Moments of probability distribution of equivalent continuous A-weighted sound pressure level (L_{AeqT})

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When a noise arises from several individual noise events, then its equivalent continuous A-weighted sound pressure level (L_{AeqT}) depends on the number of noise events (N), the A-weighted sound exposure levels (L_{AE}) and the equivalent continuous A-weighted sound pressure level of the background noise ($\tilde{L}_{\text{AeqT}}$). At least some of those quantities are random variables. Thus, the values of L_{AeqT} , measured within a constant time interval, are random as well. The results presented here make it possible to predict the mean and variance of L_{AeqT} in terms of some characteristics of N , L_{AE} , and $\tilde{L}_{\text{AeqT}}$.

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1. INTRODUCTION

Many types of environmental noise, such as industrial noise and transportation noise, arise from several individual noise events (*e.g.* forgoing operation, aircraft fly-over). Figure 1 illustrates the components of A-weighted mean square sound pressure, $p_{\text{A}}^2(t)$.

The time interval $\tau = t_2 - t_1$ is assumed to be long enough to encompass all significant noise contributions of individual events. Usually the identifiable noise events, with A-weighted mean square sound pressure, $[p_{\text{Ai}}(t)]^2$, intrude upon the background noise, $[\tilde{p}_{\text{A}}(t)]^2$, coming from a multitude of sources, none of them recognizable.

For N noise events occurring during the time interval T which is much longer than the duration of a single noise event ($T \gg \tau$), we obtain:

$$p_{\text{A}}^2(t) = \sum_{i=1}^N [p_{\text{Ai}}(t)]^2 + [\tilde{p}_{\text{A}}(t)]^2. \quad (1)$$

The recommended way of measuring environmental noise is by the equivalent continuous A-weighted sound pressure level:

$$L_{\text{AeqT}} = 10 \cdot \log \left\{ \frac{1}{T} \int_0^T [p_{\text{A}}^2(t)/p_0^2] dt \right\}, \quad (2)$$

where $p_0 = 2 \cdot 10^{-5} \text{ Nm}^{-2}$ is the reference sound pressure. Substitution of Eq. (1) into the above expression yields:

$$L_{\text{AeqT}} = 10 \cdot \log \left\{ \frac{1}{T} \sum_{i=1}^N E_{\text{Ai}} + 10^{0.1 \tilde{L}_{\text{AeqT}}} \right\}, \quad (3)$$

where $\tilde{L}_{\text{AeqT}}$ is the equivalent continuous A-weighted sound pressure level of the background noise, and

$$E_{\text{Ai}} = \int_0^T [p_{\text{Ai}}^2(t)/p_0^2] dt \quad (4)$$

expresses the A-weighted sound exposure. Assuming that beyond the time interval (t_1, t_2) the value of $p_{\text{Ai}}^2(t)$ is negligible (Fig. 1), integral (4) can be rewritten as

$$E_{\text{Ai}} = \int_{t_1}^{t_2} [p_{\text{Ai}}^2(t)/p_0^2] dt. \quad (5)$$

Making use of the above quantity we obtain the A-weighted sound exposure level,

$$L_{\text{AEi}} = 10 \cdot \log \{ E_{\text{Ai}}/t_0 \}, \quad (6)$$

where t_0 denotes the reference time, *i.e.* 1 second.

The noise events and therefore the values of L_{AEi} can usually be grouped into different categories. Under the assumption that the noise sources move along the same track, the number of categories equals the number of sources types. Traffic noise

Fig. 1 The A-weighted mean-square sound pressure of noise made up to several individual noise events.

consists mainly from two categories: passenger cars and trucks. In the case of airport noise that number increases due to the variety of aircrafts types. For two, three, *etc.* different tracks, the number of noise event categories doubles, triples, *etc.* Assuming n categories Eqs. (3) and (5) give

$$L_{AeqT} = 10 \cdot \log \left\{ \frac{t_0}{T} \sum_{k=1}^N \sum_{i=1}^{N_k} 10^{0.1 L_{AEi}^{(k)}} + 10^{0.1 \tilde{L}_{AeqT}} \right\}, \quad (7)$$

where N_k is the number of noise events of k -th category in time T : $N_1 + N_2 + \dots + N_n = N$.

In reality N_k , $L_{AEi}^{(k)}$, and $\tilde{L}_{AeqT}$ are random variables. Thus, the equivalent continuous A-weighted sound pressure level, L_{AeqT} , is random as well. The aim of this paper is to find its mean, variance, and skewness.

For convenience, the equivalent continuous A-weighted sound pressure level defined by Eq. (7) will be written as:

$$L_{AeqT} = M \cdot \ln \bar{I}_A, \quad (8)$$

where $M = 10/\ln 10$ and the time-average relative intensity $\bar{I}_A$ is given by the sum:

$$\bar{I}_A = (t_0/T) \sum_{k=1}^n \sum_{i=1}^{N_k} x_{k,i} + x_0. \quad (9)$$

The quantity x_0 denotes the time-average relative intensity of the background noise (Eq. (2)) and $x_{k,i}$ expresses the sound exposure (Eqs. (4) and (6)):

$$x_0 = 10^{0.1 \tilde{L}_{AeqT}}, \quad x_{k,i} = 10^{0.1 L_{AEi}^{(k)}}. \quad (10)$$

The time-average relative intensity, $\bar{I}_A$, is a random variable and can be described in terms of the moments of its probability distribution: $\langle \bar{I}_A \rangle$, $\langle \bar{I}_A^2 \rangle$, ...

2. MOMENTS OF THE PROBABILITY DISTRIBUTION OF L_{AeqT}

The mean, variance and skewness of the equivalent continuous A-weighted sound pressure level are defined as:

$$k_1 = \langle L_{AeqT} \rangle, \quad (11)$$

and

$$k_2 = \langle (L_{AeqT} - k_1)^2 \rangle, \quad (11)$$

$$k_3 = \langle (L_{AeqT} - k_1)^3 \rangle,$$

where $\langle u \rangle$ denotes an averaging operation with respect to u . Let us use the well-known relationships between the moment-generating function,

$$m(\phi) = \langle e^{\phi L_{AeqT}} \rangle, \quad (12)$$

and the values of k_1 , k_2 , and k_3 :

$$\ln m(\phi) = k_1 \phi + \frac{1}{2} k_2 \phi^2 + \frac{1}{6} k_3 \phi^3 + \dots \quad (13)$$

In view of the definition of equivalent continuous A-weighted sound pressure level (Eq. (8)), the moment-generating function takes the form²⁾:

$$m(\phi) = \langle \bar{I}_A^{M\phi} \rangle. \quad (14)$$

Letting $M\phi = 1, 2, 3, \dots$ the following expressions determine the moments of the probability distribution of the time-average relative intensity (Eqs. (13) and (14)):

$$\begin{aligned} \ln \langle \bar{I}_A \rangle &= \frac{1}{M} k_1 + \frac{1}{2M^2} k_2 + \frac{1}{6M^3} k_3 + \dots \\ \ln \langle \bar{I}_A^2 \rangle &= \frac{2}{M} k_1 + \frac{2}{M^2} k_2 + \frac{4}{3M^3} k_3 + \dots \\ \ln \langle \bar{I}_A^3 \rangle &= \frac{3}{M} k_1 + \frac{9}{2M^2} k_2 + \frac{9}{2M^3} k_3 + \dots \end{aligned} \quad (15)$$

By solving these linear equations, we obtain the mean (k_1), variance (k_2) and skewness (k_3) of the equivalent continuous A-weighted sound pressure level (Eqs. (11)):

$$\begin{aligned} k_1 &= M \cdot \ln \frac{\langle \bar{I}_A^3 \rangle^{1/3} \langle \bar{I}_A \rangle^2}{\langle \bar{I}_A^2 \rangle^{3/2}}, \\ k_2 &= M^2 \cdot \ln \frac{\langle \bar{I}_A^2 \rangle^4}{\langle \bar{I}_A \rangle^5 \langle \bar{I}_A^3 \rangle}, \\ k_3 &= M^3 \cdot \ln \frac{\langle \bar{I}_A^3 \rangle \langle \bar{I}_A \rangle^3}{\langle \bar{I}_A^2 \rangle^3}. \end{aligned} \quad (16)$$

3. MOMENTS OF THE PROBABILITY DISTRIBUTION OF $\bar{I}_A$

In view of Eq. (9), the time-average relative intensity, $\bar{I}_A$, depends on the number of noise events, N_k , time average energy of the background noise,

Fig. 2 The A-weighted mean-square sound pressure of noise consisting of one category of noise events.

Fig. 3 The A-weighted mean-square sound pressure of noise composed of two categories of noise events.

x_0 , and the sound exposure, x_{ki} .

To get some insight into the nature of the dependence of $\bar{I}_A$ on N_k , x_0 , and x_{ki} , we start our analysis with the simple case of noise consisting of one category of noise event, $[p_{Ai}^{(1)}]^2$ (Fig. 2).

For example, when inside a factory only one operation (e.g. forging) is performed; or within a residential area only one type of vehicle (e.g. passenger car) may pass. In these cases one can write (Eq. (9)):

$$\bar{I}_A = (t_0/T) \sum_{i=1}^N x_i + x_0, \quad (17)$$

where the following nomenclature has been used: $N_1 = N$, $x_{1i} = x_i$.

We assume that N , x_i , and x_0 are independent random variables. If $\langle \rangle_{x_i}$ denotes an averaging operation with respect to its subscript, then

$$\langle \bar{I}_A \rangle_{x_i} = (t_0/T) \sum_{i=1}^N \langle x_i \rangle + x_0. \quad (18)$$

However the operation of averaging gives the same result, $\langle x \rangle$, for the first, second and N -th event which occur in time T . Thus, expression (18) takes the form:

$$\langle \bar{I}_A \rangle_{x_i} = (t_0/T) N \langle x \rangle + x_0. \quad (19)$$

Finally, the averages with respect to variables x_0 and N yield:

$$\langle \bar{I}_A \rangle_{x_i, x_0, N} = (t_0/T) \langle N \rangle \langle x \rangle + \langle x_0 \rangle. \quad (20)$$

(The subscripts will be omitted below to simplify notation).

After a calculation similar to that in the derivation of $\langle \bar{I}_A \rangle$, we obtain:

$$\begin{aligned} \langle \bar{I}_A^2 \rangle &= (t_0/T)^2 [\langle N \rangle \langle x^2 \rangle + \langle N(N-1) \rangle \langle x \rangle^2] \\ &+ 2(t_0/T) \langle x_0 \rangle \langle N \rangle \langle x \rangle + \langle x_0^2 \rangle, \end{aligned} \quad (21)$$

$$\begin{aligned} \langle \bar{I}_A^3 \rangle &= (t_0/T)^3 [\langle N \rangle \langle x^3 \rangle + 3 \langle N(N-1) \rangle \langle x^2 \rangle \langle x \rangle \\ &+ \langle N(N-1)(N-2) \rangle \langle x \rangle^3] \\ &+ 3(t_0/T)^2 \langle x_0 \rangle [\langle N \rangle \langle x^2 \rangle + \langle N(N-1) \rangle \langle x \rangle^2] \\ &+ 3(t_0/T) \langle x_0^2 \rangle \langle N \rangle \langle x \rangle + \langle x_0^3 \rangle. \end{aligned} \quad (22)$$

In the case of two categories of noise events (Fig. 3), such as two different operations of forging (industrial noise) or two types of vehicles (traffic noise), we have (Eq. (9)):

$$\bar{I}_A = (t_0/T) \left[\sum_{i=1}^{N_1} x_{1i} + \sum_{i=1}^{N_2} x_{2i} \right] + x_0. \quad (23)$$

Proceeding along lines analogous to those used in the derivations of Eqs. (20)–(22), one gets

$$\langle \bar{I}_A \rangle = (t_0/T) [\langle N_1 \rangle \langle x_1 \rangle + \langle N_2 \rangle \langle x_2 \rangle] + \langle x_0 \rangle, \quad (24)$$

$$\begin{aligned} \langle \bar{I}_A^2 \rangle &= (t_0/T)^2 [\langle N_1 \rangle \langle x_1^2 \rangle + \langle N_1(N_1-1) \rangle \langle x_1 \rangle^2 \\ &+ \langle N_2 \rangle \langle x_2^2 \rangle + \langle N_2(N_2-1) \rangle \langle x_2 \rangle^2 \\ &+ 2 \langle N_1 \rangle \langle N_2 \rangle \langle x_1 \rangle \langle x_2 \rangle + 2(t_0/T) \langle x_0 \rangle \\ &\cdot [\langle N_1 \rangle \langle x_1 \rangle + \langle N_2 \rangle \langle x_2 \rangle] + \langle x_0^2 \rangle], \end{aligned} \quad (25)$$

$$\begin{aligned} \langle \bar{I}_A^3 \rangle &= (t_0/T)^3 [\langle N_1 \rangle \langle x_1^3 \rangle + 3 \langle N_1(N_1-1) \rangle \langle x_1^2 \rangle \langle x_1 \rangle \\ &+ \langle N_1(N_1-1)(N_1-2) \rangle \langle x_1 \rangle^3 + \langle N_2 \rangle \langle x_2^3 \rangle \\ &+ 3 \langle N_2(N_2-1) \rangle \langle x_2^2 \rangle \langle x_2 \rangle \\ &+ \langle N_2(N_2-1)(N_2-2) \rangle \langle x_2 \rangle^3 + 3 \langle N_2 \rangle \langle x_2 \rangle \\ &\cdot (\langle N_1 \rangle \langle x_1^2 \rangle + \langle N_1(N_1-1) \rangle \langle x_1 \rangle^2) \\ &+ 3 \langle N_1 \rangle \langle x_1 \rangle (\langle N_1 \rangle \langle x_1^2 \rangle + \langle N_2(N_2-1) \rangle \langle x_2 \rangle^2) \\ &+ 3(t_0/T)^2 \langle x_0 \rangle \langle N_1 \rangle \langle x_1^2 \rangle + \langle N_1(N_1-1) \rangle \langle x_1 \rangle^2 \\ &+ \langle N_2 \rangle \langle x_2^2 \rangle + \langle N_2(N_2-1) \rangle \langle x_2 \rangle^2 \\ &+ 2 \langle N_1 \rangle \langle N_2 \rangle \langle x_1 \rangle \langle x_2 \rangle + 3(t_0/T) \langle x_0^2 \rangle \\ &\cdot [\langle N_1 \rangle \langle x_1 \rangle + \langle N_2 \rangle \langle x_2 \rangle] + \langle x_0^3 \rangle], \end{aligned} \quad (26)$$

Substitution of $\langle \bar{I}_A \rangle$, $\langle \bar{I}_A^2 \rangle$, and $\langle \bar{I}_A^3 \rangle$ into Eqs. (16) yields the mean, variance, and skewness of the equivalent continuous A-weighted sound pressure level.

In the special case of one category of noise event, combination of Eqs. (16) with Eqs. (20)–(22) yields the mean and variance of L_{AeqT} as follows:

$$k_1 = M \cdot \ln(A/B) \quad (27)$$

where

$$\begin{aligned} A = & \{ (t_0/T)^3 [\langle N \rangle \langle x^3 \rangle + 3 \langle N(N-1) \rangle \langle x^2 \rangle \langle x \rangle \\ & + \langle N(N-1)(N-2) \rangle \langle x \rangle^3] + 3(t_0/T)^2 \langle x_0 \rangle \\ & \cdot [\langle N \rangle \langle x^2 \rangle + \langle N(N-1) \rangle \langle x \rangle^2] \\ & + 3(t_0/T) \langle x_0^2 \rangle \langle N \rangle \langle x \rangle + \langle x_0^3 \rangle \}^{1/3}, \\ B = & \{ (t_0/T) \langle N \rangle \langle x \rangle + \langle x_0 \rangle \}^{-2} \cdot \{ (t_0/T)^2 [\langle N \rangle \langle x^2 \rangle \\ & + \langle N(N-1) \rangle \langle x \rangle^2] + 2(t_0/T) \langle x_0 \rangle \langle N \rangle \langle x \rangle \\ & + \langle x_0^2 \rangle \}^{3/2}, \end{aligned}$$

and

$$k_2 = M^2 \cdot \ln(C/D) \quad (28)$$

where

$$\begin{aligned} C = & \{ (t_0/T)^2 [\langle N \rangle \langle x^2 \rangle + \langle N(N-1) \rangle \langle x \rangle^2] \\ & + 2(t_0/T) \langle x_0 \rangle \langle N \rangle \langle x \rangle + \langle x_0^2 \rangle \}^4 \\ & \cdot \{ (t_0/T) \langle N \rangle \langle x \rangle + \langle x_0 \rangle \}^{-5} \end{aligned}$$

with $D = \langle \bar{I}_A^3 \rangle$ given by Eq. (22).

4. CONCLUSIONS

Expressions for the average and variance (Eqs. (11)) of the equivalent continuous A-weighted sound pressure level have been derived (Eqs. (16), (24)–(26)) under the following assumptions:

—the time of observation T is constant,

—environmental noise consist of two categories

of noise events and background noise (Fig. 2),

—noise events are characterized by the A-weighted sound exposure levels ($L_{\text{AE}}^{(1)}$, $L_{\text{AE}}^{(2)}$) and their numbers (N_1 , N_2) which have occurred during the time of observation T ,

—background noise arising from nonidentifiable sources is described by its equivalent continuous A-weighted sound pressure level ($\bar{L}_{\text{AeqT}}$),

— $L_{\text{AE}}^{(1)}$, $L_{\text{AE}}^{(2)}$, N_1 , N_2 , and $\bar{L}_{\text{AeqT}}$ are independent random variables with no restriction put on their statistical distributions.

When an environmental noise is composed of one category of noise events, then Eqs. (16), (24)–(26) simplify to the form given by Eqs. (27) and (28).

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