Finite element simulation of slant cantilever beam and approximate equation formula of its resonance frequency

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The first part of this paper deals with the finite element simulation of a slant cantilever beam, and the second part contains an approximate equation formula of its resonance frequency. We investigated such vibrational behaviors of a slant cantilever beam because we had to design a large-diameter torsional vibrator with many slant slits cut into its end-face for a rod-type ultrasonic motor. Also, we thought that the simulated and investigated results on the slant cantilever beam might be very useful in other applications, for example, vibrational sensors. In this paper, therefore, we present some finite element simulations of a slant cantilever beam and we investigate the results of its approximate equation formula, as academic research material.

Keywords: Slant cantilever beam, Finite element simulation, Approximate equation formula, Vibration mode pattern, Resonance frequency

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1. INTRODUCTION

We have previously endeavored to develop several kinds of ultrasonic motors.¹⁻³⁾ One of them is the rod-type ultrasonic motor.⁴⁾ This ultrasonic motor has a large diameter and is designed for high torque. The ability of an ultrasonic motor depends on the vibrational characteristics of the stator vibrator. Therefore, we wanted to use a large-diameter torsional vibrator with many slant slits cut into its end-face, so that, in order to rotate the rotor, their flexural vibration was necessary. In this case, fundamental vibration-characteristics of the slant cantilever beam were important for good performance, and then we realised that vibration-characteristics of the slant cantilever beam, for example, its resonance frequencies and vibration modes, had to be simulated. From such practical engineering demands, we simulated the slant cantilever beam. The simulation was done by the finite element method (FEM) of the modified SAPIV Structural Analysis

Program.⁵⁾ From the simulated results, we discovered resonance frequencies and vibration modes of the slant cantilever beam. Moreover, we investigated an approximate equation formula for the flexural vibration resonance frequency of a slant cantilever beam.

This paper deals with the finite element simulation of the slant cantilever beam and the approximate equation formula of its resonance frequency. The simulated results for the cantilever beam will be very useful not only in designing a torsional stator vibrator for an ultrasonic motor, but also in other applications of the slant cantilever beam. Therefore, we present some simulated data of the slant cantilever beam as academic research material.

2. FINITE ELEMENT SIMULATION OF A SLANT CANTILEVER BEAM

2.1 Slant Cantilever Beam

The slant cantilever beam dealt with in this paper

is shown in Fig. 1, together with a normal cantilever beam. The slant cantilever beam in Fig. 1(b) is the same as the normal cantilever beam in Fig. 1(a) when inclined horizontally. The detailed form of the slant cantilever beam is shown in Fig. 2. The main problem we wanted to consider is the flexural vibration characteristic in the x - y horizontal plane; however, in the finite element simulation, some fundamental modes of flexural vibration in the x - z vertical plane and longitudinal or torsional vibration are also considered.

2.2 Finite Element Method (FEM)

The slant cantilever beam shown in Fig. 2 was divided as shown in Fig. 3 for the FEM simulation. There are 385 nodal points and 240 divided elements. It is thought that these divisions in Fig. 3 might be enough for simulation of the first and second flexural vibrations, the first longitudinal vibration, the first torsional vibration. In the FEM simulation, it was supposed that the slant cantilever

beam was composed of piezoelectric ceramics, and for its material constants, the ones of the piezoelectric ceramics C-201 of Fuji Ceramics Co. Ltd. were used. The material constants of C-201 piezoelectric ceramics are presented in Table 1.

The FEM program used in the simulation is the SAPIV structural analysis program, which was modified to deal with piezoelectric vibration and is fundamentally applicable to anisotropic material and three-dimensional vibration problems.⁵⁾ The boundary condition for the FEM simulation was that nodal points of the support part were fixed and the other nodal points were free.

Fig. 1 Normal cantilever beam and slant cantilever beam.

Fig. 2 Form of slant cantilever beam.

Fig. 3 Finite element divisions of slant cantilever beam.

Table 1 Material constants of piezoelectric ceramics C-201 (Fuji Ceramics Co. Ltd.).

Property	Symbol	Unit	
Coupling factor	k_r		0.595
	k_{31}		0.346
	k_{33}		0.714
	k_t		0.473
	k_{15}		0.735
C constant	C_{11}^E	N/m ²	14.9×10^{10}
	C_{12}^E	N/m ²	8.7×10^{10}
	C_{13}^E	N/m ²	9.1×10^{10}
	C_{33}^E	N/m ²	13.2×10^{10}
	C_{44}^E	N/m ²	2.5×10^{10}
	C_{66}^E	N/m ²	3.1×10^{10}
e constant	e_{31}	C/m ²	-4.5
	e_{33}	C/m ²	17.5
	e_{15}	C/m ²	15.2
ϵ constant	$\epsilon_{11}^S/\epsilon_0$		904
	$\epsilon_{33}^S/\epsilon_0$		818
Poisson's ratio	σ^E		0.29
Mechanical quality factor	Q_m		889
Density	ρ	kg/m ³	7.82×10^3

2.3 Simulated Results⁶⁾

In the simulation, we gave the simulated vibration modes simplified names, as shown in Table 2. Simulations of the resonance frequency f_r and mode patterns were done, for example, under such conditions that width d , thickness t , and slant angle θ of the cantilever beam were changeable, but the length l is kept constant at $l=10$ mm. Simulated vibration mode patterns of the slant cantilever beam at the slant angle $\theta=30^\circ$ are shown in Figs. 4~9, in

comparison with the vibration modes of the normal cantilever beam. It was clear that, in the case of the slant cantilever beam, the vibration mode patterns were much influenced and modified; that is, the following results were obtained:

- (1) The L1-mode was found to be modified to the H2-mode by inclining the normal cantilever beam; that is, distinction between the L1- and H2-mode vibrations was not so easy, and it can be said that the L1-mode accompanied the

Table 2 Expression of vibration modes.

Vibration mode	Simplified name
First flexural vibration (horizontal direction)	H1-mode
Second flexural vibration (horizontal direction)	H2-mode
First flexural vibration (vertical direction)	V1-mode
Second flexural vibration (vertical direction)	V2-mode
First longitudinal vibration	L1-mode
First torsional vibration	T1-mode

Fig. 4 Simulated vibration mode patterns of L1-mode.

Fig. 5 Simulated vibration mode patterns of T1-mode.

(a) Normal cantilever beam

(b) Slant cantilever beam ($\theta=30^\circ$)

Fig. 6 Simulated vibration mode patterns of H1-mode.

H2-mode in the slant cantilever beam. Therefore, the top of the cantilever beam makes an elliptic motion. It is thought that this vibration mode could be applied to an actuator due to the elliptic motion.

- (2) In the T1-mode, by inclining the normal cantilever beam, the symmetry of the form, which is regarded as the vibration axis of the torsional vibration, was lost. Therefore, an asymmetrical torsional vibration occurred in the cantilever beam. In other words, the component of the flexural vibration was largely contained in the displacements.

- (3) H1-mode and H2-mode

In both the H1-mode and the H2-mode, displacement in the longitudinal-direction was found. Therefore, an elliptic motion occurred on the top of the slant cantilever beam in the same manner as in the L1-mode, but the direction of its rotation was different. Moreover, it was found that the resonance frequencies of the

(a) Normal cantilever beam

(b) Slant cantilever beam ($\theta=30^\circ$)

Fig. 7 Simulated vibration mode patterns of H2-mode.

H1- and H2-modes decreased with the slant angle. Therefore, we concluded that the simulated results of these vibration modes could be applied to the construction of an ultrasonic motor.

- (4) V1-mode and V2-mode

Components of torsional vibration were also found in the displacement of both V1- and V2-modes. It was thought that slippage of the symmetry axis was caused by inclination. Moreover, by inclination, such simple and pure flexural vibration, as is found in the normal cantilever beam, was not found.

3. APPROXIMATE EQUATION FORMULA OF FLEXURAL VIBRATION RESONANCE FREQUENCY

3.1 Resonance Frequency of a Cantilever Beam

The resonance frequency of a normal cantilever

(a) Normal cantilever beam

(b) Slant cantilever beam ($\theta=30^\circ$)

Fig. 8 Simulated vibration mode patterns of V1-mode.

beam flexurally vibrating, as shown in Fig. 1(a) and Fig. 2 ($\theta=0^\circ$), can be expressed by the following equation

$$\begin{aligned} f_n &= \frac{\alpha_n^2}{2\pi l^2} \sqrt{\frac{EI}{\rho A}} \\ &= \frac{\alpha_n^2}{2\pi l^2} \sqrt{\frac{E}{\rho} \frac{d}{\sqrt{12}}}, \end{aligned} \quad (1)$$

where each parameter has the following meaning ;

- n : resonance mode number
- α_n : frequency constant, $\alpha_1 = 1.875$
- E : Young's modulus
- I : second moment of area
- ρ : material density
- A : cross-sectional area ($d \times t$)

In the case of the slant cantilever beam, if an equation formula such as Eq. (1) is obtained, it can be very useful for applying the slant cantilever beam to an actuator or a sensor device, and so on. There-

(a) Normal cantilever beam

(b) Slant cantilever beam ($\theta=30^\circ$)

Fig. 9 Simulated vibration mode patterns of V2-mode.

fore, we determined the approximate equation formula of resonance frequency on the slant cantilever beam flexurally vibrating in the x - y plane in Fig. 2.

3.2 Approximate Equation Formula of Slant Cantilever Beam Resonance Frequency

Before obtaining the approximate equation formula of resonance frequency, we determined Young's modulus E in Eq. (1) by following the considerations in the FEM simulation, because the piezoelectric material C-201 used in the simulation is an anisotropic elastic body : that is, in the case of the normal cantilever beam, Eq. (2) is obtained from Eq. (1) as :

$$E = f_n^2 \left(\frac{2\pi l^2}{\alpha_n^2} \right)^2 \frac{\rho A}{I}. \quad (2)$$

Therefore, substituting the simulated 1st flexural resonance frequency in Eq. (2), Young's modulus E was obtained with precision as shown in Table 3

and determined as $E=9.356 \times 10^{11}$ [gcm⁻¹s⁻²]. Therefore, using Eq. (1) and applying Young's modulus E , as determined above, the 1st flexural resonance frequency was calculated as shown in Table 4, when the length l and form-ratio t/d of a normal cantilever were 10 mm and 0.1, respectively. Consequently, it is estimated that the reliability of the FEM simulation was improved by applying the Young's modulus E , as determined above.

Using these results, we derived the approximate equation formula which could be applied to the flexural resonance frequency of a slant cantilever beam, under such a condition that the cantilever beam was made to slant with its constant length l in Fig. 2. Therefore, the width d' of the slant cantilever beam is related to the width d of a normal cantilever beam as follows :

$$d' = d \cos \theta \quad (3)$$

Substituting Eq.(3) into Eq. (1), Eq. (1) is modified to Eq. (4).

$$\begin{aligned} f_{Hn}'' &= \frac{\alpha_n^2}{2\pi l^2} \sqrt{\frac{E}{\rho}} \frac{d'}{\sqrt{12}} \\ &= \frac{\alpha_n^2}{2\pi l^2} \sqrt{\frac{E}{\rho}} \frac{d}{\sqrt{12}} \cos \theta \end{aligned} \quad (4)$$

If the width d of the cantilever beam becomes width d' of the slant cantilever beam, the resonance frequency constant α_n will be changed to a function of d' , because it depends on d'/l . The values of α_n'

were obtained on Ref.7). Therefore, Eq. (4) becomes Eq. (5).

$$f_{Hn}' = \frac{\alpha_n'^2}{2\pi l^2} \sqrt{\frac{E}{\rho}} \frac{d}{\sqrt{12}} \cos \theta \quad (5)$$

α_n' : frequency constant
(slant cantilever beam)

The f_{Hn}' calculated from Eq. (5) and the simulated resonance frequency from the FEM simulation are shown in Fig. 10. From these results, it was found that the resonance frequency was changed by the dimensional ratio d/l ; however, the theoretical resonance frequency from Eq. (5) and the simulated frequency were almost equal to each other when $d/l=0.1$, and then as the ratio d/l increases from 0.1 to 1.0, the difference between them increased. Therefore, we revised the theoretical equation of f_{Hn} to Eq. (6) in consideration of the influence of d/l ; that is,

$$\begin{aligned} f_{Hn_approximate} &= \left\{ \frac{d}{l} f_n \right\} + \left\{ \left(1 - \frac{d}{l} \right) f_{Hn}' \right\} \\ &= \left\{ \alpha_n^2 \frac{d}{l} + \alpha_n'^2 \left(1 - \frac{d}{l} \right) \cos \theta \right\} \\ &\quad \cdot \frac{d}{2\sqrt{12}\pi l^2} \sqrt{\frac{E}{\rho}} \end{aligned} \quad (6)$$

Equation (6) is called an approximate equation formula (abbreviated to AEF hereafter) for obtaining the resonance frequency of the slant cantilever beam.

3.3 Comparison between Resonance Frequencies Calculated by AEF and Simulated by FEM

Both resonance frequencies calculated by the AEF of Eq. (6) and simulated by FEM are shown in Figs. 11(a)~(e),⁸⁾ where we present the values of f_{H1}' calculated by Eq. (5), too. From these results, it is proven that the flexural resonance frequency (f_{H1}) can be approximately calculated, with enough exactness, in regard to the H1-mode. However the AEF of Eq. (6) might produce a noticeable error when calculating the H2-mode flexural resonance fre-

Table 3 Young's modulus obtained by resonance frequencies.

d/l	Resonance frequency f_{H1} [kHz]	Young's modulus E [N/m ²]
0.1	5.544	9.356×10^{10}
0.3	15.689	9.335×10^{10}
0.5	23.760	9.316×10^{10}
0.7	29.750	9.308×10^{10}
1.0	35.839	9.318×10^{10}

Table 4 Flexural resonance frequencies obtained by FEM and by theoretical calculation.

Form ratio d/l	Resonance frequency (for simulation) f_{H1} [kHz]	Resonance frequency (for theoretical calculation) f_t [kHz]	Error [%]
0.1	5.544	5.544	± 0.00
0.3	15.689	15.707	-0.11
0.5	23.760	23.811	-0.21
0.7	29.750	29.826	-0.25
1.0	35.839	35.912	-0.20

Fig. 10 Simulated and calculated flexural frequencies.

quency, because the H2-mode combines considerably with the L1-mode.

4. SUMMARY

Resonance mode patterns and resonance frequencies of the slant cantilever beam were simulated by the finite element method. Moreover, the approximate equation formula to calculate the resonance frequency of the H1-mode was investigated. These results are summarized as follows:

- (1) It became clear that the vibration mode patterns are greatly influenced by slanting.
- (2) The bending vibration component of displacement is also generated in the L1-mode, and an

elliptic motion is formed on the top of the slant cantilever beam.

- (3) The bending vibration components are generated in the T1-mode, too.
- (4) In the H1- and H2-modes also, the longitudinal displacements generate an elliptic motion.
- (5) The torsional displacements are generated in the V1-mode and the V2-mode, but in the torsional vibration mode, the displacements on the top of the slant cantilever beam are influenced little.
- (6) We obtained the approximate equation formula to calculate the H1-mode flexural resonance frequency of the slant cantilever beam.

Fig. 11 Flexural resonance frequencies by FEM simulation and by calculation with AEF.

(7) In the estimation of resonance frequencies, Young's modulus was determined so that the results simulated by FEM were made equal to the calculated results.

As mentioned above, we clarified the vibrational behavior of the slant cantilever beam. The academic research material dealt with in this paper might be applicable not only to the construction of ultrasonic motors using the slant-slit torsional vibrator, but also to other demands.

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