Notes on a Paper of Kuyk

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We assume for simplicity that all fields treated in this paper are of characteristic zero.

PROPOSITION 1. *Let k be a field, t_0 an indeterminate over k , p a (positive) prime number, K the algebraic closure of $k(t_0)$, c an element of k , the t_i ($i \in \mathbb{Z}$ integers) are elements of K satisfying*

$$t_{i+1}^p = t_i + c,$$

$$\Omega = \bigcup_{i \in \mathbb{Z}} k(t_i),$$

$A(\Omega)$ the group of automorphisms of Ω , $A(\Omega/k)$ the group of k -automorphisms of Ω , π the k -automorphism of Ω determined by

$$t_i \rightarrow t_{i+1}.$$

Then

(1) If $c \neq 0$, $A(\Omega/k)$ is an infinite cyclic group $[\pi]$ generated by π .

(2) If $c=0$ and $0 \neq \alpha_i$ ($i \in \mathbb{Z}$) are elements of k satisfying

$$\alpha_{i+1}^p = \alpha_i,$$

then k -automorphisms $\nu(\alpha)$ and $\delta(\alpha)$ are determined by

$$t_i \rightarrow \alpha_i t_i$$

and

$$t_i \rightarrow \frac{\alpha_i}{t_i}$$

respectively. And $A(\Omega/k)$ is generated by π and automorphisms of type $\nu(\alpha)$ and $\delta(\alpha)$.

(3) If $p \neq 2$, $c=0$ and $k=\mathbb{Q}$ (the rational numbers), $A(\Omega)$ is the direct product of three cyclic groups generated by ν , δ and π respectively, where

$$\nu: t_i \rightarrow -t_i,$$

$$\delta: t_i \rightarrow \frac{1}{t_i}.$$

(4) If $p=2$, $c=0$ and $k=\mathbb{Q}$, $A(\Omega)$ is the direct product of two cyclic groups generated by

$$\delta: t_i \rightarrow \frac{1}{t_i}$$

and π .

Proof. $\theta \in \Omega$ is of (algebraic) degree p^n over $k(t_0)$ if and only if

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$$\theta \in k(t_n) \setminus k(t_{n-1})$$

(that is $\theta \in k(t_n)$ and $\theta \notin k(t_{n-1})$). Let $\sigma_1 \in A(\Omega/k)$ and $\sigma_1(t_0) \in k(t_n) \setminus k(t_{n-1})$. If we set

$$\pi^{-n}\sigma_1 = \sigma_2$$

and

$$\sigma_2(t_i) = s_i' \quad (i \in Z),$$

we have

$$s_0' \in k(t_0) \setminus k(t_{-1}).$$

Let $t_0 \in k(s_m') \setminus k(s_{m-1}')$,

$$\sigma_2 \pi^m = \sigma,$$

$$\sigma(t_i) = s_i \quad (i \in Z).$$

The s_i are of the form

$$\frac{a_i t + b_i}{c_i t + d_i} \quad (a_i, b_i, c_i, d_i \in k, a_i d_i - b_i c_i \neq 0),$$

since $k(s_i) = k(t_i)$ ($i \geq 0$). We are through if we compare the leftest and the rightest sides of

$$\left(\frac{a_{i+1}t_{i+1} + b_{i+1}}{c_{i+1}t_{i+1} + d_{i+1}} \right)^p = s_{i+1}^p = s_i + c = \frac{a_i t + b_i}{c_i t + d_i} + c = \frac{(a_i + c c_i) t_{i+1}^p - a_i c + b_i - c^2 c_i + c d_i}{c_i t_{i+1}^p - c_i c + d_i}.$$

REMARK 1.

(1) The case $p=2$ and $c=1$ is treated in [2].

(2) The case $c=1$ is treated in [1].

(3) Remark 1 of [2] says that if $p=2$, $c=0$ and k not containing the imaginary unit i , $A(\Omega/k)$ is an infinite cyclic group. But this is false.

(4) Remark 2 of [2] says that if $p=3$, $c=0$ and k not containing a primitive third root of unity, $A(\Omega/k)$ is the direct product of an infinite cyclic group and a group of order 2. But this is also false.

PROPOSITION 2. Let $t_{1,0}, \dots, t_{n,0}$ are n distinct indeterminates over a field k , $p_1, \dots, p_n$ n distinct prime numbers, $c_1, \dots, c_n$ elements of k , the $t_{1,i}, \dots, t_{n,i}$ ($i \in Z$) elements of the algebraic closure of $k(t_{1,0}, \dots, t_{n,0})$ satisfying

$$t_{1,i+1}^{p_1} = t_{1,i} + c_1,$$

$$\dots\dots\dots,$$

$$t_{n,i+1}^{p_n} = t_{n,i} + c_n,$$

$$\Omega_1 = \bigcup_{i \in Z} k(t_{1,i}),$$

$$\dots\dots\dots,$$

$$\Omega_n = \bigcup_{i \in Z} k(t_{n,i}),$$

$$\Omega = \bigcup_{i \in Z} k(t_{1,i}, \dots, t_{n,i}),$$

$A(\)$ the same as proposition 1. Then $A(\Omega/k)$ is the direct product of $A(\Omega_1/k), \dots, A(\Omega_n/k)$.

Proof. Let $\pi_{i_1 \dots i_n}$ be the k -automorphism of Ω determined by

$$\begin{aligned} t_{1,m} &\rightarrow t_{1,m+i_1}, \\ &\dots\dots\dots, \\ t_{n,m} &\rightarrow t_{n,m+i_n} \quad (m \in \mathbb{Z}). \end{aligned}$$

Let

$$\begin{aligned} \sigma' &\in A(\Omega/k), \\ \sigma'(t_{1,0}) &\in k(t_{1,i_1}, \dots, t_{n,i_n}) \end{aligned}$$

and

$$\pi_{-i_1, \dots, -i_n} \sigma' = \sigma.$$

We have

$$\sigma(t_{1,0}) \in k(t_{1,0}, \dots, t_{n,0})$$

and

$$\sigma(\Omega_1) \subset \Omega_1(t_{2,0}, \dots, t_{n,0}).$$

Let $\frac{f_i}{g_i}$ be the irreducible fraction of $\sigma(t_{1,i})$, where f_i and g_i are polynomials of $n-1$ variables $t_{2,0}, \dots, t_{n,0}$ of coefficients in Ω_1 and we may assume g_i to be monic.

$$\frac{f_i}{g_i} = \sigma(t_{1,i}) = \sigma(t_{1,t}^{v_1} - c_1) = \left(\frac{f_{i+1}}{g_{i+1}} \right)^{v_1} - c_1.$$

Accordingly $g_i = 1$, $f_i \in \Omega_1$ ($i \in \mathbb{Z}$), that is

$$\sigma(\Omega_1) \subset \Omega_1$$

and

$$\sigma'(\Omega_1) \subset \Omega_1$$

also. Similarly

$$\sigma'^{-1}(\Omega_1) \subset \Omega_1.$$

Consequently

$$\sigma'(\Omega_1) = \Omega_1.$$

Similarly

$$\begin{aligned} \sigma'(\Omega_2) &= \Omega_2, \\ &\dots\dots\dots, \\ \sigma'(\Omega_n) &= \Omega_n. \end{aligned}$$

REMARK 2.

- (1) If we vary n , k and $c_1, \dots, c_n$, we get various (k -) automorphism groups of Ω .
- (2) The case $c_1 = \dots = c_n = 1$ is treated in [1].

References

- [1] S. Hirabuki and R. Matsuda, The construction of fields with free abelian automorphism group, Bull. of Yamagata Univ. (to appear).
- [2] W. Kuyk, The construction of fields with infinite cyclic automorphism group, Canad. J. of Math., **17** (1965), 665-668.