Notes on the Duality Theorem of Locally Compact Groups

Ryûki MATSUDA*

Let G_1 and G_2 be compact groups, G_1^* and G_2^* be the sets of representations of G_1 and G_2 respectively. The set F of homomorphisms of G_1 into G_2 and the set Φ of homomorphisms of G_2^* into G_1^* become topological spaces naturally and canonically homeomorphic to each other [1]. We prove the similar results in this paper for locally compact groups.

Let G_i and G_2 be locally compact (Hausdorff) groups. We mean by 'representation' of G_i ($i=1, 2$) a unitary strongly continuous representation over some complex Hilbert space, dimension of which is lower than a sufficiently large cardinal number equal to, for instance the largest among $\aleph_0$ (the smallest infinite cardinal), the square of the dimension of the left regular representation of G_1 and that of G_2 . Let G_i^* be the set of representations of G_i . A map A of G_i^* is called a representation of G_i^* if

1) $A(D)$ is a unitary operator on the representation space $\mathfrak{H}(D)$ of D ,

2) $A(P^{-1}(D_1 + D_2)P) = P^{-1}(A(D_1) + A(D_2))P$,

where '+' is direct sum and P is an isometry and

3) $A(P^{-1}(D_1 \times D_2)P) = P^{-1}(A(D_1) \times A(D_2))P$,

where ' $\times$ ' is tensor product and P is an isometry. The set G_i^{**} of representations of G_i^* is naturally a topological group and the map

$$\tau_i: G_i \ni a \rightarrow A_a \in G_i^{**}$$

is an onto isomorphism [2].

Let F be the set of (continuous) homomorphisms of G_1 into G_2 . We introduce a topology (compact-open topology) on F , taking the subsets of F of type

$$W(K, U) = \{f \in F; f(K) \subset U\}$$

as an open sub-basis, where K is a compact subset of G_1 and U is an open subset of G_2 . A map φ of G_2^* into G_1^* is called a homomorphism if

1) $\varphi(D)$ has the space $\mathfrak{H}(D)$ of D as the representation space; $\mathfrak{H}(\varphi(D)) = \mathfrak{H}(D)$, for any $D \in G_2^*$,

2) $\varphi(P^{-1}(D_1 + D_2)P) = P^{-1}(\varphi(D_1) + \varphi(D_2))P$

and

3) $\varphi(P^{-1}(D_1 \times D_2)P) = P^{-1}(\varphi(D_1) \times \varphi(D_2))P$.

*Department of Mathematics, Ibaraki University, Mito, Japan

Let Φ be the set of homomorphisms of G_2^* into G_1^* . For a compact subset K of G_1 , $D_2 \in G_2^*$, $u, v \in \mathfrak{H}(D_2)$, $\varepsilon > 0$ and $\varphi \in \Phi$ we set

$$U(\varphi, \varepsilon, D_2, K, u, v) = \{\varphi' \in \Phi; |\langle \varphi'(D_2)u - \varphi(D_2)u, v \rangle| < \varepsilon, \text{ for any } x \in K\},$$

where ' $\langle \rangle$ ' is the inner product on $\mathfrak{H}(D_2)$. We introduce a topology on Φ taking the subsets of Φ of this type as a neighbourhood sub-basis of φ . If we set

$$(\theta(f))(D_2) = D_2 \circ f$$

(' $\circ$ ' is composition of maps) for $f \in F$ and $D_2 \in G_2^*$, we get an element $\theta(f)$ of Φ .

We will show that the map

$$\theta : F \rightarrow \Phi$$

is one-to-one. If f_1 and f_2 are two distinct elements of F ,

$$f_1(a) \neq f_2(a)$$

for some $a \in G_1$. Accordingly

$$A_{f_1(a)} \neq A_{f_2(a)}$$

and

$$A_{f_1(a)}(D_2) \neq A_{f_2(a)}(D_2)$$

for some $D_2 \in G_2^*$. We have

$$D_2(f_1(a)) \neq D_2(f_2(a))$$

and

$$\theta(f_1) \neq \theta(f_2).$$

We will show next that the map θ is surjective. We see easily that the map

$$\psi : G_1^{**} \ni A \rightarrow A \circ \varphi \in G_2^{**}$$

(' $\circ$ ' is composition of maps) is product preserving for $\varphi \in \Phi$. Let $A_1 \in G_1^{**}$ and $\psi(A_1) = A_2$. Let U be an open subset of the complex numbers C containing $\langle A_2(D_2)u, v \rangle$, where $D_2 \in G_2^*$ and $u, v \in \mathfrak{H}(D_2)$, and let V be the inverse image of U by the map

$$A_2' \rightarrow \langle A_2'(D_2)u, v \rangle$$

of G_2^{**} into C . Then the inverse image W of U by the map

$$A_1' \rightarrow \langle A_1'(\varphi(D_2))u, v \rangle$$

of G_1^{**} into C is a neighbourhood of A_1 and

$$\psi(W) \subset V,$$

showing ψ to be continuous. If we set

$$\tau_2^{-1} \circ \psi \circ \tau_1 = f$$

using the duality map

$$\tau_i : G_i \simeq G_i^{**},$$

we see that $f \in F$ and

$$A_a(\varphi(D_2)) = \{\psi(A_a)\}(D_2)$$

and accordingly

$$\{\varphi(D_2)\}(a) = D_2\{f(a)\}$$

for $a \in G_1$ and $D_2 \in G_2^*$. This proves that φ is the dual map of f and $\theta(f) = \varphi$.

We will show next the continuity of θ . We consider $U(\varphi, \varepsilon, D_2, K, u, v)$, where we may assume the compact subset K of G_1 not to be empty.

$$U_{f(a)} = \left\{ b' \in G_2; | \langle D_{2b'}u - D_{2f(a)}u, v \rangle | < \frac{1}{2}\varepsilon \right\}$$

is a neighbourhood of $f(a)$, where $a \in K$. We can take a neighbourhood V_a of a such that $f(\overline{V_a}) \subset U_{f(a)}$, where '—' is topological closure. K is covered by a finite number of V_a 's, say $V_{a_1}, \dots, V_{a_n}$ and

$$W = \bigcap_{i=1}^n W(\overline{V_{a_i}} \cap K, U_{f(a_i)})$$

is a neighbourhood of f . If $f' \in W$ and $x \in K$ we have

$$f(x), f'(x) \in U_{f(a_j)}$$

for some j . Then we have

$$| \langle D_{2f(x)}u - D_{2f(a_j)}u, v \rangle | < \frac{1}{2}\varepsilon,$$

$$| \langle D_{2f'(x)}u - D_{2f(a_j)}u, v \rangle | < \frac{1}{2}\varepsilon$$

and

$$| \langle D_{2f'(x)}u - D_{2f(a)}u, v \rangle | < \varepsilon.$$

Consequently

$$\theta(f') \in U(\varphi, \varepsilon, D_2, K, u, v)$$

and we see that θ is continuous.

We will show finally that θ^{-1} is continuous. Let $f \in F$ and $f \in W(K, U)$, where

$$U = \{ b' \in G_2; | \langle D_{2b'}u - D_{2f(a)}u, v \rangle | < \varepsilon \}, a \in K.$$

We have

$$f(K) \subset U' = \{ b' \in G_2; | \langle D_{2b'}u - D_{2f(a)}u, v \rangle | < \varepsilon \lambda \}$$

for some λ ($0 < \lambda < 1$). Let

$$\varphi' = \theta^{-1}(f') \in U(\varphi, \varepsilon(1-\lambda), D_2, K, u, v)$$

and $x \in K$. We have

$$| \langle D_{2f'(x)}u - D_{2f(x)}u, v \rangle | < \varepsilon(1-\lambda),$$

$$| \langle D_{2f(x)}u - D_{2f(a)}u, v \rangle | < \varepsilon \lambda$$

and

$$| \langle D_{2f'(x)}u - D_{2f(a)}u, v \rangle | < \varepsilon.$$

Consequently

$$f'(x) \in U \text{ and } f' \in W(K, U),$$

showing θ^{-1} to be continuous.

We have proved

THEOREM. F and Φ are canonically homeomorphic to each other.

References

- [1] K. SUZUKI, Notes on the duality theorem of non-commutative topological groups, Tohoku Math. Journ., **15** (1963), 182-186.
- [2] N. TATSUUMA, A duality theorem for locally compact groups, Journ. of Math. Kyoto Univ. **8-2** (1967) 187-293.