On Certain Connected Submanifolds of Euclidean Sphere

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Let $\bar{M}^m$ be a simply connected, complete Riemannian manifold of non-positive constant curvature, that is, a Euclidean space E^m or a hyperbolic space H^m . Let M^n , $n \geq 2$, be a differentiable manifold. When f is an immersion of M^n into $\bar{M}^m$, we can define a function L_p on M^n by

$$L_p(x) = d(p, f(x)) \quad x \in M^n,$$

where d is the distance function in $\bar{M}^m$. For almost all $p \in \bar{M}^m$, L_p is a Morse function on M^n . Nomizu and Rodríguez [5] proved the following

THEOREM A. *Let M^n be a connected, complete Riemannian manifold isometrically immersed into E^m . If every Morse function L_p on M^n has index 0 or n at any of its critical points, then M^n is imbedded as a Euclidean n -subspace E^n or a Euclidean n -sphere S^n .*

This theorem includes that if M^n is compact such that every Morse function L_p has exactly two critical points, then M^n is imbedded as a Euclidean n -sphere S^n . Cecil [2] proved the following

THEOREM B. *Let M^n be a connected, compact manifold immersed into H^m . If every Morse function L_p on M^n has exactly two critical points, then M^n is imbedded as a metric n -sphere S^n .*

On the other hand, the author investigated in [3] connected, compact submanifolds M^n of E^m such that $C_n(L_p) = 1$ or $C_0(L_p) = 1$ for all Morse functions L_p on M^n , where $C_k(L_p)$ is the number of critical points of L_p of index k .

In this note, we deal with connected submanifolds M^n of a Euclidean sphere S^m .

We shall mean C^∞ differentiable by "differentiable" and an $n (\geq 2)$ -dimensional differentiable manifold M^n by a manifold M^n .

1. Preliminaries. Let M^n be a connected manifold. Let f be an immersion of M^n into a Euclidean m -sphere S^m about the origin of E^{m+1} , $m \geq n+1$. Since f induces a Riemannian metric on M^n , M^n satisfies the second axiom of countability ([4], p. 271). Therefore $K = \{y \in S^m; y = f(x) \text{ or } -f(x) \text{ for } x \in M^n\}$

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has measure zero. For a point $p \in S^m - K$, we define a function L_p on M^n by

$$L_p(x) = d(p, f(x)) \quad x \in M^n,$$

the distance in S^m from p to $f(x)$. If S^m is a unit sphere, then a geodesic $c(s)$ in S^m , parametrized by arc length s , such that $c(0) = y$ and $\dot{c}(0) = e$ is given by $c(s) = y \cos s + e \sin s$. This implies

$$\cos L_p(x) = \langle p, f(x) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in E^{m+1} . Therefore $x \in M^n$ is a critical point of L_p if and only if $p = f(x) \cos L_p(x) + e \sin L_p(x)$ and e is a unit vector normal to $f(M^n)$ at $f(x)$.

We denote by $\perp(M)$ the normal bundle to M^n . The exponential mapping $\exp: \perp(M) \rightarrow S^m$ is given by $\exp(x, se) = f(x) \cos s + e \sin s$ for $(x, se) \in \perp(M)$. If $p = \exp(x, se)$ and the Jacobian $\exp_*$ of $\exp$ has nullity $k > 0$ at (x, se) , then we shall call p a focal point of (M^n, x) of multiplicity k . For $(x, e) \in \perp(M)$, we denote by A_e a symmetric endomorphism of the tangent space $T_x M^n$ which corresponds to the second fundamental form of M^n .

We assume hereafter S^m is a unit sphere.

LEMMA 1. A point $p = \exp(x, s_0 e)$, $0 < s_0 < \pi$, is a focal point of (M^n, x) of multiplicity k if and only if $\cot s_0 = \lambda$ for an eigenvalue λ of A_e of multiplicity k .

PROOF. Let $(u^1, \dots, u^n, U)$ be a coordinate system at $x \in M^n$. Let $e_1, \dots, e_{m-n}$ be orthonormal vector fields normal to $f(U)$ such that $e_1(x) = e$. If $x' \in U$ and v is a vector normal to $f(U)$ at $f(x')$, then we have

$$v = s \left(\sqrt{1 - \sum_{i=2}^{m-n} (t^i)^2} e_1(x') + t^2 e_2(x') + \dots + t^{m-n} e_{m-n}(x') \right).$$

This implies $(u^1, \dots, u^n, s, t^2, \dots, t^{m-n})$ are coordinates at $(x, s_0 e)$ in $\perp(M)$. By the definition of mapping $\exp$, we obtain

$$\exp_*(\partial/\partial s)_{(x, s_0 e)} = -f(x) \sin s_0 + e \cos s_0$$

and

$$\exp_*(\partial/\partial t^i)_{(x, s_0 e)} = e_i(x) \sin s_0.$$

Let $X = \sum_{j=1}^n \xi^j (\partial/\partial u^j)_{(x, s_0 e)}$. Since $T_{(x, s_0 e)} \perp(M) = T_x M^n \oplus E^{m-n}$, there exists $Y \in T_x M^n$ and $(Y, 0) = X$. We assume the vector field e_1 normal to $f(U)$ such that $\nabla_{f_*(Y)} e_1 = -f_*(A_e Y)$, where ∇ is the Riemannian connection in S^m . We denote by $V(s)$ an infinitesimal variation of geodesic $c(s) = f(x) \cos s + e \sin s$, $0 \leq s \leq s_0$, such that $V(0) = f_*(Y)$ and $\nabla_e V = -f_*(A_e Y)$. Then we have

$$\exp_* X = V(s_0).$$

Therefore $\exp_* X$ is the parallel displacement of $f_*(Y) \cos s_0 - f_*(A_e Y) \sin s_0$ along the geodesic $c(s)$ from $f(x)$ to p . This implies $\exp_* X = 0$ if and only if $A_e Y = Y \cot s_0$, that is, $\cot s_0$ is an eigenvalue of A_e .

It is obvious that

$$\exp_* X' = 0$$

for $X' \in T_{(x, s_0 e)} \perp (M)$ only if $X' = \sum_{j=1}^n \xi'^j (\partial/\partial u^j)_{(x, s_0 e)}$. Thus we have proved the lemma.

LEMMA 2. Let $p \in S^m - K$ and $x \in M^n$. Then

(1) x is a degenerate critical point of L_p if and only if $\cot L_p(x) = \lambda$ for an eigenvalue λ of A_e ,

(2) if x is a non-degenerate critical point of L_p , then the index of L_p at x is equal to the number of eigenvalues λ of A_e such that $\cot L_p(x) < \lambda$, counting multiplicities.

PROOF. We choose a coordinate system $(u^1, \dots, u^n, U)$ at $x \in M^n$ such that $\{(\partial/\partial u^1)_x, \dots, (\partial/\partial u^n)_x\}$ is an orthonormal basis of $T_x M^n$. We put $y = f(x')$ for $x' \in U$. Then we have

$$\begin{aligned} & -\frac{\partial^2 L_p}{\partial u^i \partial u^j} \sin L_p(x') - \frac{\partial L_p}{\partial u^i} \frac{\partial L_p}{\partial u^j} \cos L_p(x') \\ & = \langle p, \frac{\partial^2 y}{\partial u^i \partial u^j} \rangle \end{aligned}$$

from $\cos L_p(x') = \langle p, y \rangle$. Therefore if x is a critical point of L_p , then

$$\begin{aligned} & -\frac{\partial^2 L_p}{\partial u^i \partial u^j}(x) \sin L_p(x) \\ & = \langle p, \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle \\ & = \langle f(x), \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle \cos L_p(x) + \langle e, \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle \sin L_p(x), \end{aligned}$$

because $p = f(x) \cos L_p(x) + e \sin L_p(x)$. Since $\langle y, \partial y / \partial u^i \rangle = 0$, we have

$$\langle f(x), \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle = -\delta_{ij}.$$

On the other hand, if we write $A_e(\partial/\partial u^j)_x = \sum_{i=1}^n A_{eij}(x)(\partial/\partial u^i)_x$, then we have easily that

$$\langle e, \frac{\partial^2 y}{\partial u^i \partial u^j}(x) \rangle = A_{eij}(x).$$

Therefore we obtain

$$\frac{\partial^2 L_p}{\partial u^i \partial u^j}(x) = \delta_{ij} \cot L_p(x) - A_{eij}(x),$$

that is, the Hessian H of L_p at x is given by

$$H(X, Y) = \langle (I \cot L_p(x) - A_e)X, Y \rangle \quad X, Y \in T_x M^n.$$

Here I denotes the identity mapping of $T_x M^n$ into itself. Thus the lemma follows.

By virtue of Lemmas 1 and 2, L_p is a Morse function on M^n for almost all $p \in S^m - K$. Furthermore, Lemmas 1 and 2 show the following

THEOREM 1. (Index theorem for L_p) *For $p \in S^m - K$, the index of L_p at a non-degenerate critical point $x \in M^n$ is equal to the number of focal points of (M^n, x) which lie on the minimizing geodesic $c(s) = f(x) \cos s + e \sin s$, $0 \leq s \leq L_p(x)$, from $f(x)$ to p . Each focal point is counted with its multiplicity.*

LEMMA 3. *If $x \in M^n$ is a non-degenerate critical point of L_p of index k , then there exist sequences $\{x_\alpha\}$ of points in M^n and $\{p_\alpha\}$ of points in $S^m - K$ such that*

- (1) $\{x_\alpha\}$ and $\{p_\alpha\}$ converge to x and p respectively,
- (2) L_{p_α} is a Morse function on M^n and x_α is a critical point of L_{p_α} of index k .

2. Main theorems

THEOREM 2. *Let M^n be a connected, complete Riemannian manifold isometrically immersed into S^m . If every Morse function L_p on M^n has index 0 or n at each critical point of it, then M^n is imbedded as a small n -sphere S^n .*

PROOF. Let $x \in M^n$ and let e be a unit vector normal to $f(M^n)$ at $f(x)$. We assume λ is the largest eigenvalue of A_e . If there exists the next largest eigenvalue μ of A_e , then $p = f(x) \cos L_p(x) + e \sin L_p(x)$, $\lambda > \cot L_p(x) > \mu$, is not a focal point of (M^n, x) from Lemma 1. Therefore L_p has x as a non-degenerate critical point. The index k of L_p at x is equal to the multiplicity of eigenvalue λ of A_e from Theorem 1. There exists, from Lemma 3, a Morse function L_q on M^n which has a critical point x' of index k . By the assumption, L_q has index 0 or n at x' . And so we have $k = n$ since k cannot be 0. This shows that A_e has only one eigenvalue λ of multiplicity n , which implies M^n is a totally umbilical submanifold of E^{m+1} , through the immersion f . Therefore M^n is imbedded into E^{m+1} as a Euclidean n -sphere ([1], p. 231). Hence M^n is imbedded into S^m as a small n -sphere S^n .

COROLLARY. *Let M^n be a connected, compact manifold immersed into S^m . If every Morse function L_p on M^n has exactly two critical points, then M^n is imbedded as a small n -sphere S^n .*

PROOF. Since M^n is compact, a Morse function L_p on M^n attains to the

maximal value at one critical point and the minimal value at the other. Therefore the corollary follows.

THEOREM 3. *Let M^n be a connected, compact manifold immersed into S^m . Assume that*

- (1) *there exists an umbilical point $x \in M^n$,*
- (2) *$C_n(L_p)=1$ for every Morse function L_p on M^n .*

Then M^n is imbedded as a small n -sphere S^n .

PROOF. If $x \in M^n$ is an umbilical point, then we have $A_e = \lambda(e)I$ for any vector e normal to $f(M^n)$ at $f(x)$. Let $e_1, \dots, e_{m-n}$ be orthonormal vectors normal to $f(M^n)$ at $f(x)$. Then, from Lemma 1, only one focal point p_i of (M^n, x) lies on each geodesic $c_i(s) = f(x) \cos s + e_i \sin s$, $0 \leq s < \pi$, where $1 \leq i \leq m-n$. The focal point p_i is given by $p_i = f(x) \cos L_{p_i}(x) + e_i \sin L_{p_i}(x)$ and $\cot L_{p_i}(x) = \lambda(e_i)$. For a point $p = f(x) \cos L_p(x) + e_i \sin L_p(x)$ such that $L_{p_i}(x) < L_p(x) < \pi$, x is a non-degenerate critical point of L_p of index n from Theorem 1. There exist sequences $\{x_\alpha\}$ of points in M^n and $\{p_\alpha\}$ of points in $S^m - K$ as in Lemma 3. Since $C_n(L_{p_\alpha}) = 1$ from (2), each Morse function L_{p_α} on M^n attains to the maximal value at x_α . Therefore

$$L_p(x) = \lim_{\alpha} L_{p_\alpha}(x_\alpha) \geq \lim_{\alpha} L_{p_\alpha}(x') = L_p(x')$$

for all $x' \in M^n$.

Let $\tilde{c}_i(s) = f(x) \cos s - e_i \sin s$, $0 \leq s < \pi$. Then, from $\lambda(-e_i) = -\lambda(e_i)$, the antipodal point $-p_i$ of p_i is only one focal point of (M^n, x) along the geodesic $\tilde{c}_i(s)$. Therefore if $q = f(x) \cos L_q(x) - e_i \sin L_q(x)$ such that $L_{-p_i}(x) < L_q(x) < \pi$, then L_q attains to the maximal value at x . We have

$$L_q(x') + L_{-q}(x') = \pi$$

for $x' \in M^n$. Hence L_{-q} attains to the minimal value at x . The point $-q$ can be written as $-q = f(x) \cos L_{-q}(x) + e_i \sin L_{-q}(x)$ such that $0 < L_{-q}(x) < L_{p_i}(x)$. Thus each L_{p_i} is a constant function on M^n , and hence there exists in E^{m+1} a linear subvariety E_i^m perpendicular to p_i such that $f(M^n) \subset E_i^m$. Since $e_1, \dots, e_{m-n}$ are orthonormal, $p_1, \dots, p_{m-n}$ are linearly independent. Therefore $E_1^m \cap \dots \cap E_{m-n}^m$ is a linear subvariety E^{n+1} in E^{m+1} . We know that $S^m \cap E^{n+1}$ is a Euclidean n -sphere S^n . The mapping $f: M^n \rightarrow S^n$ is an immersion. Since S^n is simply connected from $n \geq 2$, f is a diffeomorphism. Therefore M^n is imbedded as a small n -sphere S^n .

THEOREM 4. *Let M^n be a connected, compact manifold immersed into S^m . Assume that*

- (1) *there exists an umbilical point $x \in M^n$,*
- (2) *$C_0(L_p)=1$ for every Morse function L_p on M^n .*

Then M^n is imbedded as a small n -sphere S^n .

PROOF. If $x \in M^n$ is an umbilical point, then we have $A_e = \lambda(e)I$ for a unit vector e normal to $f(M^n)$ at $f(x)$. Therefore, from Lemma 1, $p' = f(x) \cos L_p(x) + e \sin L_p(x)$ such that $\cot L_{p'}(x) = \lambda(e)$ is only one focal point of (M^n, x) along a geodesic $c(s) = f(x) \cos s + e \sin s$, $0 \leq s < \pi$. For a point $p = f(x) \cos L_p(x) + e \sin L_p(x)$ such that $0 < L_p(x) < L_{p'}(x)$, x is a non-degenerate critical point of L_p of index 0 from Theorem 1. There exist sequences $\{x_\alpha\}$ of points in M^n and $\{p_\alpha\}$ of points in $S^m - K$ as in Lemma 3. Since $C_0(L_{p_\alpha}) = 1$ from (2), each Morse function L_{p_α} attains to the minimal value at x_α . Therefore

$$L_p(x) = \lim_{\alpha} L_{p_\alpha}(x_\alpha) \leq \lim_{\alpha} L_{p_\alpha}(x') = L_p(x')$$

for all $x' \in M^n$. Similarly, if $q = f(x) \cos L_q(x) - e \sin L_q(x)$ such that $0 < L_q(x) < L_{-p'}(x)$, then L_q attains to the minimal value at x , which implies L_{-q} attains to the maximal value at x . Thus L_p is a constant function on M^n . Therefore we can find a small n -sphere S^n such that $f(M^n) \subset S^n$. Since $f: M^n \rightarrow S^n$ is an immersion and S^n is simply connected from $n \geq 2$, M^n is imbedded as a small n -sphere S^n .

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