Integral Transforms and Semigroups of Linear Operators on the W^2 -space

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Introduction. Let W^2 be the totality of measurable functions such that

$$\int_{-\infty}^{\infty} \frac{|f(v)|^2}{v^2+1} dv < \infty.$$

When f and g are in W^2 we define the inner product of f and g by

$$(f, g) = \int_{-\infty}^{\infty} f(v) \overline{g(v)} \frac{dv}{v^2+1}.$$

Introducing the operations of addition and scalar multiplication in W^2 as usual, we can show that the space W^2 is the Hilbert space with norm $\|f\| = [(f, f)]^{1/2}$. Let

$$p_{\lambda}(t, u) = \frac{\lambda t}{\pi[(\lambda t)^2 + u^2]}.$$

If $t > 0$ and $\lambda > 0$, this function $p_{\lambda}(t, u)$ is the Cauchy density function. When $\lambda = r - iq$ ($r > 0$) and $t > 0$ let

$$(P_{\lambda}(t)f)(v) = \int_{-\infty}^{\infty} p_{\lambda}(t, u) f(v-u) du$$

and

$$(P_{\lambda}(0)f)(v) = f(v)$$

for f in W^2 . The purpose of this paper is to show that a family $\{P_{\lambda}(t): 0 \leq t < \infty\}$ of linear operators on W^2 is a semigroup of class (C_0) and that there exists a one-parameter strongly continuous group of linear operators $\{P_{-iq}(t): -\infty < t < \infty\}$ on W^2 .

1. Semigroups and groups of linear operators on W^2 . If $\lambda = r - iq$ ($r > 0$) and $t > 0$,

$$|p_{\lambda}(t, u)| = \frac{t(r^2 + q^2)^{1/2}}{\pi[(u^2 + t^2(r^2 - q^2))^2 + 4t^4 r^2 q^2]^{1/2}}. \quad (1.1)$$

LEMMA 1.1. Let $\lambda = r - iq$ ($r > 0$) and $t > 0$. If f is in W^2 , then $(P_{\lambda}(t)f)(v)$

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is a function in W^2 .

PROOF. Take a positive number a such that $|t^2(r^2 - q^2)| < a^2$. Then

$$\begin{aligned} |(P_\lambda(t)f)(v)| &\leq \int_{-a}^a |p_\lambda(t, u)f(v-u)|du + \int_a^\infty |p_\lambda(t, u)f(v-u)|du \\ &\quad + \int_{-\infty}^{-a} |p_\lambda(t, u)f(v-u)|du. \end{aligned}$$

Since $|p_\lambda(t, u)| \leq K_1(r, q, t) < \infty$ for $|u| \leq a$ and $|p_\lambda(t, u)| \leq K_2(r, q, t)u^{-2}$ for $|u| > a$, we have

$$\begin{aligned} |(P_\lambda(t)f)(v)| &\leq K_1(r, q, t) \int_{-a}^a |f(v-u)|du \\ &\quad + K_2(r, q, t) \int_a^\infty u^{-2} |f(v-u)|du + K_2(r, q, t) \int_{-\infty}^{-a} u^{-2} |f(v-u)|du < \infty \end{aligned}$$

for each $v \in R$ (the real line). It is known [11; Theorem 20] that if

$$\frac{1}{2T} \int_{-T}^T |g(v)|^2 dv$$

is bounded in T , then

$$\int_{-\infty}^{\infty} \frac{|g(v)|^2}{v^2 + 1} dv < \infty.$$

Hence, to show that $P_\lambda(t)f$ is in W^2 , it suffices to show that

$$\frac{1}{2T} \int_{-T}^T |(P_\lambda(t)f)(v)|^2 dv \tag{1.2}$$

is bounded in T . By the Schwartz inequality we see that

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T |(P_\lambda(t)f)(v)|^2 dv &\leq \frac{1}{2T} \int_{-T}^T \left(\int_{-\infty}^{\infty} |p_\lambda(t, v-u)f(u)| du \right)^2 dv \\ &\leq \frac{1}{2T} \int_{-T}^T \left[\int_{-\infty}^{\infty} |p_\lambda(t, v-u)|^2 (1 + (v-u)^2) du \int_{-\infty}^{\infty} \frac{|f(u)|^2}{(u-v)^2 + 1} du \right] dv. \end{aligned}$$

From (1.1) it follows that

$$\int_{-\infty}^{\infty} |p_\lambda(t, x)|^2 (1 + x^2) dx < \infty.$$

Also it is known [11; pp. 168-169] that

$$\frac{1}{2T} \int_{-T}^T \left[\int_{-\infty}^{\infty} \frac{|f(u)|^2}{(v-u)^2 + 1} du \right] dv$$

is bounded in T . From these facts we obtain that (1.2) is bounded in T . Q.E.D.

We have the following

LEMMA 1.2. Let $\lambda = r - iq$ ($r > 0$) and $t > 0$. If f is in $L^2(R)$ then

$$\int_{-\infty}^{\infty} p_{\lambda}(t, u) f(v-u) du = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (U_w f)(u) \exp(iuv - \lambda t|u|) du$$

where we let

$$(U_w f)(u) = (Uf)(u) = \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a f(w) \exp(-i w u) dw.$$

Here l.i.m denotes limit in the mean with index 2. cf. [3; Theorem 21, p. 974]

LEMMA 1.3. Let $\lambda = r - iq$ ($r > 0$) and $t > 0$. Let α be a positive number. Then

1) if $0 < tr < \alpha$,

$$\begin{aligned} (P_{\lambda}(t)f)(z) &= (z - i\alpha) (2\pi)^{-1/2} \int_{-\infty}^0 (Uf_{\lambda t})(u) \exp(\lambda t + iz)u du \\ &\quad + (z - i\alpha) (2\pi)^{-1/2} \int_0^{\infty} (Uk_{\lambda t})(u) \exp(-\lambda t + iz)u du \\ &\quad + \left(\frac{2}{\pi}\right)^{1/2} i\lambda t \int_0^{\infty} (Uk_{\lambda t})(u) \exp(\lambda t - \alpha)u du, \end{aligned} \quad (1.3)$$

2) if $\alpha < tr$,

$$\begin{aligned} (P_{\lambda}(t)f)(z) &= (z - i\alpha) (2\pi)^{-1/2} \int_{-\infty}^0 (Uf_{\lambda t})(u) \exp(\lambda t + iz)u du \\ &\quad + (z - i\alpha) (2\pi)^{-1/2} \int_0^{\infty} (Uk_{\lambda t})(u) \exp(-\lambda t + iz)u du \\ &\quad - \left(\frac{2}{\pi}\right)^{1/2} i\lambda t \int_{-\infty}^0 (Uk_{\lambda t})(u) \exp(\lambda t - \alpha)u du, \end{aligned} \quad (1.3')$$

where we let

$$f_{\lambda t}(u) = \frac{f(u)}{u - i\alpha + i\lambda t}, \quad k_{\lambda t}(u) = \frac{f(u)}{u - i\alpha - i\lambda t}.$$

PROOF. It is easy to show that

$$\begin{aligned} p_{\lambda}(t, u - z) &= \frac{1}{2\pi} \left[\frac{z - i\alpha}{(u - i\alpha + i\lambda t)(\lambda t - i(u - z))} + \frac{z - i\alpha}{(u - i\alpha - i\lambda t)(\lambda t + i(u - z))} \right. \\ &\quad \left. + \frac{2\lambda t}{(u - i\alpha - i\lambda t)(u - i\alpha + i\lambda t)} \right]. \end{aligned}$$

Hence we have

$$\begin{aligned} (P_{\lambda}(t)f)(v) &= (z - i\alpha) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(u) du}{(u - i\alpha + i\lambda t)(\lambda t + i(z - u))} \\ &\quad + (z - i\alpha) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(u) du}{(u - i\alpha - i\lambda t)(\lambda t - i(z - u))} \end{aligned}$$

$$+ \frac{\lambda t}{\pi} \int_{-\infty}^{\infty} \frac{f(u) du}{(u - i\alpha - i\lambda t)(u - i\alpha + i\lambda t)} \quad (1.4)$$

By the Parseval theorem it holds that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{f(u) du}{(u - i\alpha + i\lambda t)(\lambda t + i(z - u))} &= (2\pi)^{1/2} \int_{-\infty}^0 (Uf_{\lambda t})(u) \exp(\lambda t u + izu) du, \\ \int_{-\infty}^{\infty} \frac{f(u) du}{(u - i\alpha - i\lambda t)(\lambda t - i(z - u))} &= (2\pi)^{1/2} \int_0^{\infty} (Uk_{\lambda t})(u) \exp(-\lambda t u + izu) du, \\ \int_{-\infty}^{\infty} \frac{f(u) du}{(u - i\alpha - i\lambda t)(u - i\alpha + i\lambda t)} &= \begin{cases} (2\pi)^{1/2} i \int_0^{\infty} (Uk_{\lambda t})(u) \exp(\lambda t - \alpha) u du & \text{if } 0 < tr < \alpha, \\ -(2\pi)^{1/2} i \int_{-\infty}^0 (Uk_{\lambda t})(u) \exp(\lambda t - \alpha) u du & \text{if } \alpha < tr \end{cases} \end{aligned}$$

(cf. [1, p. 128], [11, Theorem 3]). From these equalities and from (1.4) we obtain (1.3) and (1.3'). Q. E. D.

LEMMA 1.4. *Let $\lambda = r - iq$ ($r > 0$) and $t > 0$. Then $P_{\lambda}(t)$ is a linear bounded operator on W^2 .*

PROOF. It is clear that $P_{\lambda}(t)$ is linear. Let α be a positive number such that $1 \leq \alpha$ and $0 < tr < \alpha$. By the Minkowski inequality and by Plancherel's theorem, and by the Schwartz inequality we see that

$$\begin{aligned} \|P_{\lambda}(t)f\| &\leq \left[\int_{-\infty}^{\infty} \frac{z^2 + \alpha^2}{z^2 + 1} \left| (2\pi)^{-1/2} \int_{-\infty}^0 (Uf_{\lambda t})(u) \exp(\lambda t + iz)u du \right|^2 dz \right]^{1/2} \\ &\quad + \left[\int_{-\infty}^{\infty} \frac{z^2 + \alpha^2}{z^2 + 1} \left| (2\pi)^{-1/2} \int_0^{\infty} (Uk_{\lambda t})(u) \exp(-\lambda t + iz)u du \right|^2 dz \right]^{1/2} \\ &\quad + \left[\int_{-\infty}^{\infty} \frac{1}{z^2 + 1} \left| \left(\frac{2}{\pi} \right)^{1/2} i\lambda t \int_0^{\infty} (Uk_{\lambda t})(u) \exp(\lambda t - \alpha)u du \right|^2 dz \right]^{1/2} \\ &\leq \alpha \left[\int_{-\infty}^0 |(Uf_{\lambda t})(u)|^2 du \right]^{1/2} + \alpha \left[\int_0^{\infty} |(Uk_{\lambda t})(u)|^2 du \right]^{1/2} \\ &\quad + \left(\frac{2}{\pi} \right)^{1/2} |\lambda| t \left[\int_0^{\infty} |(Uk_{\lambda t})(u)|^2 du \right]^{1/2} \\ &\quad \cdot \left[\int_0^{\infty} |\exp(\lambda t - \alpha)u|^2 du \right]^{1/2} \left[\int_{-\infty}^{\infty} \frac{dz}{z^2 + 1} \right]^{1/2} \\ &\leq \left[\alpha \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - i\alpha + i\lambda t} \right| + \alpha \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - i\alpha - i\lambda t} \right| \right] \|f\| \\ &\quad + \left(\frac{2}{\pi} \right)^{1/2} |\lambda| t \left[\sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - i\alpha - i\lambda t} \right| \right] \pi^{1/2} \left[-\frac{1}{2(tr - \alpha)} \right]^{1/2} \|f\|. \quad (1.5) \end{aligned}$$

Since

$$\sup_{u \in \mathbb{R}} \left| \frac{u-i}{u-i\alpha+i\lambda t} \right|, \sup_{u \in \mathbb{R}} \left| \frac{u-i}{u-i\alpha-i\lambda t} \right|$$

are bounded, from (1.5) it follows that $P_\lambda(t)$ is a bounded operator on W^2 .

Q. E. D.

THEOREM 1.1. *Let $\lambda = r - iq$ ($r > 0$). Let $P_\lambda(0) = I$, where I denotes the identity operator on W^2 . Then a family $\{P_\lambda(t): 0 \leq t < \infty\}$ of linear bounded operators on W^2 is a semigroup of class (C_0) .*

PROOF. *Semigroup property $P_\lambda(t)P_\lambda(s) = P_\lambda(t+s)$: Let $0 < s, t < \infty$. First let f be a continuous function with compact support. Then we have*

$$\begin{aligned} (P_\lambda(t)P_\lambda(s)f)(v) &= \int_{-\infty}^{\infty} p_\lambda(t, u) du \int_{-\infty}^{\infty} p_\lambda(s, v-u-y) f(y) dy \\ &= \int_{-\infty}^{\infty} f(y) dy \int_{-\infty}^{\infty} p_\lambda(t, u) p_\lambda(s, v-u-y) du \\ &= \int_{-\infty}^{\infty} p_\lambda(t+s, v-y) f(y) dy = (P_\lambda(t+s)f)(v) \end{aligned}$$

(cf. [8, proof of Lemma 1.3]). When f is in W^2 , taking a sequence of continuous functions f_n with compact support converging to f in the norm we obtain

$$\begin{aligned} \|P_\lambda(t)P_\lambda(s)f - P_\lambda(t+s)f\| &\leq \|P_\lambda(t)P_\lambda(s)f - P_\lambda(t)P_\lambda(s)f_n\| \\ &\quad + \|P_\lambda(t+s)f_n - P_\lambda(t+s)f\|. \end{aligned}$$

By Lemma 1.4

$$\begin{aligned} \|P_\lambda(t)P_\lambda(s)(f-f_n)\| &\leq \|P_\lambda(t)\| \|P_\lambda(s)\| \|f-f_n\| \longrightarrow 0 \quad \text{and} \\ \|P_\lambda(t+s)(f-f_n)\| &\leq \|P_\lambda(t+s)\| \|f-f_n\| \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, hence $P_\lambda(t)P_\lambda(s)f = P_\lambda(t+s)f$ for f in W^2 .

Strong continuity $P_\lambda(s)f \rightarrow P_\lambda(t)f$ ($s \rightarrow t$): By the semigroup property it suffices to prove that

(a) $P_\lambda(t)f \rightarrow f$ as $t \rightarrow +0$ in the norm for all f in W^2 .

It is known [4, Theorem 10.6.5] that (a) is equivalent to

(b) $(P_\lambda(t)f, g) \rightarrow (f, g)$ as $t \rightarrow +0$ for all f, g in W^2 .

Let us prove (b). First let f be a continuous function with compact support. Then by Lemma 1.2 and by the Parseval theorem we obtain

$$\begin{aligned} (P_\lambda(t)f, g) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} (Uf)(u) \exp(i\lambda t|u|) du \right] \overline{\frac{g(v)}{v^2+1}} dv \\ &= \int_{-\infty}^{\infty} (Uf)(u) \exp(-\lambda t|u|) \overline{U_v[g(v)(v^2+1)^{-1}]}(u) du \end{aligned}$$

$$\longrightarrow \int_{-\infty}^{\infty} (Uf)(u) \overline{U_v[g(v)(v^2+1)^{-1}]}(u) du \quad \text{as } t \longrightarrow +0.$$

By the Parseval theorem

$$\int_{-\infty}^{\infty} (Uf)(u) \overline{U_v[g(v)(v^2+1)^{-1}]}(u) du = \int_{-\infty}^{\infty} f(v) \overline{\frac{g(v)}{v^2+1}} dv,$$

hence it follows that $(P_\lambda(t)f, g) \rightarrow (f, g)$ as $t \rightarrow +0$ for the above function f . If f is in W^2 , taking a sequence of continuous functions f_n with compact support converging to f in the norm we have

$$\begin{aligned} |(P_\lambda(t)f, g) - (f, g)| &\leq |(P_\lambda(t)(f-f_n), g)| \\ &\quad + |(P_\lambda(t)f_n, g) - (f_n, g)| + |(f_n, g) - (f, g)|. \end{aligned} \quad (1.6)$$

It is seen from (1.5) that $P_\lambda(t)$ is uniformly bounded in the neighborhood at $t=0$. Hence from (1.6) it follows that

$$|(P_\lambda(t)f, g) - (f, g)| \longrightarrow 0 \quad \text{as } t \longrightarrow +0. \quad \text{Q. E. D.}$$

We have a following

REMARK 1.1. Let f be in W^2 . Then $P_\lambda(t)f$ is strongly continuous in λ on $C^+ = \{\lambda: \text{Real part of } \lambda \text{ is positive}\}$.

LEMMA 1.5. Let $q \neq 0$ be given and let f be in W^2 . Let

$$\begin{aligned} (P_{-iq}(t)f)(z) &= (z - i\alpha) \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Uf_{-iq_t})(u) \exp(-iqt + iz)u du \\ &\quad + (z - i\alpha) \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_0^a (Uk_{-iq_t})(u) \exp(iqt + iz)u du \\ &\quad + \left(\frac{2}{\pi}\right)^{1/2} qt \int_0^\infty (Uk_{-q_t})(u) \exp(iqt - \alpha)u du \end{aligned} \quad (1.7)$$

for a positive number $\alpha \geq 1$ and $t > 0$. Then $P_{r-iq}(t)f \rightarrow P_{-iq}(t)f$ as $r \rightarrow +0$ in the norm and $P_{-iq}(t)$ is a linear bounded operator on W^2 .

PROOF. Let $\lambda = r - iq$ ($r > 0$). Suppose $0 < tr < 1 \leq \alpha$. By (1.3) and by the Minkowski inequality, and by Plancherel's theorem we see that

$$\begin{aligned} \|P_\lambda(t)f - P_{-iq}(t)f\| &\leq \left[\int_{-\infty}^{\infty} \frac{z^2 + \alpha^2}{z^2 + 1} \left| (2\pi)^{-1/2} \int_{-\infty}^0 (Uf_{\lambda t})(u) \exp(\lambda t + iz)u du \right. \right. \\ &\quad \left. \left. - \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Uf_{-iq_t})(u) \exp(-iqt + iz)u du \right|^2 dz \right]^{1/2} \\ &\quad + \left[\int_{-\infty}^{\infty} \frac{z^2 + \alpha^2}{z^2 + 1} \left| (2\pi)^{-1/2} \int_0^\infty (Uk_{\lambda t})(u) \exp(-\lambda t + iz)u du \right. \right. \\ &\quad \left. \left. - \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_0^a (Uk_{-iq_t})(u) \exp(iqt + iz)u du \right|^2 dz \right]^{1/2} \end{aligned}$$

$$\begin{aligned}
& + \left[\int_{-\infty}^{\infty} \frac{1}{z^2 + 1} \left| \left(\frac{2}{\pi} \right)^{1/2} i \lambda t \int_0^{\infty} (Uk_{\lambda t})(u) \exp(\lambda t - \alpha) u \, du \right. \right. \\
& \quad \left. \left. - \left(\frac{2}{\pi} \right)^{1/2} q t \int_0^{\infty} (Uk_{-iq t})(u) \exp(-iq t - \alpha) u \, du \right|^2 dz \right]^{1/2} \\
& \leq \alpha \left[\int_{-\infty}^0 |(Uf_{\lambda t})(u) e^{\lambda t u} - (Uf_{-iq t})(u) e^{-iq t u}|^2 du \right]^{1/2} \\
& \quad + \alpha \left[\int_0^{\infty} |(Uk_{\lambda t})(u) e^{-\lambda t u} - (Uk_{-iq t})(u) e^{iq t u}|^2 du \right]^{1/2} \\
& \quad + 2^{1/2} t \left| \int_0^{\infty} \lambda (Uk_{\lambda t})(u) e^{(\lambda t - \alpha) u} du - \int_0^{\infty} q (Uk_{-iq t})(u) e^{(-iq t - \alpha) u} du \right| \\
& \longrightarrow 0 \text{ as } r \longrightarrow +0.
\end{aligned}$$

In the same manner as the proof of Lemma 1.4 we can prove that $P_{-iq}(t)$ is a bounded operator on W^2 . Q. E. D.

REMARK 1.2. If f is in $L^2(R)$,

$$(P_{-iq}(t)f)(v) = \text{l.i.m.}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a (Uf)(u) \exp(iuv + iqt|u|) du.$$

This follows from Lemma 1.2.

THEOREM 1.2. Let $q \neq 0$ be given and let α be the same number as in Lemma 1.5. When $t > 0$, let

$$\begin{aligned}
(P_{-iq}(-t)f)(z) &= (z - i\alpha) \text{l.i.m.}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Uf_{iq t})(u) \exp(iqt + iz)u \, du \\
& \quad + (z - i\alpha) \text{l.i.m.}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_0^a (Uk_{iq t})(u) \exp(-iq t + iz)u \, du \\
& \quad - \left(\frac{2}{\pi} \right)^{1/2} q t \int_0^{\infty} (Uk_{iq t})(u) \exp(iqt - \alpha)u \, du
\end{aligned} \tag{1.8}$$

for f in W^2 . Then $P_{-iq}(-t)$ is a linear bounded operator on W^2 and

$$P_{-iq}(t)P_{-iq}(-t) = P_{-iq}(-t)P_{-iq}(t) = I. \tag{1.9}$$

A family $\{P_{-iq}(t) : -\infty < t < \infty\}$ is a strongly continuous group of linear operators on W^2 , where let $P_{-iq}(0) = I$.

PROOF. In the same manner as the proof of Lemma 1.4 we can prove that $P_{-iq}(-t)$ is a linear bounded operator on W^2 . Let us prove (1.9). Let $S(R)$ be the space of rapidly decreasing and infinitely differentiable functions. It is easy to show that $S(R)$ is dense in W^2 . First let us prove that

$$(P_{-iq}(-t)f)(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Uf)(u) \exp(izu - iqt|u|) du \tag{1.10}$$

for all f in $S(R)$. Let f be in $S(R)$. By the Parseval theorem we see that

$$\begin{aligned}(Uf_{iq})(u) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{f(y)}{y - tq - i\alpha} \exp(-iyu) dy \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Uf)(y) U_w \left[\frac{1}{w - tq - i\alpha} \right] (u - y) dy \\ &= i \int_u^{\infty} (Uf)(y) \exp(itq - \alpha)y dy \exp(\alpha - itq)u\end{aligned}$$

(cf. [1, p. 128], [11, Theorem 3]). Hence we obtain by the Fubini theorem that

$$\begin{aligned}(\text{the first term of (1.8)}) &= (z - i\alpha)(2\pi)^{-1/2} \int_{-\infty}^0 \left[i \int_u^{\infty} (Uf)(y) \exp(itq - \alpha)y dy \right] \\ &\quad \cdot \exp(\alpha + iz)u du \\ &= (z - i\alpha)(2\pi)^{-1/2} i \left[\int_{-\infty}^0 (Uf)(y) \exp(itq - \alpha)y dy \int_{-\infty}^y \exp(\alpha + iz)u du \right. \\ &\quad \left. + \int_0^{\infty} (Uf)(y) \exp(itq - \alpha)y dy \int_{-\infty}^0 \exp(\alpha + iz)u du \right] \\ &= (2\pi)^{-1/2} \left[\int_{-\infty}^0 (Uf)(y) \exp(itq + iz)y dy + \int_0^{\infty} (Uf)(y) \exp(itq - \alpha)y dy \right]. \quad (1.11)\end{aligned}$$

Also by the Parseval theorem we see that

$$\begin{aligned}(Uk_{iq})(u) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{f(y)}{y + tq - i\alpha} \exp(-iyu) dy \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Uf)(y) U_w \left[\frac{1}{w + tq - i\alpha} \right] (u - y) dy \\ &= i \int_u^{\infty} (Uf)(y) \exp(-itq - \alpha)y dy \exp(itq + \alpha)u.\end{aligned}$$

Hence we obtain that

$$\begin{aligned}(\text{the second term of (1.8)}) &= (z - i\alpha)(2\pi)^{-1/2} \int_0^{\infty} \left[i \int_u^{\infty} (Uf)(y) \exp(-itq - \alpha)y dy \right] \exp(\alpha + iz)u du \\ (\text{by the Fubini theorem}) &= (z - i\alpha)(2\pi)^{-1/2} i \int_0^{\infty} (Uf)(y) \exp(-itq - \alpha)y dy \int_0^y \exp(\alpha + iz)u du \\ &= (2\pi)^{-1/2} \int_0^{\infty} (Uf)(y) \exp(-itq + iz)y dy - (2\pi)^{-1/2} \int_0^{\infty} (Uf)(y) \\ &\quad \cdot \exp(-\alpha - itq)y dy. \quad (1.12)\end{aligned}$$

By the Parseval theorem we see that

(the third term of (1.8))

$$\begin{aligned}
 &= (2\pi)^{-1} 2iqt \int_{-\infty}^{\infty} \frac{f(v)dv}{(v+tq-i\alpha)(v-tq-i\alpha)} \\
 &= -(2\pi)^{-1} i \left[\int_{-\infty}^{\infty} \frac{f(v)dv}{v+tq-i\alpha} - \int_{-\infty}^{\infty} \frac{f(v)dv}{v-tq-i\alpha} \right] \\
 &= (2\pi)^{-1/2} \int_0^{\infty} (Uf)(u) (\exp(-itq-\alpha)u - \exp(itq-\alpha)u) du. \quad (1.13)
 \end{aligned}$$

From (1.11), (1.12) and (1.13) we obtain (1.10). By (1.10) and by the fact that $P_{-iq}(-t)$ is a bounded operator, it holds that

$$(P_{-iq}(-t)f)(z) = \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a (Uf)(u) \exp(izu - iqt|u|) du \quad (1.14)$$

for f in $L^2(R)$. Hence if f is in $L^2(R)$, from Remark 1.2 and (1.14) it follows that

$$P_{-iq}(t)P_{-iq}(-t)f = P_{-iq}(-t)P_{-iq}(t)f = f. \quad (1.15)$$

By this equality and by the fact that $P_{-iq}(t)$ and $P_{-iq}(-t)$ are bounded operators, we obtain (1.15) for all f in W^2 .

From Remark 1.2, (1.14) and from the fact that $P_{-iq}(t)$, $P_{-iq}(-t)$ are bounded operators it follows that

$$P_{-iq}(t)P_{-iq}(s) = P_{-iq}(t+s) \quad (1.16)$$

for $-\infty < t, s < \infty$. In the same manner as the proof of Theorem 1.1 we can show that $\{P_{-iq}(t): 0 \leq t < \infty\}$ is a semigroup of class (C_0) . Hence by this fact and by (1.16) the family $\{P_{-iq}(t): -\infty < t < \infty\}$ is a one-parameter strongly continuous group. Q. E. D.

2. Infinitesimal generators and its domains. In what follows, for simplicity let $n(u) = f(u)/(u-i)$ for f in W^2 . In the following discussions, suppose that $\alpha = 1$ in Lemma 1.3, Lemma 1.5 and Theorem 1.2.

THEOREM 2.1. *The infinitesimal generator A_λ of the semigroup $\{P_\lambda(t): 0 \leq t < \infty\}$ in Theorem 1.1 is given by the following form;*

$$\begin{aligned}
 D(A_\lambda) &= \{f \in W^2: |u|n(u) \in L^2(R)\} \text{ and for } f \text{ in } D(A_\lambda) \\
 (A_\lambda f)(z) &= i\lambda \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a (Un)(u) (\text{sign } u) \exp(izu) du \\
 &\quad - \lambda(z-i) \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a |u| (Un)(u) \exp(izu) du. \quad (2.1)
 \end{aligned}$$

PROOF. Let $D(A_\lambda)$ denote the domain of the infinitesimal generator A_λ . When f is in $D(A_\lambda)$, let $A_\lambda f = g$. Let us put

$$H(\lambda t, u) = \begin{cases} (Uf_{\lambda t})(u)e^{\lambda t u} & \text{for } u \leq 0, \\ (Uk_{\lambda t})(u)e^{-\lambda t u} & \text{for } u > 0, \end{cases}$$

and

$$L(\lambda t, z) = \left(\frac{2}{\pi}\right)^{1/2} i\lambda \int_0^\infty (Uk_{\lambda t})(u) \exp(\lambda t - 1)u \, du \frac{1}{z-i}.$$

Note that

$$f(z) = (z-i) \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a (Un)(u) \exp(izu) du.$$

Then by definition of the infinitesimal generator A_λ we have

$$\begin{aligned} & \|t^{-1}(P_\lambda(t)f - f) - A_\lambda f\| \\ &= \left[\int_{-\infty}^\infty \left| t^{-1} \left\{ (2\pi)^{-1/2} \int_{-\infty}^\infty H(\lambda t, u) \exp(izu) du \right. \right. \right. \\ & \quad \left. \left. - \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a (Un)(u) \exp(izu) du \right\} \right. \\ & \quad \left. + L(\lambda t, z) - \frac{g(z)}{z-i} \right|^2 dz \Big]^{1/2} \longrightarrow 0 \quad \text{as } t \longrightarrow +0 \end{aligned} \quad (2.2)$$

Let us put

$$K(z) = \left(\frac{2}{\pi}\right)^{1/2} q \int_0^\infty (Un)(u) \exp(-u) du \frac{1}{z-i}.$$

It is easy to show that $L(\lambda t, z) \rightarrow K(z)$ as $t \rightarrow +0$ in the L^2 -norm. Hence, if we let

$$g_1(z) = \frac{g(z)}{z-i}, \quad g_2(z) = g_1(z) - K(z),$$

then $g_2(z) \in L^2(R)$, and by Plancherel's theorem and by (2.2) we see that

$$\begin{aligned} & \left[\int_{-\infty}^\infty \left| t^{-1} \left\{ (2\pi)^{-1/2} \int_{-\infty}^\infty H(\lambda t, u) \exp(izu) du \right. \right. \right. \\ & \quad \left. \left. - \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a (Un)(u) \exp(izu) du \right\} - g_2(z) \right|^2 dz \Big]^{1/2} \\ &= \left[\int_{-\infty}^\infty |(H(\lambda t, u) - (Un)(u))t^{-1} - (Ug_2)(u)|^2 du \right]^{1/2} \\ &= \left[\int_0^\infty |\{(Uk_{\lambda t})(u) \exp(-\lambda t u) - (Un)(u)\}t^{-1} - (Ug_2)(u)|^2 du \right. \\ & \quad \left. + \int_{-\infty}^0 |\{(Uf_{\lambda t})(u) \exp(\lambda t u) - (Un)(u)\}t^{-1} - (Ug_2)(u)|^2 du \right]^{1/2} \\ & \longrightarrow 0 \quad \text{as } t \longrightarrow +0. \end{aligned}$$

Hence

$$\left[\int_0^\infty | \{ (Uk_{\lambda t})(u) \exp(-\lambda tu) - (Un)(u) \} t^{-1} - (Ug_2)(u) |^2 du \right]^{1/2} \longrightarrow 0 \quad \text{as } t \longrightarrow +0, \quad (2.3)$$

$$\left[\int_{-\infty}^0 | \{ (Uf_{\lambda t})(u) \exp(\lambda tu) - (Un)(u) \} t^{-1} - (Ug_2)(u) |^2 du \right]^{1/2} \longrightarrow 0 \quad \text{as } t \longrightarrow +0 \quad (2.4)$$

Let us show that (2.3) and (2.4) hold if and only if $|u|(Un)(u)$ is L^2 -integrable.

Suppose that (2.3) and (2.4) hold. We see that

$$\begin{aligned} & \int_0^\infty | [(Uk_{\lambda t})(u) \exp(-\lambda tu) - (Un)(u)] t^{-1} - (Ug_2)(u) |^2 du \\ &= \int_0^\infty | [(Uk_{\lambda t} - Un)(u) \exp(-\lambda tu) + (\exp(-\lambda tu) - 1)(Un)(u)] t^{-1} - (Ug_2)(u) |^2 du \\ &= \int_0^\infty | \exp(-\lambda tu) U(t^{-1}(k_{\lambda t} - n))(u) + t^{-1}(\exp(-\lambda tu) - 1)(Un)(u) - (Ug_2)(u) |^2 du. \end{aligned} \quad (2.5)$$

Also

$$\begin{aligned} & \int_{-\infty}^0 | [(Uf_{\lambda t})(u) \exp(\lambda tu) - (Un)(u)] t^{-1} - (Ug_2)(u) |^2 du \\ &= \int_{-\infty}^0 | \exp(\lambda tu) U(t^{-1}(f_{\lambda t} - n))(u) + t^{-1}(\exp(\lambda tu) - 1)(Un)(u) - (Ug_2)(u) |^2 du. \end{aligned} \quad (2.6)$$

It is seen that for some function $Q(u)$ in $L^2([0, \infty))$

$$\int_0^\infty | \exp(-\lambda tu) U(t^{-1}(k_{\lambda t} - n))(u) - Q(u) |^2 du \longrightarrow 0 \quad \text{as } t \longrightarrow +0.$$

Hence by (2.3) and (2.5) we see that for some function $R_1(u) = (Ug_2)(u) - Q(u)$ in $L^2([0, \infty))$

$$\int_0^\infty | t^{-1}(\exp(-\lambda tu) - 1)(Un)(u) - R_1(u) |^2 du \longrightarrow 0 \quad \text{as } t \longrightarrow +0. \quad (2.7)$$

On the otherhand, clearly

$$t^{-1}(\exp(-\lambda tu) - 1)(Un)(u) \longrightarrow -\lambda u(Un)(u) \quad \text{as } t \longrightarrow +0 \quad (2.8)$$

for almost all $u \in [0, \infty)$. By (2.7) and (2.8) we obtain that $R_1(u) = -\lambda u(Un)(u)$ for almost all $u \in [0, \infty)$. In the same manner, from (2.4) and (2.6) we see that for some function $R_2(u)$ in $L^2((-\infty, 0])$

$$\int_{-\infty}^0 |t^{-1}(\exp(\lambda tu) - 1)(Un)(u) - R_2(u)|^2 du \longrightarrow 0 \quad \text{as } t \longrightarrow +0$$

and $R_2(u) = \lambda u(Un)(u)$ for almost all $u \in (-\infty, 0]$. Hence if (2.3) and (2.4) hold, $|u|(Un)(u)$ is in $L^2(R)$.

Conversely if f is a function in W^2 such that $|u|(Un)(u) \in L^2(R)$, from the above argument it is clear that f is in $D(A_\lambda)$. Lastly we obtain

$$D(A_\lambda) = \{f \in W^2: |u|(Un)(u) \in L^2(R)\}.$$

Next let us show that if f is in $D(A_\lambda)$, $A_\lambda f$ is given by (2.1). Let $B_\lambda f$ be the right hand side of (2.1). By the Minkowski inequality we see that

$$\begin{aligned} & \|t^{-1}(P_\lambda(t)f - f) - B_\lambda f\| \\ & \leq \left[\int_{-\infty}^{\infty} \left| \left[(z-i) \left\{ (2\pi)^{-1/2} \int_{-\infty}^0 (Uf_{\lambda t})(u) \exp(\lambda tu + izu) du \right. \right. \right. \right. \\ & \quad \left. \left. \left. - \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Un)(u) \exp(izu) du \right\} t^{-1} \right. \right. \\ & \quad \left. \left. + i\lambda \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Un)(u) \exp(izu) du \right. \right. \\ & \quad \left. \left. + \lambda(z-i) \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 |u|(Un)(u) \exp(izu) du \right. \right. \\ & \quad \left. \left. + i\lambda (2\pi)^{-1/2} \int_0^{\infty} (Un)(u) \exp(-u) du \right] \right|^2 \frac{dz}{z^2+1} \Big]^{1/2} \\ & + \left[\int_{-\infty}^{\infty} \left| \left[(z-i) \left\{ (2\pi)^{-1/2} \int_0^{\infty} (Uk_{\lambda t})(u) \exp(-\lambda tu + izu) du \right. \right. \right. \right. \\ & \quad \left. \left. \left. - \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_0^a (Un)(u) \exp(izu) du \right\} t^{-1} \right. \right. \\ & \quad \left. \left. - i\lambda \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_0^a (Un)(u) \exp(izu) du \right. \right. \\ & \quad \left. \left. + \lambda(z-i) \lim_{a \rightarrow \infty} \int_0^a |u|(Un)(u) \exp(izu) du \right. \right. \\ & \quad \left. \left. + i\lambda (2\pi)^{-1/2} \int_0^{\infty} (Un)(u) \exp(-u) du \right] \right|^2 \frac{dz}{z^2+1} \Big]^{1/2} \\ & + \left[\int_{-\infty}^{\infty} \left| \left(\frac{2}{\pi} \right)^{1/2} i\lambda \left\{ \int_0^{\infty} (Uk_{\lambda t})(u) \exp(\lambda t - 1)u du \right. \right. \right. \\ & \quad \left. \left. \left. - \int_0^{\infty} (Un)(u) \exp(-u) du \right\} \right|^2 \frac{dz}{z^2+1} \right]^{1/2}. \end{aligned} \quad (2.9)$$

By the Minkowski inequality we have

(the first term of (2.9))

$$\begin{aligned}
&\leq \left[\int_{-\infty}^{\infty} \left| (2\pi)^{-1/2} \int_{-\infty}^0 ((Uf_{\lambda t})(u) - (Un)(u)) t^{-1} \exp(\lambda tu + izu) du \right. \right. \\
&\quad + \frac{i\lambda}{z-i} \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Un)(u) \exp(izu) du \\
&\quad + \left. \frac{i\lambda}{z-i} (2\pi)^{-1/2} \int_0^{\infty} (Un)(u) \exp(-u) du \right|^2 dz \Big]^{1/2} \\
&\quad + \left[\int_{-\infty}^{\infty} \left| \left\{ (2\pi)^{-1/2} \int_{-\infty}^0 (Un)(u) \exp(\lambda tu + izu) du \right. \right. \right. \\
&\quad - \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Un)(u) \exp(izu) du \Big\} t^{-1} \\
&\quad + \left. \left. \lambda \lim_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 |u| (Un)(u) \exp(izu) du \right|^2 dz \right]^{1/2}. \tag{2.10}
\end{aligned}$$

By the Parseval theorem it holds that

$$\begin{aligned}
&[(Uf_{\lambda t})(u) - (Un)(u)] t^{-1} \\
&= -i\lambda (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{f(v)}{(v-i+i\lambda t)(v-i)} \exp(-ivu) dv \\
&= -i\lambda (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Un)(v) U_y \left[\frac{1}{y-i+i\lambda t} \right] (u-v) dv \\
&= \lambda \int_u^{\infty} (Un)(v) \exp(1-\lambda t)(u-v) dv.
\end{aligned}$$

Hence by the Fubini theorem we see that

$$\begin{aligned}
&(2\pi)^{-1/2} \int_{-\infty}^0 ((Uf_{\lambda t})(u) - (Un)(u)) t^{-1} \exp(\lambda tu + izu) du \\
&= \lambda (2\pi)^{-1/2} \int_{-\infty}^0 \left[\int_u^{\infty} (Un)(v) \exp(-1+\lambda t)v dv \right] \exp(1+iz)u du \\
&= \lambda (2\pi)^{-1/2} \int_{-\infty}^0 (Un)(v) \exp(-1+\lambda t)v dv \int_{-\infty}^v \exp(1+iz)u du \\
&\quad + \lambda (2\pi)^{-1/2} \int_0^{\infty} (Un)(v) \exp(-1+\lambda t)v dv \int_{-\infty}^0 \exp(1+iz)u du \\
&= -\frac{i\lambda}{z-i} (2\pi)^{-1/2} \left[\int_{-\infty}^0 (Un)(u) \exp(\lambda t + iz)v dv \right. \\
&\quad \left. + \int_0^{\infty} (Un)(v) \exp(-1+\lambda t)v dv \right].
\end{aligned}$$

From this equality it follows that

$$\left[\int_{-\infty}^{\infty} \left| (2\pi)^{-1/2} \int_{-\infty}^0 ((Uf_{\lambda t})(u) - (Un)(u)) t^{-1} \exp(\lambda tu + izu) du \right. \right.$$

$$\begin{aligned}
& + \frac{i\lambda}{z-i} \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Un)(u) \exp(izu) du \\
& + \frac{i\lambda}{z-i} (2\pi)^{-1/2} \int_0^\infty (Un)(u) \exp(-u) du \Big|^2 dz \Big]^{1/2} \\
& = |\lambda| \left[\int_{-\infty}^\infty \frac{1}{z^2+1} (2\pi)^{-1} \left| - \int_{-\infty}^0 (Un)(u) \exp(\lambda t + iz) u du \right. \right. \\
& \quad \left. \left. - \int_0^\infty (Un)(u) \exp(-1 + \lambda t) u du + \text{l.i.m}_{a \rightarrow \infty} \int_{-a}^0 (Un)(u) \exp(izu) du \right. \right. \\
& \quad \left. \left. + \int_0^\infty (Un)(u) \exp(-u) du \right|^2 dz \right]^{1/2} \longrightarrow 0 \quad \text{as } t \longrightarrow +0.
\end{aligned}$$

Also by Plancherel's theorem we see that

$$\begin{aligned}
& \left[\int_{-\infty}^\infty \left| \left\{ (2\pi)^{-1/2} \int_{-\infty}^0 (Un)(u) \exp(\lambda t u + izu) du \right. \right. \right. \\
& \quad \left. \left. - \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 (Un)(u) \exp(izu) du \right\} t^{-1} \right. \\
& \quad \left. \left. + \lambda \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^0 |u| (Un)(u) \exp(izu) du \right|^2 dz \right]^{1/2} \\
& = \left[\int_{-\infty}^0 |(Un)(u) (\exp(\lambda t u) - 1) t^{-1} - \lambda |u| (Un)(u)|^2 du \right]^{1/2} \longrightarrow 0 \quad \text{as } t \longrightarrow +0.
\end{aligned}$$

Hence from (2.10) we obtain that (the first term of (2.9)) $\rightarrow 0$ as $t \rightarrow +0$. In the same manner we can prove that (the second term of (2.9)) $\rightarrow 0$ as $t \rightarrow +0$. It is easily seen that (the third term of (2.9)) $\rightarrow 0$ as $t \rightarrow +0$. From these facts it follows that $A_\lambda f = B_\lambda f$. Q. E. D.

THEOREM 2.2. *The infinitesimal generator A_{-iq} of the group $\{P_{-iq}(t): -\infty < t < \infty\}$ in Theorem 1. 2 is given by the following form;*

$$\begin{aligned}
D(A_{-iq}) &= \{f \in W^2: |u|(Un)(u) \in L^2(R)\} \text{ and for } f \text{ in } D(A_{-iq}) \\
(A_{-iq}f)(z) &= q \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a (Un)(u) (\text{sign } u) \exp(izu) du \\
& \quad + iq(z-i) \text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a |u| (Un)(u) \exp(izu) du. \quad (2.11)
\end{aligned}$$

PROOF. In the same manner as the proof of Theorem 2.1 we obtain

$$D(A_{-iq}) = \{f \in W^2: |u|(Un)(u) \in L^2(R)\}.$$

If f is in $D(A_{-iq})$, then

$$n(u) = (u-i)(Un)(u) \cdot \frac{1}{u-i} \in L^1(R).$$

By the Parseval theorem we see that

$$\begin{aligned}
t^{-1}(Uf_{-iq_t} - Un)(u) &= -(2\pi)^{-1/2}q \int_{-\infty}^{\infty} \frac{f(v)}{(v-i)(v-i+qt)} \exp(-iuv) dv \\
&= -(2\pi)^{-1/2}q \int_{-\infty}^{\infty} (Un)(w) U_v \left[\frac{1}{v-i+qt} \right] (u-w) dw,
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
t^{-1}(Uk_{-iq_t} - Un)(u) &= (2\pi)^{-1/2}q \int_{-\infty}^{\infty} \frac{f(v)}{(v-i)(v-i-qt)} \exp(-iuv) dv \\
&= (2\pi)^{-1/2}q \int_{-\infty}^{\infty} (Un)(w) U_v \left[\frac{1}{v-i-qt} \right] (u-w) dw.
\end{aligned} \tag{2.13}$$

Since

$$(Un)(w), U_v \left[\frac{1}{v-i+qt} \right] (w), U_v \left[\frac{1}{v-i-qt} \right] (w)$$

are in $L^1(R)$, (2.12) and (2.13) are functions in $L^1(R)$ and in $L^2(R)$. From these facts and from the same argument as the proof of Theorem 2.1 we obtain (2.11).

Q. E. D.

Note that if f is in $D(A_{-iq})$,

$$\begin{aligned}
&\text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a (Un)(u) (i \text{ sign } u) \exp(izu) du \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Un)(u) (i \text{ sign } u) \exp(izu) du.
\end{aligned}$$

REMARK 2.1. In Theorem 2.1 and Theorem 2.1

$$1) \quad \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{n(u)}{u-z} du = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Un)(u) (i \text{ sign } u) \exp(izu) du,$$

2) if $n(u)$ is absolutely continuous on each bounded closed interval and $n'(u)$ is absolutely integrable over R ,

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{n'(u)}{u-z} du = -\text{l.i.m}_{a \rightarrow \infty} (2\pi)^{-1/2} \int_{-a}^a |u| (Un)(u) \exp(izu) du.$$

Here

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{n(u)}{u-z} du, \quad \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{n'(u)}{u-z} du$$

denote the Hilbert transforms of $n(u)$ and $n'(u)$.

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