Bifurcations in a two-mass model of the vocal folds

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1. Introduction

The dynamics of the vocal fold oscillation was analyzed in a previous work¹⁾ using a simplified version of the two-mass model.²⁾ Three equilibrium positions were found, and a bifurcation diagram was derived using a coupling stiffness coefficient and the lung pressure as control parameters. It was shown that the oscillation region can be divided into two subregions, which correspond to oscillation around different equilibrium positions. This result was a novel finding for the oscillation dynamics, and has been confirmed in later works.³⁾ However, no interpretation of its meaning in terms of phonation has been offered yet. In this letter we will attempt an interpretation, showing that each subregion could be related to a different vocal register, one for the chest and the other for the falsetto.

The present analysis has also the purpose of updating the previous results where the original equations of the two-mass model were used, incorporating improvements to the aerodynamic equations of the two-mass model suggested recently in the literature.^{3,4)}

2. Two-mass model equations

We neglect nonlinearities in the mechanical properties of the tissues, and the influence of the vocal tract and subglottal system. It has been shown that these simplifications do not alter significantly the basic dynamics of the oscillation.^{1,3,5)} The glottal aerodynamical equations are modified from the original equations of the model,²⁾ assuming Bernoulli flow below the glottal constriction, and a free downstream jet above it.^{3,4)} With the above assumptions, the equations of motion are

$$\begin{cases} m_1 \ddot{x}_1 + r_1 \dot{x}_1 + s_1 + k_c(x_1 - x_2) = F_1 \\ m_2 \ddot{x}_2 + r_2 \dot{x}_2 + s_2 + k_c(x_2 - x_1) = 0 \end{cases} \quad (1)$$

where x_i ($i=1, 2$) is the displacement of mass m_i from its rest position, r_i is the damping coefficient, k_c is the coupling stiffness, s_i is the elastic restoring force given by

$$s_i = \begin{cases} k_i x_i, & \text{for } x_i > -x_0 \\ k_i x_i + h_i(x_i + x_0), & \text{for } x_i \leq -x_0 \end{cases} \quad (i=1, 2) \quad (2)$$

where k_i is the stiffness coefficient, h_i is the extra stiffness coefficient introduced by the collision between the opposite folds, x_0 is the mass displacement where

collision occurs; and F_1 is the aerodynamic force on mass m_1 , given by

$$F_1 = \begin{cases} d_1 l_g P_s \left[1 - \frac{(x_2 + x_0)^2}{(x_1 + x_0)^2} \right], & \text{for } x_1 \geq x_2 > -x_0 \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

where d_1 is the width of mass m_1 , l_g is its length, and P_s is the subglottal pressure.

3. Bifurcation diagram

As shown in the previous work,¹⁾ the equilibrium positions appear only in the open glottis condition $x_1 > -x_0$, $x_2 > -x_0$. Setting the time derivatives to zero in the equations of motion (1), we obtain two solutions for the equilibrium positions: the rest position $x_1=0$, $x_2=0$, and a second position given by the simultaneous equations

$$x_2 = \alpha x_1 \quad (4)$$

$$\begin{aligned} k_1 \left[1 + \alpha \frac{k_2}{k_1} \right] (x_1 + x_0)^2 - d_1 l_g P_s (1 - \alpha^2) (x_1 + x_0) \\ - d_1 l_g P_s x_0 (1 - \alpha^2) = 0 \end{aligned} \quad (5)$$

under the condition $x_1 \geq x_2$. In Eq. (4), α is the coupling coefficient

$$\alpha = \frac{k_c}{k_2 + k_c} \quad (6)$$

Note that Eq. (5) has two solutions for x_1 ; however, only one of them belongs to the open glottis condition. Also, the second equilibrium position requires a rectangular or convergent glottis ($x_1 \geq x_2$).

The two equilibrium positions may become coincident at $x_1=0$, $x_2=0$. Introducing these values in Eq. (5), we obtain for the subglottal pressure

$$P_s = \frac{k_1 x_0 (1 + \alpha k_2 / k_1)}{2 d_1 l_g (1 - \alpha)} \quad (7)$$

For a subglottal pressure lower than this value, the only equilibrium position is the rest position. At and above it, both equilibrium positions exist.

In our previous analysis,¹⁾ three equilibrium positions were found. The third one was associated to an additional pressure drop at the glottal entry due to the formation of a vena contracta in the airstream, which had not been included in the present analysis. This equilibrium position was judged as not significant to

the overall dynamics; its absence here would confirm that conclusion. Also, the second equilibrium position was found to exist in both convergent and divergent glottis. The present assumption of a jet above the glottal constriction eliminates this position in the divergent case.

The stability of each equilibrium position may be next determined linearizing the equations of motion around each equilibrium position, and the examining the roots of the corresponding characteristic equation.

We consider the standard configuration given by the parameters $l_g = 1.4$ cm, $d_1 = 0.25$ cm, $x_0 = 0.02$ cm, $m_1 = 0.125$ g, $m_2 = 0.025$ g, $k_1 = 80,000$ dyn/cm, $k_2 = 8,000$ dyn/cm, $r_1 = 20$ dyn s/cm.

The results are illustrated by the bifurcation diagram in Fig. 1. Curves BC and BE are Hopf bifurcations for the rest position and the second equilibrium position, respectively, and delimit the oscillation region EBC. Curve AD is given by Eq. (7), and indicates the coincidence of both equilibrium positions. Below the curve,

Fig. 1 Bifurcation diagram of the two-mass model.

Fig. 2 State space plot for $\alpha=0.75$ and $P_s=8,000$ dyn/cm² (point P_1 in Fig. 1).

Fig. 3 State space plot for $\alpha=0.3$ and $P_s=35,000$ dyn/cm² (point P_2 in Fig. 1). Point A is the location of the second equilibrium position.

the only equilibrium position is the rest position; whereas at and above it, both equilibrium position exist. As discussed in the previous analysis,¹⁾ this curve has the effect of exchanging the equilibrium positions involved in the two Hopf bifurcations.

Two state space plots are shown in Figs. 2 and 3, which corresponds to points P_1 and P_2 in Fig. 1, respectively. Note that in the case of Fig. 2, the oscillation takes place around the rest position; whereas in the case of Fig. 3, it does it around the second equilibrium position (point A). This may be explained by the bifurcation diagram noting that point P_1 is below curve AD, in the region where the only equilibrium position is the rest position. Point P_2 is above curve AD, near the Hopf bifurcation involving the second equilibrium position.

4. Discussion

Figure 1 shows that the oscillation region EBC is divided by curve AD into two subregions, in which the oscillation occurs around different equilibrium positions: the rest position in subregion DBC, which is associated to a rectangular glottal ($x_1=x_2$), and the second equilibrium position in subregion EBD, which is associated to a convergent glottal shape ($x_1>x_2$). We may compare this analytical result with previous experimental observations of the oscillation, where the same glottal shapes have been observed during oscillation at the chest and falsetto registers, respectively.⁹⁾ This suggests the possibility of relating subregion DBC to vocal fold oscillation in the chest register, and subregion EBD to the falsetto register.

Further analytical efforts on more sophisticated models of the vocal folds are desirable as a next step to confirm this conclusion, which would explain vocal registers as a consequence of vocal fold oscillation around different equilibrium positions.

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