Propagation of sound waves in a multi-layered viscoelastic tube

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Theoretical and experimental investigation are given on the propagation of sound waves in a cylindrical viscoelastic tube of multi-layered material. In the analysis, the locally reacting model is applied for the tube wall vibration. Simplified theory is presented in the case of single-layered and two-layered tubes, and the influence of the tube wall thickness on the resonance of the wall vibration is discussed. The phase velocity and absorption of sound waves are measured by using silicone and natural rubber tubes. The experimental results are in good agreement with the theoretical prediction. The same measurements for two-layered tube which is made by inserting the natural rubber tube into the silicone rubber tube are done and are compared with the theory.

Keywords: Viscoelastic tube, Locally reacting model, Complex Young's modulus, Velocity dispersion

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1. INTRODUCTION

When sound waves propagate in an elastic tube of rubber-like material, the tube wall vibrates in response to inner pressure, and the consequent motion causes the generation of velocity dispersion and sound energy dissipation. Such geometrical dispersion has been investigated theoretically and experimentally by many researchers up to this time. Recently a series of works were done by Guelke and Bunn. They used electrical-acoustical analogy and transmission line theory to simulate sound wave propagation in tubes, and derived an equivalent circuit of tube using the distributed element model. Based on their study, Capper et al. estimated the compliance of a tube wall using the acoustic input impedance method, and proposed a possible application to measurements of dynamical properties of the large airways of the lungs in human being. In the above-mentioned works, they treated single-layered tube and assumed that the tube has thin wall for theoretical simplicity. However, the actual airway has complex organization and has a finite thickness.

To approach the problem of sound propagation in complex organized tube, the present paper deals with the wave mechanism in a viscoelastic tube of multi-layered material. Coupling between sound waves and tube wall vibration is thoroughly discussed. In the analysis, it is assumed that the tube wall reacts to local sound pressure only and plane waves propagate in the tube. Simplified theory is presented in the case where the tube is made of single-layered and two-layered material. Measurements of the phase velocity and sound absorption in silicone and natural rubber tubes are carried out. The elastic moduli and mechanical constants of the tubes are estimated from the experimental data. Two-layered tube is compounded by the silicone and natural rubber tubes. The same measurements are done for the velocity dispersion and are compared with the theory.
2. WAVE EQUATION

Consider a cylindrical elastic tube filled with an ideal gas. When sound waves propagate in such tube, the tube wall vibrates in response to internal pressure. Plane wave motion in the tube is governed by the following basic equations:

\[
\frac{\partial (\rho S)}{\partial t} + \frac{\partial}{\partial x} (\rho S u) = 0 , \tag{1}
\]

\[
\rho \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x} , \tag{2}
\]

\[
P = P_0 \left( \frac{\rho}{\rho_0} \right) . \tag{3}
\]

where \( u \) is the particle velocity, \( P \) is the internal pressure, \( t \) is the time, \( x \) is the axial coordinate along the tube, \( S \) is the inner cross-sectional area, and \( \rho, \gamma \) are the density and the specific heats ratio of the gas, respectively. Subscript zero designates the value at static pressure. If it can be assumed that the motion of the tube wall obeys locally reacting model,\(^5\) the wall vibrates in response to the inner pressure. In this case a wave equation is obtained from Eqs. (1)~(3),\(^6\) and it takes the form

\[
\frac{\partial^2 \phi}{\partial x^2} - \left( \frac{1}{c_0^2} + \rho \frac{S'}{S} \right) \frac{\partial^2 \phi}{\partial t^2} = 0 , \tag{4}
\]

where \( \phi \) is the velocity potential, \( c_0 = \sqrt{\gamma P_0 / \rho_0} \) is the sound speed, and \( S' = dS/dP \). For simplicity, we assume that wave shape changes slowly with propagation. The introduction of the retarded time \( \tau = t - x/c_0 \) and a slow spatial variable \( x' = \varepsilon x \) (\( \varepsilon \) is small parameter) allow us lower the order of Eq. (4),

\[
\frac{\partial \phi}{\partial x} + \frac{\rho c_0}{2} \frac{S'}{S} \frac{\partial \phi}{\partial \tau} = 0 , \tag{5}
\]

where the terms of order \( \varepsilon^2 \) is neglected because of its smallness and the original variable \( x \) is used. Since \( S'/S \) has a frequency-dependent characteristic, phase velocity changes with frequency and sound absorption occurs simultaneously.

3. COUPLING OF TUBE WALL VIBRATION AND SOUND WAVES

3.1 Formalization

Let us suppose that a cylindrical tube has inner radius \( r_0 \). Assuming that the tube wall vibration due to inner sound pressure \( p (= P - P_0) \) is uniform along the \( x \) axis, the cross-sectional area \( S(p) \) becomes

\[
S(p) = \pi [r_0 + \xi_r(p)]^2 , \tag{6}
\]

where \( \xi_r(p) \) is the radial displacement of the inner wall. Generally, the condition of \( |\xi_r| < r_0 \) is satisfied, so it follows that

\[
S' \approx \frac{2}{r_0} \left( \frac{d \xi_r}{dp} \right) . \tag{7}
\]

Elastic wave motion in a homogeneous isotropic medium is described by\(^7\)

\[
\rho \frac{\partial^2 \xi}{\partial t^2} = (\lambda + 2\mu) \nabla \cdot \nabla \cdot \xi - \mu \nabla \times \nabla \times \xi , \tag{8}
\]

where \( \xi \) is the displacement vector, \( \rho \) is the tube density, and \( \lambda, \mu \) are Lamé's constants. \( \lambda \) and \( \mu \) are expressed in terms of Young's modulus \( E \) and Poisson's ratio \( \sigma \) as follows,

\[
\lambda = \frac{E \sigma}{(1+\sigma)(1-2\sigma)} , \quad \mu = \frac{E}{2(1+\sigma)} . \tag{9}
\]

Since the wall vibration is axially symmetric and locally reacting, \( \xi \) is reduced to the radial component \( \xi_r \) only. Then substitution of harmonic excitation \( \xi_r = \tilde{\xi}_r(r) e^{i\omega t} \) into Eq. (8) yields

\[
\frac{d^2 \tilde{\xi}_r}{dr^2} + \frac{1}{r} \frac{d \tilde{\xi}_r}{dr} + \left( \kappa^2 \frac{\rho c_0^2}{S} \right) \tilde{\xi}_r = 0 , \tag{10}
\]

in which \( \kappa = \omega c_0 / \rho_0 \) is the wave number of longitudinal wave propagating with the speed \( c_l = \sqrt{(\lambda + 2\mu)/\rho_0} \).

Equation (10) has the solution

\[
\tilde{\xi}_r = a J_1(\kappa r) + b N_1(\kappa r) , \tag{11}
\]

where \( a \) and \( b \) are the constants which are determined by appropriate boundary conditions, and \( J_1, N_1 \) are the first-order Bessel and Neumann functions, respectively. Using the cylindrical coordinate system, the relation between radial stress \( T \) and displacement \( \xi_r \) is given by\(^7\)

\[
T = (\lambda + 2\mu) \frac{\partial \xi_r}{\partial r} + \lambda \frac{\xi_r}{r} . \tag{12}
\]

Now we turn to a discussion of sound propagation in a multi-layered tube. Figure 1 shows a model of \( n \)-layered tube. The inner radius is \( r_n \), the radii to the interface between the \( i \)th and \( (i+1) \)th layer is \( r_i \) (\( i = 1, 2, \cdots, n-1 \)), and the outer radius is \( r_n \). Furthermore Lamé's constants and the density in each layer are \( \lambda_i, \mu_i \) and \( \rho_i \) (\( i = 1, 2, \cdots, n \)), respectively. \( \xi_r \) and \( T \) in the \( i \)th layer satisfy the following equations

\[
If internal pressure $P_0 e^{i\omega t}$ acts on the unit area of the wall and the continuity of stress is imposed at each interface, the radial stress is given by

$$T(i) = (\lambda_i + 2\mu_i) \frac{\partial \xi(i)}{\partial r} + \lambda_i \frac{\xi(i)}{r}.$$  

The last layer terminates in a free surface, therefore

$$T|_{r=r_i} = 0.$$  

Furthermore the continuity condition of the displacement leads to

$$\xi(i)|_{r=r_{i-1}} = \xi(i)|_{r=r_{i+1}}.$$  

The constants $a_i$ and $b_i$ are obtained from Eqs. (13), (14), and (15),

$$a_i = \frac{1}{D_i} \left[ -P_i r_{i-1} \left( (\lambda_i + 2\mu_i) k_i r_i N_i ' (\kappa, r_i) 
+ \lambda_i N_i (\kappa, r_i) \right) 
+ P_i r_i \left( (\lambda_i + 2\mu_i) k_i r_i N_i ' (\kappa, r_{i-1}) 
+ \lambda_i N_i (\kappa, r_{i-1}) \right) \right].$$

$$b_i = \frac{1}{D_i} \left[ P_i r_{i-1} \left( (\lambda_i + 2\mu_i) k_i r_i J_i ' (\kappa, r_i) 
+ \lambda_i J_i (\kappa, r_i) \right) 
- P_i r_i \left( (\lambda_i + 2\mu_i) k_i r_i J_i ' (\kappa, r_{i-1}) 
+ \lambda_i J_i (\kappa, r_{i-1}) \right) \right],$$

where

$$D_i = (\lambda_i + 2\mu_i) k_i r_{i-1} \left( J_i ' (\kappa, r_{i-1}) N_i (\kappa, r_i) 
- J_i (\kappa, r_i) N_i ' (\kappa, r_{i-1}) \right).$$

By use of Eqs. (13) and (18), it is possible to derive the displacement at $r=r_0$. The result is

$$\xi(i)|_{r=r_0} = -\frac{P_r r_0}{D_i} \left[ (\lambda_i + 2\mu_i) k_i r_i J_i (\kappa, r_0) 
- J_i (\kappa, r_0) N_i (\kappa, r_i) \right].$$

and a prime denotes differentiation with respect to $r$. Substituting Eq. (18) into Eq. (13) and taking the boundary conditions of Eq. (17) into account, a recurrence formula for $P_i$ is obtained

$$P_i = \frac{2}{\pi G_{\text{r}}} \left[ D_{i+1} r_{i+1} (\lambda_i + 2\mu_i) P_{i+1} \right],$$

where

$$G_i = D_i \left[ (\lambda_i + 2\mu_i) k_i r_{i+1} \left( J_i (\kappa, r_{i+1}) N_i (\kappa, r_{i+1}) 
- J_i (\kappa, r_{i+1}) N_i (\kappa, r_{i-1}) \right) 
+ \lambda_i \left( J_i (\kappa, r_{i+1}) N_i (\kappa, r_{i-1}) 
- J_i (\kappa, r_{i-1}) N_i (\kappa, r_{i+1}) \right) \right]$$

and $R_i$ is the ratio of $P_i$ to $P_{i-1}$ and is related to the following formula,

$$R_i = \frac{P_i}{P_{i-1}} = \frac{2D_{i+1} r_{i+1} (\lambda_i + 2\mu_i)}{\pi G_{\text{r}} - 2D_{i+1} (\lambda_i + 2\mu_i) R_{i+1}},$$

and $R_n$ is obtained from the boundary condition Eq. (16),

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Since \( \frac{d\xi_t}{d\rho} \) is equal to the ratio of \( \xi_t|_{r=r_0} \) to \( P_0 \), \( S'/S \) is obtained from Eqs. (7) and (20),

\[
\frac{S'}{S} = -\frac{2}{D_1} \left[ (\lambda_1 + 2\mu_1)k_0 r_1 (J_1(\kappa_0 r_0) K_0(\kappa_0 r_0) - J_0(\kappa_0 r_0) K_1(\kappa_0 r_0)) + \lambda_1 J_0(\kappa_0 r_0) K_0(\kappa_0 r_0) - J_0(\kappa_0 r_0) K_1(\kappa_0 r_0) - \frac{2r_1}{\pi r_0} (\lambda_1 + 2\mu_1) R_1 \right].
\]

Equations (5) and (23) tell us that the elasticity of tube determines apparently wave motion in the tube.

### 3.2 Single-Layer Tube

If the tube is made of single-layered material, Eq. (23) can be simplified. Assuming the wavelength of longitudinal wave in the material is much longer than the tube dimensions \( r_0 \) and \( r_1 \); i.e., \( \kappa_0 r_0, \kappa_0 r_1 \ll 1 \), and then using asymptotic expansions of the Bessel and Neumann functions, Eq. (23) becomes

\[
\frac{S'}{S} = \frac{(1 - \sigma_1^2)(1 + 2\sigma_1 H_1/(1 - \sigma_1) + H_1^2)}{1 - (\omega/\omega_0)^2 + f_0 \eta_1/\gamma_1},
\]

where

\[
\omega_0 = \frac{1}{d_1} \sqrt{\frac{8(1 - \sigma_1)H_1\gamma_1}{\rho_0 (1 + \sigma_1) (2 - 2\sigma_1)(3 - 2\sigma_1)(1 + H_1^2)H_1 + (1 - H_1^2)\ln \left( \frac{(1 + H_1)/(1 - H_1)}{1 + H_1 \gamma_1} \right)}}
\]

is a resonance frequency of the tube wall vibration, \( d_1 = (r_0 + r_1)/2 \) is the mean radius, \( H_1 = (r_1 - r_0)/2d_1 \) is the normalized thickness, \( \sigma_1, \eta_1 \) are the dynamic modulus and viscosity, respectively. The complex Young’s modulus is expressed as \( E_1 = \gamma_1 + j\omega \eta_1 \). The expression of \( \omega_0 \) shows that the resonance frequency depends on not only the elastic modulus but also the tube thickness. According to the electrical-acoustical analogy, wall vibration can be modeled as a single degree of freedom resonator whose equivalent circuit is the series of three mechanical elements; i.e., compliance \( C_m \), resistance \( R_m \), and mass \( M \) per unit axial length. Then it follows that

\[
\frac{S'}{S} = \frac{4\pi C_m}{1 - (\omega/\omega_0)^2 + f_0 R_m C_m},
\]

where \( \omega_0 = 1/\sqrt{MC_m} \). The relations between the mechanical and elastic constants are obtained by comparing Eq. (24) with Eq. (25), that is

\[
C_m = \frac{(1 - \sigma_1^2)(1 + 2\sigma_1 H_1/(1 - \sigma_1) + H_1^2)}{4\pi H_1 \gamma_1},
\]

\[
R_m = \frac{4\pi H_1 \eta_1}{(1 - \sigma_1^2)(1 + 2\sigma_1 H_1/(1 - \sigma_1) + H_1^2)},
\]

\[
M = 4\pi \rho_0 d_1^2 (2 - 2\sigma_1)(3 - 2\sigma_1)(1 + H_1^2)H_1 + (1 - H_1^2)\ln \left( \frac{(1 + H_1)/(1 - H_1)}{1 + H_1 \gamma_1} \right),
\]

If \( H_1 \leq 1/100 \) is satisfied, Eq. (26) becomes

\[
C_m \simeq \frac{(1 - \sigma_1^2)}{4\pi H_1 \gamma_1},
\]

\[
R_m \simeq \frac{4\pi H_1 \eta_1}{(1 - \sigma_1^2)},
\]

\[
M \simeq 4\pi \rho_0 d_1^2 H_1,
\]

\[
\omega_0 \simeq \frac{1}{d_1} \sqrt{\frac{\gamma_1}{\rho_0 (1 - \sigma_1^2)}}.
\]

In this case, the difference between Eq. (26) and Eq. (27) is the order of \( 10^{-8} \), and the resonance frequency is independent of the tube thickness. However, such thin viscoelastic tube of \( H_1 \leq 1/100 \) is apt to yield and is not able to maintain its circular cross section.

In addition to sound energy dissipation due to the wall vibration, other dissipations such as heat conduction loss at the wall and in air have to be taken into account in exact description of the wave equation.\(^9\) These losses \( \alpha(\omega) \) may be included in an ad hoc manner by rewriting Eq. (5) in the form

\[
\frac{\partial \phi}{\partial x} + \frac{\rho_0 c_0 S'}{S} \frac{\partial \phi}{\partial \tau} = -\alpha(\omega) \phi.
\]
This equation contains the mixed terms which are described in both time and frequency domains. To be strictly accurate, we should express such terms by means of the convolution integral. For example, the second term in the left-hand side corresponds to the expression
\[
\frac{\rho_c c_0 S'}{2} \frac{\partial \phi}{\partial \tau} \rightarrow \int_{-\infty}^{\infty} g(\tau') \frac{\partial \phi(\tau - \tau')}{\partial \tau'} d\tau'.
\]
When \( \xi = \omega R_0 C_m < 1 \), \( g(\tau) \) is given by the inverse Fourier transform of \( \rho_c c_0 S'/2S \),
\[
g(\tau) = 2\pi \rho_c c_0 C_m \int_{-\infty}^{\infty} e^{j\omega \tau} \frac{1}{1 - (\omega/\omega_0)^2 + j\omega R_0 C_m} d\omega
\]
for \( \tau < 0 \)
\[
= \frac{2\pi \rho_c c_0 C_m}{\sqrt{1 - \xi^2}} e^{-\omega_0 t} \sin(\omega \sqrt{1 - \xi^2} \tau) \text{ for } \tau > 0.
\]
From Eq. (28), the phase velocity \( c_p \) and absorption coefficient \( \alpha \) are easily obtained,
\[
c_p = \frac{c_0}{1 + \rho_c c_0 \sqrt{S'/2S}}
\]
\[
\alpha = \frac{c_0}{2} \left( \frac{S'}{S} \right)^\alpha,
\]
where \( \Re[ ] \) and \( \Im[ ] \) are the real and imaginary parts of \( S'/S \), respectively.

### 3.3 Two-Layer Tube

Now we suppose a two-layered tube of thin wall; i.e., \( H_1, H_2 \leq 1/100 \), where \( H_i = (r_i - r_0)/2d_1 \) and \( H_s = (r_s - r_0)/2d_2 \) are the normalized thickness, and \( d_1 = (r_0 + r_1)/2 \) and \( d_2 = (r_1 + r_2)/2 \) are the mean radii (The equations for finite thickness of the tube wall are described in the Appendix.). Then mechanical elements of the first and second layers are given by Eq. (27),
\[
C_{m1,2} = \frac{1 - \sigma_s}{4\pi H_{s1,2}}
\]
\[
R_{m1,2} = \frac{4\pi H_{s1,2} \gamma_{1,2}}{(1 - \sigma_s)}
\]
\[
M_{s1,2} = 4\rho r d_{1,2}^2 H_{s1,2}.
\]

In this case, two-layered system is expressed as the series connection of two single-layered equivalent circuits. Hence the system has only one resonance frequency. The new mechanical elements \( C_m, R_m, \) and \( M \) are

\[
C_m = \frac{C_{m1} C_{m2}}{C_{m1} + C_{m2}}
\]
\[
(1 - \sigma_s) \left( 1 - \sigma_s \right)
\]
\[
R_m = R_{m1} + R_{m2}
\]
\[
= 4\pi \left( 1 - \sigma_s \right) H_{s1} \gamma_1 + (1 - \sigma_s) H_{s2} \gamma_2
\]
\[
M = M_1 + M_2 = 4\pi [\rho d_1^2 H_1 + \rho d_2^2 H_2].
\]

In the same way as the single-layered case, it is possible to calculate the phase velocity and absorption coefficient using Eqs. (25) and (31).

### 4. EXPERIMENT AND DISCUSSION

Figure 2 shows a block diagram of the experimental apparatus. Two types of rubber tubes were used. The tube #1 is made of silicone rubber material, and the tube #2 is natural rubber. Each tube is 10 m in length. The dimensions of the tubes are shown in Table 1. Sound source is a small speaker of earphone type, and is jointed with the tubes directly. Tone-burst sinusoids of frequency range from 100 Hz~16 kHz were used as an input signal. A prove tube microphone of 1 mm in diameter was inserted in the holes at the outer side of the tubes. Measuring points are A, B, and C whose distances are 0.25, 1.25, and 3.25 m from the source, respectively. The phase velocity

![Fig. 2 Block diagram of the experimental apparatus.](image)

Table 1 Dimensions and density of the rubber tubes.

<table>
<thead>
<tr>
<th>Tube</th>
<th>Inner diameter 2r0 (mm)</th>
<th>Thickness h (mm)</th>
<th>Density ( \rho_0 ) (kg/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>10.0</td>
<td>1.5</td>
<td>1.150</td>
</tr>
<tr>
<td>#2</td>
<td>7.9</td>
<td>1.6</td>
<td>1.010</td>
</tr>
</tbody>
</table>
was measured by the transit time of signal between the points A and C. The absorption coefficient was also measured by the ratio of pressure amplitudes at the points A and B.

The phase velocity and absorption coefficient of both tubes are shown in Fig. 3. Solid and short-dashed lines are the best-fitted curves based on Eqs. (25), (31), and the experimental data, which are represented by symbols. The theory and experiment are in good agreement as a whole. The main resonance frequencies due to axially symmetric vibration of each tube are 3.3 kHz and 2.1 kHz, respectively. The phase velocities change abruptly near the resonance from 290 to 410 m/s for the tube #1, and from 270 to 420 m/s for the tube #2. The absorption coefficients are attained to 17 and 21 neper/m at the respective resonance frequency. It should be noted that there exists another small resonance at near 600 Hz for the tube #2. This is because flexural vibration mode causes the change of the tube cross-sectional area. The present theory includes the effect of the flexural vibration on the dispersion in accordance with the previous analysis.

The elastic and mechanical constants of the tubes were estimated and the results are shown in Table 2. Poisson’s ratio of rubber material is typically 0.48. For both silicone and natural rubber materials, \( \eta \) and \( R_m \) are almost the same magnitudes, however the other constants are obviously different. The dynamical modulus \( \gamma \) of the silicone is several times larger than that of the natural rubber. It is important to distinguish between actual and effective mass. The former is a static mass which is determined by the density and volume, whereas the other is a dynamical mass dependent on the tube geometry and thickness. The effective mass is generally smaller than the actual mass. For example, the mass of the tube #1 is actually 0.062 kg/m, whose value corresponds to the thin tube theory of Eq. (27). However, the effective mass is estimated as 0.048 kg/m, and is reasonably agreement with the calculated value using Eq. (26). The effective mass \( M \) of the tube #2 is a little smaller than that of the other tube because of its different size and density. Incidentally in the figures the best-fitted curves are shown in long-dashed lines based on the thin wall assumption.

Two-layered tube was made by inserting the tube #2 into #1. Figure 4 shows the dispersion curves of the compound tube. Solid line is the theoretically predicted curve based on Eqs. (31) and (A.1). The tube has a single resonance at 2.5 kHz which ranges between two resonance frequencies in Fig. 3, however a little close to the frequency of the inner tube #2. The change of phase velocity near the resonance becomes a little small in comparison with those in Fig. 3. The same trend also can be seen for the absorption data. The agreement between the theory and experiment is not so good as we expected. The reason which such deviation

| Table 2 Elastic constants \( \gamma \) and \( \eta \), and mechanical constants \( C_m \), \( R_m \), and \( M \) of the rubber tubes. |
|---|---|---|---|---|
| Tube | \( \gamma \) (Pa) | \( \eta \) (Pa·s) | \( C_m \) (ms²/kg) | \( R_m \) (kg/ms) | \( M \) (kg/m) |
| #1 | 1.2 \times 10^2 | 73 | 4.9 \times 10^{-9} | 120 | 0.048 |
| #2 | 2.9 \times 10^2 | 74 | 1.7 \times 10^{-7} | 150 | 0.034 |
appears is probably an insufficient contact of two tubes. The insertion process of tubes might change their elastic properties. Adequate experimental arrangement should be needed for perfect comparison with the theory.

5. CONCLUSION

Coupling mechanism between sound waves in a viscoelastic tube and the tube wall vibration due to internal pressure has been investigated theoretically and experimentally. The phase velocity and sound absorption in silicone and natural rubber tubes were measured and were compared with the theory. They were in relatively good agreement for single-layered tube. However, satisfactory agreement was not observed for two-layered tube probably because of insufficient arrangement of the experiment.

In general, an assumption of thin tube wall is made for simple analysis and theory of sound propagation in the tube. Within the framework of the present study, however, such thin wall assumption is not enough satisfied for theoretical description even if the wavelength is much larger than the radius and thickness.

REFERENCES

7) P. M. Morse and H. Feshbach, Methods of Theoretical Physics. Part I (McGraw-Hill Book Company, New York, 1953), Sec. 2.2.

A. APPENDIX

Likewise in Sec. 3.2, Eq. (23) can be rewritten for two-layered system. Assuming $K_f r_0$, $K_f r_1$, $K_f r_2$, $K_f R_2 \ll 1$, Eq. (23) becomes
where

\[
\frac{S'}{S} = 4\pi C_m \frac{1 - (\omega/\omega_0)^2 + j\omega \tau_1}{[1 - (\omega/\omega_0)^2 + j\omega \tau_2][1 - (\omega/\omega_0)^2 + j\omega \tau_3]},
\]

Equation (A.1) have one zero and two poles whose resonance frequencies correspond to \(\omega_1\), \(\omega_2\), and \(\omega_3\), respectively. The phase velocity and absorption coefficient can be calculated by using Eqs. (25) and (31). If \(H_1, H_2, \ldots \leq 1/100\) are satisfied, the constants \(C_n, \tau_1, \tau_2, \tau_3\) in Eq. (A.1) can be simplified, and

\[
\omega_1 = \sqrt{\frac{8(1 - \sigma_1)(1 - \sigma_3)K_1 H_1 \gamma_1 + K_3 H_2 \gamma_2}{(1 - \sigma_1)K_1 L_1 + (1 - \sigma_3)K_3 L_2}},
\]

\[
\omega_2 = \sqrt{\frac{8(1 - \sigma_3)H_1 \gamma_1}{L_1}},
\]

\[
\omega_3 = \sqrt{\frac{8(1 - \sigma_2)(1 - \sigma_3)K_1 H_2 \gamma_1 + K_3 H_1 \gamma_2}{(1 - \sigma_2)K_1 L_1 + (1 - \sigma_3)K_3 L_2}},
\]

Equation (A.1) have one zero and two poles whose resonance frequencies correspond to \(\omega_1\), \(\omega_2\), and \(\omega_3\), respectively. The phase velocity and absorption coefficient can be calculated by using Eqs. (25) and (31). If \(H_1, H_2, \ldots \leq 1/100\) are satisfied, the constants \(C_n, \tau_1, \tau_2, \tau_3\) in Eq. (A.1) can be simplified,

\[
C_n = \frac{K_1 K_2 H \gamma_1 + K_3 H \gamma_2}{4\pi \gamma_1 [K_1 H \gamma_1 + K_1 H \gamma_2]},
\]

\[
\tau_1 = \frac{K_1 K_2 \eta_1 + K_3 H \eta_2}{K_1 K_2 \gamma_1 + K_3 H \gamma_2},
\]

\[
\tau_2 = \frac{\eta_1}{\gamma_1},
\]

\[
\tau_3 = \frac{K_1 H \eta_1 + K_2 H \eta_2}{K_1 H \gamma_1 + K_2 H \gamma_2},
\]

and

\[
K_1 = (1 + \sigma_1)(1 - \sigma_3)(1 + H_1)^2 + 2\sigma_1 H_1,
\]

\[
K_2 = (1 + \sigma_2)(1 - \sigma_2)(1 + H_2)^2 + 2\sigma_2 H_2,
\]

\[
K_3 = 4(1 - \sigma_3)^2(1 - 2\sigma_2)H_1,
\]

\[
L_1 = \rho d_1^4(1 + \sigma_1)[2(1 - 2\sigma_2)(1 - 2\sigma_3)(1 + H_2)^2 + (1 - H_1)^2 \ln{(1 + H_1)(1 - H_1)}],
\]

\[
L_2 = \rho d_2^4(1 + \sigma_2)[2(1 - 2\sigma_2)(1 - 2\sigma_3)(1 + H_1)^2 + (1 - H_2)^2 \ln{(1 + H_2)(1 - H_2)}].
\]

Equation (A.1) have one zero and two poles whose resonance frequencies correspond to \(\omega_1\), \(\omega_2\), and \(\omega_3\), respectively. The phase velocity and absorption coefficient can be calculated by using Eqs. (25) and (31). If \(H_1, H_2, \ldots \leq 1/100\) are satisfied, the constants \(C_n, \tau_1, \tau_2, \tau_3\) in Eq. (A.1) can be simplified,

\[
C_n \approx \frac{(1 - \sigma_2)^2(1 - \sigma_3)^2}{4\pi[(1 - \sigma_2)^2 H_1 \gamma_1 + (1 - \sigma_3)^2 H_2 \gamma_2]},
\]

\[
\tau_1 \approx \tau_2 = \frac{\eta_1}{\gamma_1},
\]

\[
\tau_3 \approx \frac{(1 - \sigma_2)^2 H_1 \eta_1 + (1 - \sigma_3)^2 H_2 \eta_2}{(1 - \sigma_2)^2 H_1 \gamma_1 + (1 - \sigma_3)^2 H_2 \gamma_2},
\]

and

\[
\omega_1 \approx \omega_2 \approx \frac{1}{d_1} \sqrt{\frac{\gamma_1}{\rho_1(1 - \sigma_2)^2}},
\]

\[
\omega_3 \approx \sqrt{\frac{(1 - \sigma_2)^2 H_1 \gamma_1 + (1 - \sigma_3)^2 H_2 \gamma_2}{(1 - \sigma_1)^2[\rho_1 d_2^2 H_1 + \rho_2 d_2^2 H_2]}}.
\]

Consequently, \(\frac{S'}{S}\) becomes

\[
\frac{S'}{S} \approx \frac{4\pi C_n}{1 - (\omega/\omega_0)^2 + j\omega \tau_3},
\]

where \(\omega_0 = 1/\sqrt{MC_n}\) and \(\tau_3 = R_m C_n\). These simplified equations agree with the results in Sec. 3.3.