Two kinds of parameter estimation methods based on the information theory and an application to the acoustical measurement

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(Received 7 February 1981)

In this paper, two kinds of parameter estimation methods for overcoming the obstruction of the background noise co-existing in many actual acoustical or vibrational measurements are studied. As it is very important in an estimation problem for the measurement with the background noise of non-Gaussian type to grasp first the statistical properties of noise from all viewpoints as precise as possible, two kinds of information quantities which enable us to evaluate the distance between two different types of distribution forms or the uncertainty between two random physical quantities (e.g., the input and the output of the acoustical system) are newly introduced as the estimation criteria, and two kinds of sequential estimation algorithms which can be applied to the estimation for the general non-linear acoustical system with the background noise of arbitrary distribution type are explicitly derived. Furthermore, the legitimacy and the effectiveness of these algorithms are confirmed by applying to the actual parameter estimation problem of the reverberation room under the background noise from an acoustical viewpoint.

PACS number: 43. 50. Yw, 43. 60. Cg, 43. 55. Br

1. INTRODUCTION

It is well-known that various types of measurement methods have been studying in the field of environmental noise or vibration.¹⁾ But these are limited to the ideal measuring situation without the fluctuating background noise or vibration. So we cannot often obtain the accurate data of an objective noise or vibration signal by using these methods under the actual measuring situations with random background noise or vibration. In this paper, from this viewpoint we study the statistical technique for overcoming the obstruction of the background noise or vibration co-existing in the various kinds of actual acoustical or vibrational measurements.

Generally speaking, the measurement with the random background noise or vibration must be

treated as one of stochastic estimation problem in a case where an effective method of directly or deterministically removing the noise or vibration cannot be found. Though the instantaneous value of the randomly fluctuating background noise or vibration is originally difficult to be predicted, there exist some stable informations (e.g., the statistical properties like expectation, variance and probability distribution form) which can be utilized in the parameter estimation. As it is not essential from the viewpoint of functional complexity (like non-linearity) on the deterministic relation of the objective system between input and output, but from the viewpoint of existence of this background noise or vibration, that brings the difficulty to the above estimation problem, it can be said that an accuracy of the parameter estimation depends originally on

how precisely the statistical property on this background noise or vibration is grasped. Although many types of statistical study on the parameter estimation method can be already found in the stochastic system analysis,²⁾ most of all are very often established under the three assumptions, i.e., (i) the linearity of internal mechanism of the phenomena and their observation systems, (ii) the Gaussian distribution property of the noise or vibration fluctuation, and (iii) the choice of estimation criterion with mean square error type. But the statistical property of the background noise or vibration which we meet very often in actual acoustical or vibrational measurements are fairly far from the typical properties of Gaussian distribution adopted in the usual stochastic system analysis. So in estimating the unknown parameters characterizing the objective acoustical or vibrational system under the randomly fluctuating background noise or vibration by use of an information of the stationary random test input and its corresponding output, if the sufficient accuracy of estimation is required for the actual observed data with arbitrary background noise or vibration of non-Gaussian type, and the estimated result is especially used for the prediction of the noise or vibrational evaluation index L_x (i.e., the $(100-x)$ percentile level) as well as the L_{eq} level (i.e., the equivalent noise level), we must establish the estimation method based on the introduction of new error criterion matched with the arbitrary shape of error distribution, in place of the well-known estimation methods used very often in the system analysis (e.g., based on the criterion of minimum mean square error).

In this paper, we establish two kinds of parameter estimation methods for the general non-linear system under the background noise or vibration of non-Gaussian type by precisely grasping the estimation error with the measure of information. Concretely, these two methods are derived by (i) minimizing the Kullback's information³⁾ which can evaluate the distance between the error probability distribution form and the probability distribution form of background noise or vibration, and gives the estimated value coinciding with the true value in an idealized special case, and (ii) minimizing the measure of information defined as a conditional entropy⁴⁾ between input and output which can evaluate the measure of uncertainty between input and output originally brought by the background

noise or vibration.

Furthermore, we apply these two new types of estimation method to an actual estimation problem of the reverberation time in the field of architectural acoustics and confirm the legitimacy and the effectiveness of our estimation methods experimentally.

2. THEORETICAL CONSIDERATION

First of all, we describe the steady state of the acoustical or vibrational system polluted by the background noise or vibration with a linear or nonlinear input-output model of Eq. (1):

$$y = f(\mathbf{x}, \mathbf{a}, v), \quad (1)$$

where the vector $\mathbf{x}$ ($\triangleq (x_1, x_2, \dots, x_m)$) denotes the stationary random input composed of m test excitation factors, the vector $\mathbf{a}$ ($\triangleq (a_1, a_2, \dots, a_n)$) denotes the unknown parameter vector, v (: scalar) denotes the stationary background noise or vibration with already-known statistics and y denotes the observed stationary output measured under the background noise or vibration v . In addition, the function $f(\cdot)$ denotes the relationship between input and output which can be preestablished based on the information about the physical laws governing the objective phenomenon and is assumed to have the inverse function on v (i.e., $v = f^{-1}(\mathbf{x}, y, \mathbf{a})$).

2.1 Parameter Estimation Based on the Kullback's Information

It is clear that the precise grasping of the noise distribution from input and output data leads to the accurate parameter estimation. So we introduce the error e based on Eq. (1) as

$$e \triangleq f^{-1}(\mathbf{x}, y, \mathbf{a}). \quad (2)$$

The error e_0 in the case when the estimated parameter $\mathbf{a}$ coincides with the true value $\mathbf{a}^*$ is clearly equal to the background noise v itself:

$$e_0 \triangleq f^{-1}(\mathbf{x}, y, \mathbf{a}^*) = v. \quad (3)$$

So we can estimate precisely the system parameters through searching the value $\hat{\mathbf{a}}$ of $\mathbf{a}$ which gives the minimum of the distance in some sense between the error distribution $P(e|\mathbf{a})$ and the background noise distribution $P_v(v)$. This error probability density $P(e|\mathbf{a})$ is generally considered to have various types of distribution form owing to both the non-linearity of mechanism of system and the non-Gaussian distribution property of the arbitrary background

noise. But if an information on the noise probability density $P_v(v)$ is grasped with the moment statistics in the generalized distribution form of statistical Laguerre orthonormal expansion series (it is already proposed in Ref. (5) that this distribution form is effective to express the distribution of the random quantity (like energy) fluctuating only in the positive region[†]), we have

$$P_v(v) = \frac{v^{m_v-1} \exp(-v/s_v)}{\Gamma(m_v)s_v^{m_v}} \left\{ 1 + \sum_{n=3}^{\infty} A_n L_n^{(m_v-1)} \left(\frac{v}{s_v} \right) \right\} \quad (4-a)$$

with

$$\left. \begin{aligned} m_v &\triangleq \frac{\mu_v^2}{\sigma_v^2}, & s_v &\triangleq \frac{\sigma_v^2}{\mu_v} \\ (\mu_v, \sigma_v^2: \text{mean and variance of the noise}) \\ \text{and} \\ A_n &\triangleq \frac{\Gamma(m_v)n!}{\Gamma(m_v+n)} \left\langle L_n^{(m_v-1)} \left(\frac{v}{s_v} \right) \right\rangle_v, \end{aligned} \right\} \quad (4-b)$$

where $\Gamma(\cdot)$ denotes a Gamma function and $\langle \cdot \rangle_v$ shows an expectational operation. It is reasonable that the same expression form as Eq. (4) is given as an error probability density $P(e|a)$ which should be coincided with the probability density expression $P_v(v)$ of background noise:

$$P(e|a) = \frac{e^{m-1} \exp(-e/s)}{\Gamma(m)s^m} \left\{ 1 + \sum_{n=3}^{\infty} B_n L_n^{(m-1)} \left(\frac{e}{s} \right) \right\} \quad (6-a)$$

with

[†] If the noise probability density is expressed in the general form of Gram-Charlier A type series⁶⁾ (suitable to the probability expression of random noise fluctuating in the both positive and negative region $(-\infty, \infty)$):

$$\left. \begin{aligned} P_v(v) &= \frac{1}{\sqrt{2\pi} \sigma_v} \exp \left\{ -\frac{(v-\mu_v)^2}{2\sigma_v^2} \right\} \\ &\quad \cdot \left\{ 1 + \sum_{n=3}^{\infty} A_n H_n \left(\frac{v-\mu_v}{\sigma_v} \right) \right\} \\ \text{with} \\ A_n &\triangleq \frac{1}{n!} \left\langle H_n \left(\frac{v-\mu_v}{\sigma_v} \right) \right\rangle_v. \end{aligned} \right\} \quad (5)$$

It is also reasonable that the same expression form as Eq. (5) is chosen as an error probability density $P(e|a)$ and the same theoretical consideration can be developed as in the case of Eq. (4).

$$\left. \begin{aligned} m &\triangleq \frac{\mu^2(a)}{\sigma^2(a)}, & s &\triangleq \frac{\sigma^2(a)}{\mu(a)}, \\ B_n &\triangleq \frac{\Gamma(m)n!}{\Gamma(m+n)} \left\langle L_n^{(m-1)} \left(\frac{e}{s} \right) \right\rangle_e, \\ \mu(a) &\triangleq \langle e|a \rangle_e, & \sigma^2(a) &\triangleq \langle (e-\mu(a))^2 | a \rangle_e. \end{aligned} \right\} \quad (6-b)$$

Next, we introduce the Kullback's information defined as a measure of distance between arbitrary two probability density functions $P_i(x)$ and $P_j(x)$ ³⁾:

$$\left. \begin{aligned} I &\triangleq \left\langle \log_e \left(\frac{P_i(x)}{P_j(x)} \right) \right\rangle_x \\ &= \int P_i(x) \log_e \frac{P_i(x)}{P_j(x)} dx. \end{aligned} \right\} \quad (7)$$

There is a theorem on this Kullback's information as follows:

[Theorem]³⁾ For any of two probability density functions $P_i(x)$ and $P_j(x)$, Kullback's information I is always more than zero, that is, $I \geq 0$, and the equality holds if and only if $P_i(x)$ is equal to $P_j(x)$ for the arbitrary value of x .

If we adopt this measure of distance between $P(e|a)$ and $P_v(v)$ as an error criterion in the parameter estimation, we can directly have

$$\begin{aligned} I(a) &\triangleq \left\langle \log_e \frac{P(e|a)}{P_v(e)} \right\rangle_e \\ &= \left\langle \log_e \frac{\Gamma(m_v)s_v}{\Gamma(m)s} + \frac{m-1}{2} \log_e \left(\frac{e}{s} \right)^2 \right. \\ &\quad \left. - \frac{m_v-1}{2} \log_e \left(\frac{e}{s_v} \right)^2 - \left(\frac{1}{s} - \frac{1}{s_v} \right) e \right. \\ &\quad \left. + \log_e \frac{1 + \sum_{n=3}^{\infty} B_n L_n^{(m-1)} \left(\frac{e}{s} \right)}{1 + \sum_{n=3}^{\infty} A_n L_n^{(m_v-1)} \left(\frac{e}{s_v} \right)} \right\rangle_e. \end{aligned} \quad (8)$$

So the estimated optimal value which gives the minimum of Eq. (8) can be obtained by solving the equations shown below:

$$\left\{ \begin{aligned} \nabla_a I(a) &= 0 \quad (a: \text{unknown vector}), \end{aligned} \right. \quad (9-a)$$

$$\left\{ \begin{aligned} \text{or} \\ I(a) &= 0 \quad (a: \text{unknown scalar}), \end{aligned} \right. \quad (9-b)$$

where ∇_a means the differential operator on vector a .

As these are the equations of regression function type, we can easily solve (9-a) or (9-b) by use of the well-known stochastic approximation method⁷⁾ (see APPENDIX). If we show the concrete algorithm

corresponding to only (9-b) for simplicity, it can be expressed as follows:

$$\begin{aligned} \hat{a}_{K+1} = & \hat{a}_K + \gamma_K \left\{ \log_e \frac{\Gamma(m_v) s_v}{\Gamma(m_K) s_K} \right. \\ & + \frac{m_K - 1}{2} \log_e \left(\frac{f^{-1}(x_K, y_K, \hat{a}_K)}{s_K} \right)^2 \\ & - \frac{m_v - 1}{2} \log_e \left(\frac{f^{-1}(x_K, y_K, \hat{a}_K)}{s_v} \right)^2 \\ & - \left(\frac{1}{s_K} - \frac{1}{s_v} \right) f^{-1}(x_K, y_K, \hat{a}_K) \\ & \left. + \log_e \frac{1 + \sum_{n=3}^{\infty} B_n L_n^{(m_K-1)} \left(\frac{f^{-1}(x_K, y_K, \hat{a}_K)}{s_K} \right)}{1 + \sum_{n=3}^{\infty} A_n L_n^{(m_v-1)} \left(\frac{f^{-1}(x_K, y_K, \hat{a}_K)}{s_v} \right)} \right\}, \end{aligned} \quad (10)$$

where the $\{\gamma_K\}$ ($K=1, 2, \dots$) denotes the gain sequence which satisfies certain conditions of ensuring a convergence of the stochastic approximation algorithm, and m_K , s_K , B_n denote the corresponding statistics in Eq. (6) where the parameter a is replaced to $\hat{a}_K$. Furthermore, x_K and y_K denote the input and output data at the K -th time point respectively. So we can obtain an estimated optimal value of a in Eq. (1) which makes the distance measure between the noise probability density form of Eq. (4) and the error probability density form of Eq. (6) equal to zero by calculating Eq. (10) under a selection of an appropriate initial value $\hat{a}_1$.

2.2 Parameter Estimation Based on the Entropy Criterion

In section 2.1, we have derived concretely the parameter estimation algorithm by taking the Kullback's information defined for the distance measure between the noise and the error probability density forms as an error criterion of estimation. In this section, we introduce the idea of conditional entropy on the input and output in Eq. (1) as the another criterion reflecting the effect of background noise with arbitrary distribution type, and derive concretely the estimation algorithm based on this new type of criterion.

It is noteworthy that minimizing this conditional entropy is equivalent to minimizing the influence of the background noise and so this fact leads to the accurate parameter estimation. As is well-known in the information theory, the conditional

entropy $H(y|x, a)$ on the input x and output y in Eq. (1) can be expressed with the arbitrary background noise distribution as follows⁴⁾:

$$\begin{aligned} H(y|x, a) & \triangleq \langle \langle -\log_e P(y|x, a) | x, a \rangle \rangle_y \\ & = \langle \langle -\log_e [P_v(f^{-1}(x, y, a)) \\ & \quad \cdot |\nabla_y f^{-1}(x, y, a)|] | x, a \rangle \rangle_x, \end{aligned} \quad (11)$$

where the notation $|\cdot|$ denotes the absolute value. So the estimated optimal value $\hat{a}$ which gives the minimum of this conditional entropy given in Eq. (11) is obtained by solving the equation:

$$\begin{aligned} \nabla_a H(y|x, a) = & -\langle \langle \nabla_a \log_e P_v(f^{-1}(x, y, a)) \\ & + \nabla_a |\nabla_y f^{-1}(x, y, a)| | x, a \rangle \rangle_x = 0. \end{aligned} \quad (12)$$

The probability density function $P_v(v)$ of the background noise may have various types of expression, but it is natural to adopt the distribution form of Eq. (4) as the generalized expression form of $P_v(v)$ for the noise fluctuating only in the positive region like an energy fluctuation (see footnote[†]).

By substituting Eq. (4) into Eq. (12), the concrete expression of the equation can be easily obtained as follows:

$$\begin{aligned} & \left\langle \left(1 - m_v \right) \frac{\nabla_a f^{-1}(x, y, a)}{f^{-1}(x, y, a)} + \frac{1}{s_v} \nabla_a f^{-1}(x, y, a) \right. \\ & \quad \cdot \left\{ 1 + \frac{\sum_{n=3}^{\infty} A_n L_{n-1}^{(m_v)} \left(\frac{f^{-1}(x, y, a)}{s_v} \right)}{1 + \sum_{n=3}^{\infty} A_n L_n^{(m_v-1)} \left(\frac{f^{-1}(x, y, a)}{s_v} \right)} \right\} \\ & \quad \left. - \nabla_a |\nabla_y f^{-1}(x, y, a)| \right| a \rangle_{x, y} = 0. \end{aligned} \quad (13)$$

This Eq. (13) shows also the equation of regression function type, and the estimation algorithm can be concretely derived by applying again the same stochastic approximation method as in section 2.1:

$$\begin{aligned} \hat{a}_{K+1} = & \hat{a}_K + \gamma_K \left[\left(1 - m_v \right) \frac{\nabla_a f^{-1}(x_K, y_K, \hat{a}_K)}{f^{-1}(x_K, y_K, \hat{a}_K)} \right. \\ & + \frac{\nabla_a f^{-1}(x_K, y_K, \hat{a}_K)}{s_v} \\ & \cdot \left\{ 1 + \frac{\sum_{n=3}^{\infty} A_n L_{n-1}^{(m_v)} \left(\frac{f^{-1}(x_K, y_K, \hat{a}_K)}{s_v} \right)}{1 + \sum_{n=3}^{\infty} A_n L_n^{(m_v-1)} \left(\frac{f^{-1}(x_K, y_K, \hat{a}_K)}{s_v} \right)} \right\} \\ & \left. - \nabla_a |\nabla_y f^{-1}(x_K, y_K, \hat{a}_K)| \right]. \end{aligned} \quad (14)$$

The estimated optimal value $\hat{a}$ which gives the minimum of Eq. (11) is obtained by calculating this algorithm under the selection of an appropriate initial value $\hat{a}_1$.

3. EXPERIMENTAL CONSIDERATION

We have derived two parameter estimation algorithms based on the two kinds of the criteria (i.e., the Kullback's information and the conditional entropy) in the previous chapter. In this chapter, we confirm the legitimacy of these algorithms experimentally by concretely applying them to the actual estimation problem of the reverberation time under the background noise energy in the field of architectural acoustics. First of all, we introduce the input and output model of Eq. (1) for the reverberation phenomenon with the background noise energy. It is well-known that the sound intensity $I(t)$ in a room without the background noise is governed by the differential equation i.e., Eq. (15),⁸⁾ if the sound intensity field can be regarded as the reverberant field, that is,

$$\frac{dI(t)}{dt} = FI(t) + X(t) \left(F \triangleq -\frac{6 \log_e 10}{T} \right), \quad (15)$$

where T and $X(t)$ denote the reverberation time and the input sound intensity respectively. By introducing the sampling interval Δt to get the simplicity of data handling, the instantaneous value of the sound intensity at the K -th time point $I_K (\triangleq I(K\Delta t))$ can be approximately expressed as follows:

$$\begin{aligned} I_K &= \int_0^{K\Delta t} K_1 X(K\Delta t - \tau) e^{F\tau} d\tau \\ &= \sum_{i=1}^K \int_{(i-1)\Delta t}^{i\Delta t} K_1 X(K\Delta t - \tau) e^{F\tau} d\tau \sim a x_K \\ &\quad + a(1-a)x_{K-1}, \end{aligned} \quad (16)$$

where

$$a \triangleq 1 - e^{F\Delta t}, \quad x_K \triangleq cX(K\Delta t), \quad c \triangleq \frac{K_1}{F}.$$

So, considering the intensity fluctuation of background noise $v_K (\triangleq v(K\Delta t))$, the input and output model for the reverberation phenomenon with the background noise is expressed as follows:

$$\begin{aligned} y_K &= f(x_K, a, v_K) \\ &= ax_K + a(1-a)x_{K-1} + v_K. \end{aligned} \quad (17)$$

We estimate the reverberation time $T (\triangleq -6\Delta t \cdot \log_e 10 / (\log_e (1-a)))$ using concretely the above two algorithms (i.e., Eq. (10) and (14)) to the input and output data sequence $\{x_K\}$ and $\{y_K\}$ ($K=1, 2, \dots$). These input and output data are experimentally observed by the block diagram in a reverberation room shown in Fig. 1. Concretely, producing the stationary highway noise recorded in Hiroshima city as the background noise $v(t)$ from the first speaker and the white noise as a input $X(t)$ from the second speaker at the same time, we record the input and output sound intensity fluctuations and obtain the time-discrete version $\{x_K\}$, $\{y_K\}$ ($K=1, 2, \dots, 150$) by sampling the actual reverberation data at sampling interval $\Delta t = 1/3$ second. The estimated result of the reverberation time T by use of an estimation algorithm of Eq. (10) is shown in Fig. 2.

This result is obtained by simply approximating both the error and the noise distribution expression forms: $P(e|a)$ and $P_v(v)$ only by Gamma distribution form (this means $A_n = B_n = 0$ ($n \geq 3$) in Eqs. (4) and (6)) for practical use. For the various selection of initial values, estimated results show a good agreement with the true value of T which is measured in advance without the background noise and the effectiveness of this algorithm can be clearly confirmed.

Fig. 1 Block diagram of the measuring system with non-directional speakers for the reverberation time under the background noise.

Fig. 2 Estimated results of the reverberation time by the Kullback's information criterion (by Eq. (10)).

Fig. 3 Comparison between the estimated results by the minimum entropy criterion based on Gamma and non-Gamma approximation of the noise distribution.

Next, the estimated result by an algorithm of Eq. (14) is shown in Figs. 3 and 4. This estimation is carried out by using repeatedly the input and

output sound intensity data sequence eight times to get the sufficient convergence of the estimation algorithm. Figure 3 shows the comparison between

Fig. 4 Comparison between the estimated results by the minimum entropy criterion for the various initial values.

the estimated results based on the noise distribution expression form approximated by Gamma distribution and by non-Gamma distribution (including the coefficient A_s in Eq. (4)). It can be proved from Fig. 3 that considering this non-Gamma property makes a good improvement that the estimated value approaches surely to the true value, and from Fig. 4 that this algorithm converges to the same value for the various selection of initial values. These two figures show the legitimacy of this estimation algorithm based on the conditional entropy criterion.

4. CONCLUSION

In this paper, especially from the methodological viewpoint we have studied the estimation technique for overcoming the obstruction of the background noise co-existing in the various kinds of the actual acoustical measurements. Concretely, taking the non-linearity of the general acoustical system, the arbitrary non-Gaussian distribution property of the background noise and the necessity of the prediction for the noise evaluation indices L_x and L_{eq} etc. into consideration, we have newly and concretely derived two kinds of estimation algorithms based on the introduction of estimation criteria reflecting

the arbitrariness of the error distribution form, i.e., the Kullback's information and the conditional entropy.

Furthermore, we have shown two experimental examples for the confirmation of the legitimacy and the effectiveness on these two estimation algorithms by applying especially to the actual estimation problem of the reverberation time under the background noise of arbitrary distribution type, by use of the stationary input and output sound intensity fluctuation data.

ACKNOWLEDGEMENTS

We would like to express our cordial thanks to Hajime Kimura and Shizuma Yamaguchi for their helpful assistances.

APPENDIX: SUMMARY OF STOCHASTIC APPROXIMATION METHOD⁷⁾

Let $Y(X)$ be a random variable generated by another random variable X , and $Y(X)$ is statistically characterized by the conditional distribution $P(Y|X) \triangleq P(Y(X) \leq y)$. Then the regression function $M(X)$ is defined as the expected value of $Y(X)$ for the given X as follows:

$$M(X) \triangleq \langle Y(X)|X \rangle_Y \\ = \int_{-\infty}^{\infty} y dP(y|X). \quad (\text{A-1})$$

Given an equation of this regression function:

$$M(X) = \alpha \quad (\alpha: \text{the constant value}) \quad (\text{A-2})$$

which has a unique root $X = \theta$, it can be solved as the converged value of stochastic approximation algorithm of Eq. (A-3) with probability one.

$$X_{K+1} = X_K + \gamma_K \{\alpha - y_K(X_K)\} \\ (K=1, 2, \dots), \quad (\text{A-3})$$

where the $\{y_K(X_K)\}$ denotes the successive observations on $Y(X)$ at level $X_1, X_2, \dots$, and $\{\gamma_K\}$ denotes a fixed sequence of positive constants which satisfies the conditions as follows:

$$\sum_{K=1}^{\infty} \gamma_K = \infty, \quad \sum_{K=1}^{\infty} \gamma_K^2 < \infty. \quad (\text{A-4})$$

For example, we can adopt the sequence $\{1/K\}$ as the most simple one satisfying these conditions. Furthermore let the regression function satisfy the conditions shown below:

$$\text{i) } \int_{-\infty}^{\infty} [y - M(X)]^2 dP(y|X) < \infty, \quad (\text{A-5})$$

$$\text{ii) } |M(X)| \leq c + d|X| \quad (-\infty < X < \infty), \quad (\text{A-6})$$

$$\text{iii) } M(X) < \alpha \quad (X < \theta), \quad M(X) > \alpha \quad (X > \theta), \\ (\text{A-7})$$

and

$$\text{iv) } \inf_{\delta_1 \leq |X - \theta| \leq \delta_2} |M(X) - \alpha| > 0 \\ (\forall \delta_1, \delta_2: 0 < \delta_1 < \delta_2 < \infty). \quad (\text{A-8})$$

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