An Application of Theorems of Harish-Chandra and Ado

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Let G_1 and G_2 be compact groups, G_1^* and G_2^* the sets of representations of G_1 and G_2 respectively. Then the set of homomorphisms of G_1 into G_2 and the set of homomorphisms of G_2^* into G_1^* are canonically homeomorphic to each other [4]. The similar results are also valid for locally compact groups [3].

Now let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field K of characteristic zero, $\mathfrak{g}^*$ be the set of representations of $\mathfrak{g}$. A map μ which assigns to every $D \in \mathfrak{g}^*$ a matrix $\mu(D)$ of the same degree as D is called a representation of $\mathfrak{g}^*$ if it preserves similarity, direct sum and Kronecker sum. The set $\mathfrak{g}^{**}$ of representations of $\mathfrak{g}^*$ canonically becomes a Lie algebra (not necessarily finite dimensional). For $X \in \mathfrak{g}$ and $D \in \mathfrak{g}^*$, we set $(\tau(X))(D) = D(X)$. τ is a homomorphism of $\mathfrak{g}$ into $\mathfrak{g}^{**}$. If $\mathfrak{g}$ is semi-simple, the map τ is surjective [2]. On the other hand, as $\mathfrak{g}$ has a faithful representation (theorem of Ado, [1]), we see that τ is a one-to-one mapping. Therefore $\mathfrak{g}$ and $\mathfrak{g}^{**}$ are isomorphic to each other.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be semi-simple Lie algebras over K , $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ be the sets of representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. A map φ of $\mathfrak{g}_2^*$ into $\mathfrak{g}_1^*$ is called a homomorphism if it preserves degree, similarity, direct sum and Kronecker sum. Let F be the set of homomorphisms of $\mathfrak{g}_1$ into $\mathfrak{g}_2$, Φ the set of homomorphisms of $\mathfrak{g}_2^*$ into $\mathfrak{g}_1^*$. For $f \in F$ and $D \in \mathfrak{g}_2^*$, we set $(\theta(f))(D) = D \circ f$, where ' $\circ$ ' is the composition of maps. Then the map $\theta(f)$ belongs to Φ . And we have a map θ of F into Φ .

Let φ be any element of Φ . We have a homomorphism ψ of $\mathfrak{g}_1^{**}$ into $\mathfrak{g}_2^{**}$ if we set $\psi(\mu) = \mu \circ \varphi$ for $\mu \in \mathfrak{g}_1^{**}$. Then $f = \tau_2^{-1} \circ \psi \circ \tau_1$ is an element of F and we have $\theta(f) = \varphi$, where τ_1 and τ_2 are the canonical isomorphisms of $\mathfrak{g}_1$ onto $\mathfrak{g}_1^{**}$ and $\mathfrak{g}_2$ onto $\mathfrak{g}_2^{**}$ respectively. Therefore the map θ is surjective.

Now if f_1 and f_2 are two distinct elements of F , we have $f_1(X) \neq f_2(X)$ for some $X \in \mathfrak{g}_1$ and $\tau_2(f_1(X)) \neq \tau_2(f_2(X))$ and accordingly $D(f_1(X)) \neq D(f_2(X))$ for some $D \in \mathfrak{g}_2^*$. We have therefore $\theta(f_1) \neq \theta(f_2)$.

THEOREM *Let K be an algebraically closed field of characteristic zero, $\mathfrak{g}$ a semi-simple Lie algebra over K , $\mathfrak{g}^{**}$ the representations of representations of $\mathfrak{g}$. Then $\mathfrak{g}$ and $\mathfrak{g}^{**}$ are canonically isomorphic to each other.*

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Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be semi-simple Lie algebras over K , $\mathfrak{g}_1^$ and $\mathfrak{g}_2^*$ be the sets of representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Then the set F of homomorphisms of $\mathfrak{g}_1$ into $\mathfrak{g}_2$ and the set Φ of homomorphisms of $\mathfrak{g}_2^*$ into $\mathfrak{g}_1^*$ are canonically bijective to each other.*

References

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