed to permit the use of inductors which always contain resistance components so that the simulation of a string becomes more realistic.

It is not easy to change many actual components in the circuit representing a string, while it is no problem to change L_H and C_f in the switch circuit representing a hammer. Therefore, the coefficients between the mechanical system and the electrical system, including that of time and impedance, are changed for each case by changing L_H and C_f . The time transformation coefficient k_t , and the impedance transformation coefficient k_z are

$$k_t = \frac{t_o}{t_m} = \frac{f_{om}}{f_{oe}}, \quad k_z = \frac{Z_{oe}}{Z_{om}} = \sqrt{\frac{L}{C}} / \sqrt{\sigma T} \quad (10)$$

respectively, where f_{om} is the fundamental frequency of the mechanical system, and f_{oe} is that of the electrical system. The equivalent constants of the hammer are

$$L_H = k_t \cdot k_z \cdot M_H, \quad C_f = k_t / k_z \cdot K_f. \quad (11)$$

The distributed-consent transmission line representing a piano string is approximated by a cascade-connected discrete-component circuit using

Fig. 4 Actual simulator for piano string.

$$\Delta L = L \cdot \Delta x = L \frac{l}{n}, \quad \Delta C = C \cdot \Delta x = C \frac{l}{n} \quad (12)$$

where $\Delta x = l/n$, l being the length of the string. For the experimental set, $n = 60$ (60 sections), $\Delta L = 5 \text{ mH}$, $\Delta C = 555.6 \text{ pF}$ were used so that the fundamental frequency of 5 kHz is obtained. Each inductor was made by using an adjustable pot-type ferrite core with a Lits-wire winding so that a high Q value is obtained. For each capacitor, a silvered-mica type was used. Figure 4 shows the set of the circuit.

Figure 5 shows a simplified diagram of the simulator for a string and hammer system. The part shown in the left-hand side of *ab* in Fig. 5 represents the action of the hammer striking the string. A pair of multiple switches S_1 and S_2 of which close-and-open phases are opposite is used. The opened S_2 represents a state where the hammer contacts with the string; and the closed S_2 represents a state where the hammer separated from it. The action of the circuit starts with S_1 opened (S_2 closed). When the switch phase is reversed, a voltage appears between *ab* representing the contact of the hammer with the string. When the voltage between

ab is decayed to zero, the phases of S_1 and S_2 are reversed representing the hammer separated from the string. The contact time of the hammer with the string depends on their interaction, and this is not pre-determined. Figure 6 shows the actual circuit to carry out this action. Tr_1 and Tr_3 correspond to S_1 and S_2 in Fig. 5 respectively. A pulse generator was used to generate pulses with a pre-set pulsewidth according to the contact time. The pulse was repeatedly generated so that waveforms in the circuit could be observed by an oscilloscope.

Figure 7 shows the transformation coefficients k_z and k_t against key numbers for three types of piano models;—a small upright piano, a middle-size upright piano and a small grand piano. Since the characteristic impedance of the electrical system, $\sqrt{\Delta L / \Delta C}$, is constant, k_z is reversely proportional to the characteristic impedance of the mechanical system. k_z of a single string, and that of a double string proportionally increase with their frequency. k_z of a triple string is almost constant with its frequency. k_t increases proportionally with frequency in all the cases. Figure 8 shows the inductance L_H and capacitance C_f of the hammer system against the key number. These are calculated by using Eq. (11), M_H being the inertance of the hammer. This is greater than the mass of the hammer alone by about 10%. The values of M_H for each string in the double and triple strings are obtained by dividing their values by 2 and 3 respectively. K_f is obtained from static measurements. The wire-wound string and plain string have different trends in the increase of L_H and C_f with their frequency. The trend in the middle-size upright piano almost coincides with that of the small grand piano, but that of small upright piano has a minor difference in its middle and low frequency regions compared with the other pianos.

Fig. 5 Simulator representing a string and hammer system.

Fig. 6 Detail of hammer system shown in Fig. 5.

Fig. 7 Transformation coefficients for key numbers.

Fig. 8 Inductance and capacitance for key numbers.

The constants of the simulator can more easily be changed than a mechanical system to represent various cases. However, the simulator is not suitable to measure phenomena over a long period, due to the errors in Q , the discrete representation of a continuous string, and the neglect of the elasticity of a string (in the appendix of the original Japanese paper). Since the phenomenon of a struck string occurs in short time, the simulator is efficient to investigate vibrations in the piano in detail.

4. VIBRATION WAVEFORMS

The voltages and currents in the simulator representing the string and hammer systems of all the keys (88 notes) of each of above-described three types of pianos were measured. The voltage measured between a-b in Fig. 5 represents the contact force, and that measured between d-e represents the vertical tension at the bridge point (the end of the longer part of a string). Figure 9 shows the typical waveforms of the vertical tension (solid lines) and the contact force (dotted lines) in a single cycle of vibrations, taking five keys as examples. The waveform of the vertical tension has a phase lag by b/c_0 with reference to that of the contact force, where b is the distance between the striking point and the bridge point, and c_0 is the propagation velocity of the wave. Ignoring this phase lag, waveforms for key numbers 1 to 20 are the same. Note, the scales of the amplitude axis and the time axis of Fig. 9 are common so that this makes the comparison of two types of waveforms easier. Figures 10 and 11 show the contact time of the hammer and string, and

Fig. 9 Waveforms of vertical tension (solid line) and contact force (dotted line) for various key numbers.

Fig. 10 Contact time of string and hammer for key number.

Fig. 11 Contact force of string and hammer for key number.

the maximum contact force for each key of three types of pianos.

Figures 9, 10 and 11 are now explained dividing into several frequency regions. In the low frequency region, the contact force is a well-damped half-sinusoidal wave with an almost constant amplitude and pulse-width. The time required for the reflection of a pulse from the shorter end of a string decreases with the key number. When a point about 1/8 of the string length of key number 20 is struck by its

hammer, the reflected pulse returns immediately after the contact is over. The time required for the travelling of the pulse from the shorter end of the string to the contact point is given by $\tau_a = 2a/c_0$, where a is the distance between the two points. τ_a for a small upright piano is shown by a dotted line in Fig. 10. In the frequency region below key number 20, the hammer separates from the string at an instant when the string is most displaced and stopped. Just after key number 20, the contact time suddenly increases due to the influence of the reflected pulse. However this influence on the amplitude is not great, since the amplitude of the tail of the vertical tension waveform is small. If necessary, the contact time can be adjusted by changing the contact point or the value of K_r . In the region around key number 20 to 40, the waveform of each contact force is not very different from a single cycle of a damped sinusoidal wave, although it has a hump in a half cycle. The waveform of the vertical tension for each key in this region is explained as a superposition of two damped half sinusoidal waves with different phases. One is the initial force and the other one is its reflection from the shorter end of the string arrived at the contact point within the contact time. The waveform of key number 40 in Fig. 9 is an example of this type. In the region from key numbers 40 to 60, the resultant vertical tension decreases with the key number (Fig. 9), in spite of the increase of the contact force with the key number (Fig. 11). In this key region, the hammer separates from the string at the maximum velocity of string in the opposite direction. The reflection time from the longer string end, $\tau_b = 2b/c_0$, at key number 60 of a small upright piano coincides with its contact time (Fig. 10). Therefore, pulse reflected from the longer end of a string also must be taken into account for the vibration of strings higher than key number 60.

Throughout all the keys, the waveform of the vertical tensions of a string at its bridge is given by

$$e'(t) = e(t) - e\left(t - \frac{2a}{c_0}\right) 1\left(t - \frac{2a}{c_0}\right) \quad (13)$$

where $e(t)$ is the waveform of the contact force, and time for $e'(t)$ is referred to a moment after $e(t)$ by b/c_0 .

The formation of the velocity waveform at the struck string point is now discussed. Referring to Fig. 12, the waveform of an electric current O_1-O_1' (solid line) is obtained as the resultant of two electric current waveforms (dotted lines). One of the current

Fig. 12 Striking-velocity waveforms formed by superposition of original and reflected contact force waveforms.

waveforms (one of the dotted lines) is obtained by $e(t)/2R_0$, where $e(t)$ is a damped half-sinusoidal wave representing the contact force, and $2R_0$ is the sum of the characteristic impedances of two sides of the line seen from the struck string point. The other current waveform (the other dotted line) is the current reflected from the shorter end of the string, and this has a phase lag by $2a/c_0$. This waveform has the same pattern as that of the above-mentioned vertical tension. The second wave O_2-O_2' (solid line) in Fig. 12 is obtained by the resultant of two waveforms; the reflection of the waveform O_1-O_1' from the shorter end of the string with a reversed phase, and another reflection of this wave from the shorter end of the string (both dotted line). The waveforms after the second wave will be almost the same. The first and second waveforms are expressed by

$$i_1(t) = \frac{1}{2R_0} e(t) - \frac{1}{2R_0} e\left(t - \frac{2a}{c_0}\right) 1\left(t - \frac{2a}{c_0}\right) \quad (14)$$

$$i_2(t) = -\frac{1}{2R_0} \left\{ e(t) - 2e\left(t - \frac{2a}{c_0}\right) 1\left(t - \frac{2a}{c_0}\right) + e\left(t - \frac{4a}{c_0}\right) 1\left(t - \frac{4a}{c_0}\right) \right\} \quad (15)$$

respectively, where the reference time of Eq. (15) lags from that of Eq. (14) by $2b/c_0$. Waves after the third wave have almost the same waveform as the second one, but each wave delays from its preceding wave by one cycle. Figure 13 shows typical waveforms obtained by the simulator and an actual small upright piano respectively; key number 10 for an example of a single string, key number 21 for a double string, and key numbers 35 and 45 for a triple string. The waveform for the 21st key is an example of a wide pulse width due to the reflection from the shorter end of the string. The constants used for

Fig. 13 Waveforms of string velocity and contact force at a struck point. (a)–(d): Simulation. (a)–(c) upper V: 0.4 mA/div. lower V: 2 V/div. H: 50 μ s/div. (d) upper V: 0.4 mA/div. lower V: 5 V/div. H: 50 μ s/div. (a')–(d'): Actual measurements. (a')–(b') V: 0.3 m/s/div. H: 3 ms/div. (c')–(d') V: 0.3 m/s/div. H: 1 ms/div.

the simulator are obtained from general measurements of pianos (*i.e.* not the constants of a particular piano in the comparison). The simulator has already mentioned errors, and errors due to neglect of the nonlinearity of the hammer. The waveforms of the actual piano might be affected by the frequency characteristic of the sound measurement system. Despite these factors, the simulated waveforms are reasonably accurate compared with the measured ones.

The waveforms of the contact force and the vertical tension are now examined in keys higher than key number 60. Figure 14 (a–d) shows waveforms of key numbers 60, 65, 70 and 80 respectively obtained by using the simulator. All the waveforms of the contact force are almost half-sinusoidal. Figure

Fig. 14 (a)–(d): Waveforms of vertical tension before adjustment and contact force. (a) upper V: 2 V/div., lower V: 5 V/div., H: 50 μ s/div. (b)–(d) upper V: 2 V/div., lower V: 10 V/div., H: 100 μ s/div. (a')–(d'): Vertical tension waveforms formed by superposition of original and reflected waves.

14 (a'–d') shows that the waveform of the vertical tension in each key is a resultant (solid line) of two-waves (dotted lines): one is a single-cycle wave generated, and the other one is a wave delayed by one cycle which is reflected from the shorter end of the string. Therefore, Eq. (13) still holds in this key region. The ratio of the contact time τ_0 to the fundamental period T_0 is indicated under each waveform in Fig. 14 (a'–d'). It is generally said that there is a difficulty in producing a high-quality sound around key number 70. Referring to Fig. 14 (b, b'), the waveform for key number 65 contains many higher harmonics. Referring to Fig. 14 (c, c'), key number 70 consists of almost entirely even-number harmonics without its fundamental wave and odd-number harmonics. These undesirable phenomena can be removed by adjusting the stiffness of felt on the hammer, or by shifting its striking position on the string as shown in Fig. 15. K_f/K_{f0} in Fig. 15

Fig. 15 Waveforms of vertical tension after adjustment. (a)–(c): Key number 65. (a')–(c'): Key number 70.

indicates the change of the stiffness of felt on the hammer with reference to its initial value. a/l in Fig. 15 indicates the relative striking position of a string (see Fig. 2(b)). The contact time of a string is determined by K_f/K_{f0} and a/l . If τ_0/T_0 is $3/2$, $5/2$. . . , then a string vibrates at a frequency twice of the intended fundamental frequency. To avoid this state, therefore, the piano system should be designed so that τ_0/T_0 becomes an integer. τ_0 can be reduced by increasing the stiffness of felt on the hammer, or by reducing 'a' in a/l . A similar phenomenon occurs in keys higher than key number 85. For example, if key number 88 is adjusted to $K_f/K_{f0}=2$ and $a/l=1/20$, then $\tau_0/T_0=2$; and if $K_f/K_{f0}=0.7$ and $a/l=1/8.6$, then $\tau_0/T_0=3$ respectively.

The stiffness of felt on the hammer is usually adjusted when the sound of a piano is regulated in its manufacturing process. Some manufacturers design their hammers for the said key region to reduce the stiffness of felt, based on their experience.

5. EFFICIENCY OF STRING STRIKING

Part of the kinetic energy of a hammer is transmitted to a string when the hammer strikes the string in the piano. When a hammer having inertia M_H strikes a string at an initial velocity of V_H , and the hammer is bounced back at a velocity of V'_H , the difference of the two kinetic energies is supplied to the string. The ratio of this energy supplied to the

Fig. 16 Equivalent lumped constant circuit models of a hammer and string system at a struck point. (a) Model for low frequency region. (b) Model for high frequency region. (c) Simplified model for (b).

string to the initial kinetic energy, R_E , is

$$R_E = \frac{\frac{1}{2} M_H V_H^2 - \frac{1}{2} M_H V_H'^2}{\frac{1}{2} M_H V_H^2} = 1 - \left(\frac{V_H'}{V_H} \right)^2. \quad (16)$$

Let us now examine R_E for a low frequency region and a region above the middle frequency.

Assuming that the hammer separates from the string before the arrival of the reflected waves from the two ends, the impedances of the string system seen from the hammer contact point towards its two ends are equally represented by the characteristic impedance of the line, R_0 , where $R_0 = \sqrt{\sigma T}$ for the mechanical system. This assumption is correct for a low-frequency region. Figure 16(a) shows this equivalent system. If $\sqrt{L_H/C_f} < 4R_0$, then the system is oscillatory, and the electrical current flowing through L_H (representing the velocity of the hammer), $i_H(t)$, is

$$i_H(t) = I_H \exp\left(-\frac{t}{4R_0 C_f}\right) \cos \sqrt{\frac{1}{L_H C_f}} t \times \sqrt{1 - \left(\frac{\sqrt{L_H/C_f}}{4R_0}\right)^2} t. \quad (17)$$

The voltage, $e(t)$, representing the contact force is

$$e(t) = I_H \frac{\sqrt{L_H/C_f}}{\sqrt{1 - \left(\frac{\sqrt{L_H/C_f}}{4R_0}\right)^2}} \exp\left(-\frac{t}{4R_0 C_f}\right) \times \sin \sqrt{\frac{1}{L_H C_f}} \sqrt{1 - \left(\frac{\sqrt{L_H/C_f}}{4R_0}\right)^2} t. \quad (18)$$

Since the contact time τ_0 is a moment when the sine term in Eq. (18) becomes π ,

$$\tau_0 = \pi \sqrt{L_H C_f} / \sqrt{1 - \left(\frac{\sqrt{L_H/C_f}}{4R_0}\right)^2}. \quad (19)$$

The velocity of the hammer at the time of separation from the string is given by Eq. (17). This is rewritten using the constants in the mechanical system as

$$V_H' = -V_H \exp\left(-\frac{K_f}{4\sqrt{\sigma T}} \tau_0\right). \quad (20)$$

Therefore

$$R_E = 1 - \exp\left(-\frac{2\pi}{\sqrt{\left(\frac{4\sqrt{\sigma T}}{\sqrt{M_H K_f}}\right)^2 - 1}}\right). \quad (21)$$

The equivalent circuit of the contact force above the middle frequency region is now considered. Assuming that the shorter part of a string (from the striking point) vibrates rectilinearly, this can be represented by a capacitance $C_a = aC$, and this can be changed by altering the ratio a/l . Figure 16(b) shows the equivalent circuit. This can be simplified to Fig. 16(c), since the condition

$$1/j\omega \frac{C_f C_a}{C_f + C_a} \gg R_0 \quad (22)$$

is always satisfied in the middle and high frequency region of the piano. Since this satisfies the condition

$$\sqrt{\frac{L_H}{C_f + C_a}} > \frac{1}{2} \frac{R_0}{(1 + C_f/C_a)^2}, \quad (23)$$

the circuit oscillates. The current $i_H(t)$ flowing through $L_H(t)$, representing the hammer velocity, is

$$i_H(t) = I_H \exp\left(-\frac{R_0 t}{2L_H(1 + C_f/C_a)}\right) \times \cos \frac{t}{\sqrt{L_H(C_f + C_a)}}. \quad (24)$$

The voltage $e(t)$ representing the contact force is

$$e(t) = I_H \sqrt{\frac{L_H}{C_f + C_a}} \exp\left(-\frac{R_0 t}{2L_H(1 + C_f/C_a)}\right) \times \sin \frac{t}{\sqrt{L_H(C_f + C_a)}}. \quad (25)$$

The contact time is, therefore

$$\tau_0 = \pi \sqrt{L_H(C_f + C_a)}. \quad (26)$$

The velocity of the hammer at its separation from the string is expressed, using the constants of the mechanical system, by

$$V_H' = +V_H \exp\left(-\frac{\sqrt{\sigma T}}{2M_H(1 + K_a/K_f)^2} \tau_0\right) \quad (27)$$

where K_a is the stiffness which corresponds to the reciprocal of C_a , and this is given by

$$K_a = \frac{T}{a} = \frac{2\sqrt{\sigma T}}{n_0 t_0}. \quad (28)$$

This can also be expressed by using $n_0 = a/l$, the period $t_0 = 2/\sqrt{\sigma/T}$, the characteristic impedance $\sqrt{\sigma T}$. R_E is given by

$$R_E = 1 - \exp\left(-\frac{\pi\sqrt{\sigma T}}{\sqrt{M_H K_a}(1 + K_a/K_f)^3}\right). \quad (29)$$

Figure 17 shows the values of R_E for each key of three types of pianos. The values are not available for the region between Eq. (21) and Eq. (29), and therefore they are indicated by dotted lines in Fig. 17. In the high frequency regions in Eqs. (28) and (29), the nearer the string end is struck by the hammer (i.e. the smaller n_0), the greater the energy is injected into the string. Figure 17 also shows the normalized mean power per cycle, M. P., of each type of piano. These are obtained by dividing the power by that of tone A_0 (27.5 Hz). Since the mean power is low in the low frequency range, the piano must be designed to make the energy-injection efficiency high in this region. The power is highest in the small grand piano among the three examples compared. The power in the middle-size upright piano is relatively uniform with their frequency. The duration of sustenance of sound in the region above the middle frequency is proportional to the frequency.¹¹⁾ Therefore the mean power is a useful indicator to evaluate the quality of a piano. The

Fig. 17 Energy ratio and normalized mean power for key numbers.

Fig. 18 Energy ratio and wave ratio for impedance ratio.

mean power varies a little with the frequency in the low frequency region, and a uniform value is not available. However, this discussion will offer some information on the energy dissipation in the piano.

The efficiency in the low frequency region depends on the characteristic impedance of each string and that of the hammer as shown by Eq. (21). R_E in Fig. 18 shows this relationship. Referring to Eqs. (18) and (19), the pulse width and amplitude of the contact force are also affected by this quantity. When a solid wall is struck, the pulse width τ_∞ is given by

$$\tau_\infty = \pi\sqrt{\frac{M_H}{K_f}}. \quad (30)$$

Let us take the ratio of τ_∞ to the pulse width of a struck string τ_0 and use a symbol R_W for the ratio. Then

$$R_W = \tau_\infty/\tau_0 = \sqrt{1 - \left(\frac{\sqrt{M_H K_f}}{4\sqrt{\sigma T}}\right)^2}. \quad (31)$$

This ratio can be regarded as the ratio of the amplitude in a general case to the amplitude of a struck solid wall. R_W can be called 'waveform ratio,' and this is indicated in Fig. 18. The value of $4\sqrt{\sigma T}/\sqrt{M_H K_f}$ is about 2 in the lower frequency region. This value is appropriate to choose a very large value of R_E without choosing the value of R_W too small.

From Figs. 10, 11 and 17, the small upright piano shows the poorest characteristics among the three types, notably in its low frequency region.

6. CONCLUSION

1) This paper describes hammer-string interactions of the piano over the whole key range consistently, although many other published papers describe some parts of the phenomenon or those on

limited keys separately. An analogue simulator with equivalent circuits of the string-hammer system of the piano was used as a tool of the analyses.

The contact time of a hammer on a string varies widely, about 1/20 of the period of string vibration at key number 1, and twice to triple at key number 88. The force acting between a string and a hammer in each key is a half-sinusoidal waveform pulse, throughout the whole key range, due to their interaction (Fig. 9). The vibration of a string is essentially formed by the resultant of the half-sinusoidal waveform pulse, and its reflections from the two ends of the string (Fig. 12). The energy on a vibrating string is transmitted to the soundboard through its bridge so that the energy is emitted into the air as sound. The characteristic of vibrations in the string-hammer system over the whole key range is analysed by referring to the contact time and the maximum contact force of each key (Figs. 10 and 11).

2) In a key range higher than key number 60, a reflected pulse from the bridge arrives at the string-hammer contact point before they separate from each other, making a complex interference (Fig. 14). This explains the difficulty in the sound generation at a key range near number 70 (Fig. 15), which has been recognized empirically. Solutions of this problem are suggested.

3) The characteristic of energy injected from a hammer to a string, which is important in the intensity of sound, is calculated for the whole key range (Fig. 17). This also explains the recognized fact that a striking of a high-note string at a point near its end makes a better sound.

4) It is desired to inject a large amount of energy, with a high efficiency in its transmission, into a lower-frequency string to produce a sustained sound. This paper describes the relationship between the characteristic impedance of the string and that of hammer (Fig. 18), which determines the efficiency.

Most problems of the string-hammer interaction in the piano, except for those of the nonlinearity of a hammer with felt, have been solved in this paper. This research mainly aims at finding the mechanism of sound generation in the piano, but this will be useful not only for the design and adjustment of the

piano, but also for syntheses of piano sound with electronic musical instruments.

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