Introduction. Let \( W^2 \) be the totality of measurable functions such that
\[
\int_{-\infty}^{\infty} \frac{|f(v)|^2}{v^2+1} dv < \infty.
\]
When \( f \) and \( g \) are in \( W^2 \) we define the inner product of \( f \) and \( g \) by
\[
(f, g) = \int_{-\infty}^{\infty} f(v)\overline{g(v)} \frac{dv}{v^2+1}.
\]
Introducing the operations of addition and scalar multiplication in \( W^2 \) as usual, we can show that the space \( W^2 \) is the Hilbert space with norm \( \|f\| = [(f,f)]^{1/2} \).

Let
\[
p_{\lambda}(t, u) = \frac{\lambda^t}{\pi[(\lambda^2+u^2)^{1/2}]}.
\]
If \( t > 0 \) and \( \lambda > 0 \), this function \( p_{\lambda}(t, u) \) is the Cauchy density function. When \( \lambda = r - iq \) \((r > 0)\) and \( t > 0 \) let
\[
(P_{\lambda}(t)f)(v) = \int_{-\infty}^{\infty} p_{\lambda}(t, u)f(v-u)du
\]
and
\[
(P_{\lambda}(0)f)(v) = f(v)
\]
for \( f \) in \( W^2 \). The purpose of this paper is to show that a family \( \{P_{\lambda}(t) : 0 \leq t < \infty\} \) of linear operators on \( W^2 \) is a semigroup of class \((C_0)\) and that there exists a one-parameter strongly continuous group of linear operators \( \{P_{-t}(t) : -\infty < t < \infty\} \) on \( W^2 \).

1. Semigroups and groups of linear operators on \( W^2 \). If \( \lambda = r - iq \) \((r > 0)\) and \( t > 0 \),
\[
|p_{\lambda}(t, u)| = \frac{t(r^2+q^2)^{1/2}}{\pi[(u^2+t^2(r^2-q^2))^2+4t^2r^2q^2]^{1/2}}.
\]  

**Lemma 1.1.** Let \( \lambda = r - iq \) \((r > 0)\) and \( t > 0 \). If \( f \) is in \( W^2 \), then \((P_{\lambda}(t)f)(v)\)
is a function in $W^2$.

**Proof.** Take a positive number $a$ such that $|t^2(r^2-q^2)|<a^2$. Then

$$|(P_2(t)f)(v)| \leq \int_{-\sigma}^{\sigma} |p_2(t, u) f(v-u)| du + \int_{\sigma}^{\infty} |p_2(t, u) f(v-u)| du + \int_{-\infty}^{-\sigma} |p_2(t, u) f(v-u)| du.$$

Since $|p_2(t, u)| \leq K_1(r, q, t) < \infty$ for $|u| \leq a$ and $|p_2(t, u)| \leq K_2(r, q, t) u^{-2}$ for $|u| > a$, we have

$$|(P_2(t)f)(v)| \leq K_1(r, q, t) \int_{-\sigma}^{\sigma} |f(v-u)| du + K_2(r, q, t) \int_{-\infty}^{\sigma} u^{-2} |f(v-u)| du + K_2(r, q, t) \int_{-\sigma}^{-\sigma} u^{-2} |f(v-u)| du < \infty$$

for each $v \in R$ (the real line). It is known [11; Theorem 20] that if

$$\frac{1}{2T} \int_{-T}^{T} |g(v)|^2 dv$$

is bounded in $T$, then

$$\int_{-\infty}^{\infty} \frac{|g(v)|^2}{v^2 + 1} dv < \infty.$$

Hence, to show that $P_2(t)f$ is in $W^2$, it suffices to show that

$$\frac{1}{2T} \int_{-T}^{T} |(P_2(t)f)(v)|^2 dv \quad (1.2)$$

is bounded in $T$. By the Schwartz inequality we see that

$$\frac{1}{2T} \int_{-T}^{T} |(P_2(t)f)(v)|^2 dv \leq \frac{1}{2T} \int_{-T}^{T} \left( \int_{-\infty}^{\infty} |p_2(t, v-u) f(u)| du \right)^2 dv$$

$$\leq \frac{1}{2T} \int_{-T}^{T} \left( \int_{-\infty}^{\infty} |p_2(t, v-u)|^2 (1 + (v-u)^2) du \int_{-\infty}^{\infty} \frac{|f(u)|^2}{(u-v)^2 + 1} du \right) dv.$$

From (1.1) it follows that

$$\int_{-\infty}^{\infty} |p_2(t, x)|^2 (1 + x^2) dx < \infty.$$

Also it is known [11; pp. 168–169] that

$$\frac{1}{2T} \int_{-T}^{T} \left( \int_{-\infty}^{\infty} \frac{|f(u)|^2}{(v-u)^2 + 1} du \right) dv$$

is bounded in $T$. From these facts we obtain that (1.2) is bounded in $T$. Q.E.D.

We have the following
**Lemma 1.2.** Let $\lambda = r - iq$ $(r > 0)$ and $t > 0$. If $f$ is in $L^2(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} p_\lambda(t, u)f(v - u)du = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (U_\lambda f)(u) \exp(iuv - \lambda t|u|)du$$

where we let

$$(U_\lambda f)(u) = (Uf)(u) = \lim_{\alpha \to \infty} \int_{-\infty}^{\infty} f(w) \exp(-iwu)dw.$$ 

Here $\lim$ denotes limit in the mean with index 2. cf. [3, Theorem 21, p. 974]

**Lemma 1.3.** Let $\lambda = r - iq$ $(r > 0)$ and $t > 0$. Let $\alpha$ be a positive number.

Then

1) if $0 < tr < \alpha$,

$$(P_\lambda(t)f)(z) = (z - i\alpha)(2\pi)^{-1/2} \int_{-\infty}^{0} (Uf_{\lambda t})(u) \exp(\lambda t + iz)u du$$

$$+ (z - i\alpha)(2\pi)^{-1/2} \int_{0}^{\infty} (Uk_{\lambda t})(u) \exp(-\lambda t + iz)u du$$

$$\left(\frac{2}{\pi}\right)^{1/2} i\lambda t \int_{-\infty}^{\infty} (Uk_{\lambda t})(u) \exp(\lambda t - \alpha)u du, \quad (1.3)$$

2) if $0 < \alpha < tr$,

$$(P_\lambda(t)f)(z) = (z - i\alpha)(2\pi)^{-1/2} \int_{-\infty}^{0} (Uf_{\lambda t})(u) \exp(\lambda t + iz)u du$$

$$+ (z - i\alpha)(2\pi)^{-1/2} \int_{0}^{\infty} (Uk_{\lambda t})(u) \exp(-\lambda t + iz)u du$$

$$- \left(\frac{2}{\pi}\right)^{1/2} i\lambda t \int_{-\infty}^{\infty} (Uk_{\lambda t})(u) \exp(\lambda t - \alpha)u du, \quad (1.3')$$

where we let

$$f_{\lambda t}(u) = \frac{f(u)}{u - i\alpha + i\lambda t}, \quad k_{\lambda t}(u) = \frac{f(u)}{u - i\alpha - i\lambda t}.$$

**Proof.** It is easy to show that

$$p_{\lambda}(t, u - z) = \frac{1}{2\pi} \left[ \frac{z - i\alpha}{(u - i\alpha + i\lambda t)(\lambda t - i(u - z))} + \frac{z - i\alpha}{(u - i\alpha - i\lambda t)(\lambda t + i(u - z))} \right] + \frac{2\lambda t}{(u - i\alpha - i\lambda t)(u - i\alpha + i\lambda t)}.$$

Hence we have

$$(P_\lambda(t)f)(v) = (z - i\alpha) \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)du \frac{(u - i\alpha + i\lambda t)(\lambda t + i(z - u))}{(u - i\alpha - i\lambda t)(\lambda t - i(z - u))}$$

$$+ (z - i\alpha) \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)du \frac{(u - i\alpha - i\lambda t)(\lambda t + i(z - u))}{(u - i\alpha - i\lambda t)(\lambda t - i(z - u))}$$
By the Parseval theorem it holds that

\[
\int_{-\infty}^{\infty} \frac{f(u)du}{(u - i\alpha + i\lambda t)(u - i\alpha - i\lambda t)} = (2\pi)^{1/2} \int_{-\infty}^{0} (Uf\lambda_t)(u) \exp (\lambda t + i\alpha u)du,
\]

\[
\int_{-\infty}^{\infty} \frac{f(u)du}{(u - i\alpha - i\lambda t)(u - i\alpha + i\lambda t)} = (2\pi)^{1/2} \int_{0}^{\infty} (Uk\lambda_t)(u) \exp (-\lambda t + i\alpha u)du,
\]

\[
\int_{-\infty}^{\infty} \frac{f(u)du}{(u - i\alpha - i\lambda t)(u - i\alpha + i\lambda t)} = \begin{cases} (2\pi)^{1/2} \int_{0}^{\infty} (Uk\lambda_t)(u) \exp (\lambda t - \alpha)u du & \text{if } 0 < tr < \alpha, \\ -(2\pi)^{1/2} \int_{-\infty}^{0} (Uk\lambda_t)(u) \exp (\lambda t - \alpha)u du & \text{if } \alpha < tr \end{cases}
\]

(cf. [1, p. 128], [11, Theorem 3]). From these equalities and from (1.4) we obtain (1.3) and (1.3'). Q.E.D.

**LEMMA 1.4.** Let \(\lambda = r - iq\ (r > 0)\) and \(t > 0\). Then \(P_{\lambda}(t)\) is a linear bounded operator on \(W^2\).

**PROOF.** It is clear that \(P_{\lambda}(t)\) is linear. Let \(\alpha\) be a positive number such that \(1 \leq \alpha\) and \(0 < tr < \alpha\). By the Minkowski inequality and by Plancherel's theorem, and by the Schwartz inequality we see that

\[
\|P_{\lambda}(t)f\| \leq \left[ \int_{-\infty}^{\infty} \left( \frac{z^2 + \alpha^2}{z^2 + 1} \right) (2\pi)^{-1/2} \left( \int_{-\infty}^{0} (Uf\lambda_t)(u) \exp (\lambda t + iz)u du \right)^2 dz \right]^{1/2}
\]

\[+ \left[ \int_{-\infty}^{\infty} \frac{1}{z^2 + 1} \left( \frac{2}{\pi} \right)^{1/2} \left( \int_{0}^{\infty} (Uk\lambda_t)(u) \exp (-\lambda t + iz)u du \right)^2 dz \right]^{1/2}
\]

\[\leq \alpha \left[ \int_{-\infty}^{0} |(Uf\lambda_t)(u)|^2 du \right]^{1/2} + \alpha \left[ \int_{0}^{\infty} |(Uk\lambda_t)(u)|^2 du \right]^{1/2}
\]

\[+ \left( \frac{2}{\pi} \right)^{1/2} |\lambda t| \left[ \int_{0}^{\infty} |(Uk\lambda_t)(u)|^2 du \right]^{1/2}
\]

\[\cdot \left[ \int_{0}^{\infty} \exp (\lambda t - \alpha)u^2 du \right]^{1/2} \left[ \int_{-\infty}^{\infty} \frac{dz}{z^2 + 1} \right]^{1/2}
\]

\[\leq \alpha \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - i\alpha + i\lambda t} \right| + \alpha \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - i\alpha - i\lambda t} \right| \|f\|
\]

\[+ \left( \frac{2}{\pi} \right)^{1/2} |\lambda t| \left[ \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - i\alpha + i\lambda t} \right| \right]^{1/2} \left[ \int_{-\infty}^{\infty} \frac{dz}{z^2 + 1} \right]^{1/2} \|f\|. \quad (1.5)
\]
Since
\[ \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - i \alpha + i \lambda t} \right|, \sup_{u \in \mathbb{R}} \left| \frac{u - i}{u - i \alpha - i \lambda t} \right| \]
are bounded, from (1.5) it follows that $P_A(t)$ is a bounded operator on $W^2$.

Q.E.D.

**THEOREM 1.1.** Let $\lambda = r - iq$ ($r > 0$). Let $P_A(0) = I$, where $I$ denotes the identity operator on $W^2$. Then a family $\{P_A(t): 0 \leq t < \infty\}$ of linear bounded operators on $W^2$ is a semigroup of class $(C_0)$.

**PROOF.** Semigroup property $P_A(t)P_A(s) = P_A(t+s)$: Let $0 < s, t < \infty$. First let $f$ be a continuous function with compact support. Then we have
\[
(P_A(t)P_A(s)f)(v) = \int_{-\infty}^{\infty} p_A(t, u)du \int_{-\infty}^{\infty} p_A(s, v-u-y)f(y)dy
\]
\[
= \int_{-\infty}^{\infty} f(y)dy \int_{-\infty}^{\infty} p_A(t, u)p_A(s, v-u-y)du
\]
\[
= \int_{-\infty}^{\infty} p_A(t+s, v-y)f(y)dy = (P_A(t+s)f)(v)
\]
(cf. [8, proof of Lemma 1.3]). When $f$ is in $W^2$, taking a sequence of continuous functions $f_n$ with compact support converging to $f$ in the norm we obtain
\[
\|P_A(t)P_A(s)f - P_A(t+s)f\| \leq \|P_A(t)P_A(s)f - P_A(t)P_A(s)f_n\|
\]
\[
+ \|P_A(t+s)f_n - P_A(t+s)f\|.
\]
By Lemma 1.4
\[
\|P_A(t)P_A(s)(f-f_n)\| \leq \|P_A(t)\| \|P_A(s)\| \|f-f_n\| \xrightarrow{n \to \infty} 0
\]
and
\[
\|P_A(t+s)(f-f_n)\| \leq \|P_A(t+s)\| \|f-f_n\| \xrightarrow{n \to \infty} 0
\]
as $n \to \infty$, hence $P_A(t)P_A(s)f = P_A(t+s)f$ for $f$ in $W^2$.

**Strong continuity $P_A(t)f \to P_A(t)f$ ($s \to t$):** By the semigroup property it suffices to prove that
(a) $P_A(t)f \to f$ as $t \to +0$ in the norm for all $f$ in $W^2$.
It is known [4, Theorem 10.6.5] that (a) is equivalent to
(b) $(P_A(t)f, g) \to (f, g)$ as $t \to +0$ for all $f, g$ in $W^2$.
Let us prove (b). First let $f$ be a continuous function with compact support. Then by Lemma 1.2 and by the Parseval theorem we obtain
\[
(P_A(t)f, g) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} (Uf)(u) \exp(iu \lambda t|u|)du \right] \frac{g(v)}{v^2 + 1}dv
\]
\[
= \int_{-\infty}^{\infty} (Uf)(u) \exp(-\lambda t|u|)U_v[g(v)(v^2 + 1)^{-1}](u)du
\]
By the Parseval theorem
\[ \int_{-\infty}^{\infty} (Uf)(u) \overline{U_u[g(v)(v^2+1)^{-1}]}(u) du = \int_{-\infty}^{\infty} f(v) \overline{\frac{g(v)}{v^2+1}} dv, \]
hence it follows that \((P_\lambda(t)f, g)\to (f, g)\) as \(t\to +0\) for the above function \(f\). If \(f\) is in \(W^2\), taking a sequence of continuous functions \(f_n\) with compact support converging to \(f\) in the norm we have
\[|(P_\lambda(t)f, g) - (f, g)| \leq |(P_\lambda(t)(f-f_n), g)| + |(P_\lambda(t)f_n, g) - (f_n, g)| + |(f_n, g) - (f, g)|. \] (1.6)
It is seen from (1.5) that \(P_\lambda(t)\) is uniformly bounded in the neighborhood at \(t=0\). Hence from (1.6) it follows that
\[ |(P_\lambda(t)f, g) - (f, g)| \to 0 \quad \text{as} \quad t \to +0. \] Q. E. D.

We have the following

**REMARK 1.1.** Let \(f\) be in \(W^2\). Then \(P_\lambda(t)f\) is strongly continuous in \(\lambda\) on \(\mathbb{C}^+ = \{\lambda: \text{Real part of } \lambda \text{ is positive}\}\).

**LEMMA 1.5.** Let \(q \neq 0\) be given and let \(f\) be in \(W^2\). Let
\[ (P_{-iq}(t)f)(z) = (z - i\alpha) \lim_{\alpha \to \infty} (2\pi)^{-1/2} \int_{-\alpha}^{\alpha} (Uf_{-iq})(u) \exp(-iqt + iz)u du \]
\[ + (z - i\alpha) \lim_{\alpha \to \infty} (2\pi)^{-1/2} \int_{-\alpha}^{\alpha} (Uk_{-iq})(u) \exp(iqt + iz)u du \]
\[ + \left( \frac{2}{\pi} \right)^{1/2} q^t \int_{0}^{\infty} (Uk_{-iq})(u) \exp(iqt - \alpha)u du \] (1.7)
for a positive number \(\alpha \geq 1\) and \(t > 0\). Then \(P_{r-\lambda q}(t)\to P_{-iq}(t)f\) as \(r \to +0\) in the norm and \(P_{-iq}(t)\) is a linear bounded operator on \(W^2\).

**PROOF.** Let \(\lambda = r - iq\ (r > 0)\). Suppose \(0 < tr < 1 \leq \alpha\). By (1.3) and by the Minkowski inequality, and by Plancherel's theorem we see that
\[
\|P_\lambda(t)f - P_{-iq}(t)f\| \leq \left[ \int_{-\infty}^{\infty} \frac{z^2 + \alpha^2}{z^2 + 1} \left(2\pi\right)^{-1/2} \int_{-\infty}^{0} (Uf_{\lambda t})(u) \exp(\lambda t + iz)u du \right]^{1/2} dz \\
- \left[ \int_{-\infty}^{\infty} \frac{z^2 + \alpha^2}{z^2 + 1} \left(2\pi\right)^{-1/2} \int_{0}^{\infty} (Uk_{\lambda t})(u) \exp(-\lambda t + iz)u du \right]^{1/2} dz \\
- \left[ \int_{-\infty}^{\infty} \frac{z^2 + \alpha^2}{z^2 + 1} \left(2\pi\right)^{-1/2} \int_{0}^{\infty} (Uk_{-q})(u) \exp(iqt + iz)u du \right]^{1/2} dz
\]
In the same manner as the proof of Lemma 1.4 we can prove that $P_iq(t)$ is a bounded operator on $W^2$. Q.E.D.

REMARK 1.2. If $f$ is in $L^2(R)$, this follows from Lemma 1.2.

THEOREM 1.2. Let $q'=0$ be given and let $a$ be the same number as in Lemma 1.5. When $t>0$, let

$$\left(1.8\right)$$

for $f$ in $W^2$. Then $P_{-iq}(t)$ is a linear bounded operator on $W^2$ and

$$\left(1.9\right)$$

for $f$ in $W^2$. Then $P_{-iq}(-t)$ is a linear bounded operator on $W^2$ and

$$\left(1.10\right)$$

A family $\{P_{-iq}(t): -\infty < t < \infty\}$ is a strongly continuous group of linear operators on $W^2$, where let $P_{-iq}(0)=I$.

PROOF. In the same manner as the proof of Lemma 1.4 we can prove that $P_{-iq}(-t)$ is a linear bounded operator on $W^2$. Let us prove (1.9). Let $S(R)$ be the space of rapidly decreasing and infinitely differentiable functions. It is easy to show that $S(R)$ is dense in $W^2$. First let us prove that

$$\left(1.11\right)$$

In the same manner as the proof of Lemma 1.4 we can prove that $P_{-iq}(t)$ is a bounded operator on $W^2$.
for all $f$ in $S(R)$. Let $f$ be in $S(R)$. By the Parseval theorem we see that

$$(U_{f_{i_{q_{i}}}})(u) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(y) \exp(-iyu) dy$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Uf)(y) U_{w} \left[ \frac{1}{w - t q - i \alpha} \right] (u - y) dy$$

$$= i \int_{w}^{\infty} (Uf)(y) \exp(itq - \alpha)y dy \exp(\alpha - itq) u$$

(cf. [1, p. 128], [11, Theorem 3]). Hence we obtain by the Fubini theorem that

(1.11)

Also by the Parseval theorem we see that

$$(U_{k_{i_{q_{i}}}})(u) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{f(y)}{y + t q - i \alpha} \exp(-iyu) dy$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Uf)(y) U_{w} \left[ \frac{1}{w + t q - i \alpha} \right] (u - y) dy$$

$$= i \int_{w}^{\infty} (Uf)(y) \exp(- itq - \alpha)y dy \exp(itq + \alpha) u.$$

Hence we obtain that

(the second term of (1.8))

$$(z - i \alpha)(2\pi)^{-1/2} \int_{0}^{\infty} \left[ \int_{-\infty}^{\infty} (Uf)(y) \exp(- itq - \alpha)y dy \right] \exp(\alpha + iz) u du$$

(by the Fubini theorem)

$$= (z - i \alpha)(2\pi)^{-1/2} \int_{0}^{\infty} (Uf)(y) \exp(- itq - \alpha)y dy \exp(\alpha + iz) u du$$

$$= (2\pi)^{-1/2} \int_{0}^{\infty} (Uf)(y) \exp(- itq + iz)y dy - (2\pi)^{-1/2} \int_{0}^{\infty} (Uf)(y)$$

$$\cdot \exp(- \alpha - itq)y dy.$$  (1.12)
From (1.11), (1.12) and (1.13) we obtain (1.10). By (1.10) and by the fact that $P_{i}q(-t)$ is a bounded operator, it holds that
\begin{equation}
(1.14)
\end{equation}
for $f$ in $L^{2}(R)$. Hence if $f$ is in $L^{2}(R)$, from Remark 1.2 and (1.14) it follows that
\begin{equation}
(1.15)
\end{equation}
By this equality and by the fact that $P_{i}q(t)$ and $P_{i}q(-t)$ are bounded operators, we obtain (1.15) for all $f$ in $W^{2}$.

From Remark 1.2, (1.14) and from the fact that $P_{i}q(t)$, $P_{i}q(-t)$ are bounded operators it follows that
\begin{equation}
(1.16)
\end{equation}
for $-\infty < t, s < \infty$. In the same manner as the proof of Theorem 1.1 we can show that $\{P_{i}q(t): 0 \leq t < \infty\}$ is a semigroup of class $(C_{0})$. Hence by this fact and by (1.16) the family $\{P_{i}q(t): -\infty < t < \infty\}$ is a one-parameter strongly continuous group. Q.E.D.

2. Infinitesimal generators and its domains. In what follows, for simplicity let $n(u)=f(u)/(u-i)$ for $f$ in $W^{2}$. In the following discussions, suppose that $\alpha=1$ in Lemma 1.3, Lemma 1.5 and Theorem 1.2.

THEOREM 2.1. The infinitesimal generator $A_{\lambda}$ of the semigroup $\{P_{\lambda}(t): 0 \leq t < \infty\}$ in Theorem 1.1 is given by the following form;
\begin{equation}
(2.1)
\end{equation}

PROOF. Let $D(A_{\lambda})$ denote the domain of the infinitesimal generator $A_{\lambda}$. When $f$ is in $D(A_{\lambda})$, let $A_{\lambda}f=g$. Let us put
\[
H(\lambda t, u) = \begin{cases} 
(\mathcal{U}f_{\lambda}) (u) e^{\lambda tu} & \text{for } u \leq 0, \\
(\mathcal{U}k_{\lambda}) (u) e^{-\lambda tu} & \text{for } u > 0,
\end{cases}
\]
and
\[
L(\lambda t, z) = \left( \frac{2}{\pi} \right)^{1/2} i \lambda \int_{0}^{\infty} (\mathcal{U}k_{\lambda})(u) \exp (\lambda t - 1) u \frac{1}{z - i} du.
\]
Note that
\[
f(z) = (z - i) \text{i.m.} (2\pi)^{-1/2} \int_{-a}^{a} (\mathcal{U}n)(u) \exp (izu) du.
\]
Then by definition of the infinitesimal generator \(A_\lambda\) we have
\[
\left\| t^{-1}(P_\lambda(t)f - f) - A_\lambda f \right\| = \int_{-\infty}^{\infty} \left\| t^{-1} \left\{ (2\pi)^{-1/2} \int_{-\infty}^{\infty} H(\lambda t, u) \exp (izu) du \
- \text{i.m.} (2\pi)^{-1/2} \int_{-a}^{a} (\mathcal{U}n)(u) \exp (izu) du \right\} 
+ L(\lambda t, z) - \frac{g(z)}{z - i} \int_{-\infty}^{\infty} \left\| d z \right\|^{1/2} \right\| \to 0 \text{ as } t \to +0 \quad (2.2)
\]
Let us put
\[
K(z) = \left( \frac{2}{\pi} \right)^{1/2} q \int_{-\infty}^{\infty} (\mathcal{U}n)(u) \exp (-u) du \frac{1}{z - i}.
\]
It is easy to show that \(L(\lambda t, z) \to K(z)\) as \(t \to +0\) in the \(L^2\)-norm. Hence, if we let
\[
g_1(z) = \frac{g(z)}{z - i}, \quad g_2(z) = g_1(z) - K(z),
\]
then \(g_2(z) \in L^2(R)\), and by Plancherel's theorem and by (2.2) we see that
\[
\int_{-\infty}^{\infty} \left\| t^{-1} \left\{ (2\pi)^{-1/2} \int_{-\infty}^{\infty} H(\lambda t, u) \exp (izu) du \
- \text{i.m.} (2\pi)^{-1/2} \int_{-a}^{a} (\mathcal{U}n)(u) \exp (izu) du \right\} 
- g_2(z) \int_{-\infty}^{\infty} \left\| d z \right\|^{1/2} \right\| \to 0 \text{ as } t \to +0.
\]
Hence
\[
\left[ \int_{0}^{\infty} \left| \{(Uk_{n})(u) \exp(-\lambda tu) - (Un)(u)\}t^{-1} \right|^2 du \right]^{1/2} \to 0 \quad \text{as} \quad t \to +0 , \quad (2.3)
\]
\[
\left[ \int_{-\infty}^{0} \left| \{(Uf_{n})(u) \exp(\lambda tu) - (Un)(u)\}t^{-1} \right|^2 du \right]^{1/2} \to 0 \quad \text{as} \quad t \to +0 \quad (2.4)
\]

Let us show that (2.3) and (2.4) hold if and only if \( |u| (Un)(u) \) is \( L^2 \)-integrable.

Suppose that (2.3) and (2.4) hold. We see that
\[
\left[ \int_{0}^{\infty} \left| \{(Uk_{n})(u) \exp(-\lambda tu) - (Un)(u)\}t^{-1}-(Ug_{2})(u)\|^2 du \right| \right]^{1/2} \to 0 \quad \text{as} \quad t \to +0 .
\]

Also
\[
\left[ \int_{-\infty}^{0} \left| \{(Uf_{n})(u) \exp(\lambda tu) - (Un)(u)\}t^{-1}-(Ug_{2})(u)\|^2 du \right| \right]^{1/2} \to 0 \quad \text{as} \quad t \to +0 .
\]

It is seen that for some function \( Q(u) \) in \( L^2([0, \infty)) \)
\[
\left[ \int_{0}^{\infty} \left| \exp(-\lambda tu)U(t^{-1}(k_{n}-n))(u) - Q(u)\|^2 du \right| \right]^{1/2} \to 0 \quad \text{as} \quad t \to +0 .
\]

Hence by (2.3) and (2.5) we see that for some function \( R_1(u) = (Ug_{2})(u) - Q(u) \) in \( L^2([0, \infty)) \)
\[
\left[ \int_{0}^{\infty} \left| t^{-1}(\exp(-\lambda tu)-1)(Un)(u) - R_1(u)\|^2 du \right| \right]^{1/2} \to 0 \quad \text{as} \quad t \to +0 . \quad (2.7)
\]

On the other hand, clearly
\[
t^{-1}(\exp(-\lambda tu)-1)(Un)(u) \to -\lambda Un(n)(u) \quad \text{as} \quad t \to +0 \quad (2.8)
\]
for almost all \( u \in [0, \infty) \). By (2.7) and (2.8) we obtain that \( R_1(u) = -\lambda Un(n)(u) \) for almost all \( u \in [0, \infty) \). In the same manner, from (2.4) and (2.6) we see that for some function \( R_2(u) \) in \( L^2((-\infty, 0]) \)
\[\int_{-\infty}^{0} |t^{-1}(\exp(\lambda tu) - 1)(Un)(u) - R_2(u)|^2 du \to 0 \quad \text{as} \quad t \to +0\]

and \(R_2(u) = \lambda u(Un)(u)\) for almost all \(u \in (-\infty, 0]\). Hence if (2.3) and (2.4) hold, \(|u|(Un)(u)\) is in \(L^2(R)\).

Conversely if \(f\) is a function in \(W^2\) such that \(|u|(Un)(u) \in L^2(R)\), from the above argument it is clear that \(f\) is in \(D(A_a)\). Lastly we obtain

\[D(A_a) = \{ f \in W^2 : |u|(Un)(u) \in L^2(R) \}.\]

Next let us show that if \(f\) is in \(D(A_a)\), \(A_a f\) is given by (2.1). Let \(B_a f\) be the right hand side of (2.1). By the Minkowski inequality we see that

\[\|t^{-1}(P_2(t)f - f) - B_a f\| \leq \int_{-\infty}^{a} \left| \left( z - i \right) \left( (2\pi)^{-1/2} \int_{-\infty}^{0} (Uf_u)(u) \exp(\lambda tu + izu) du \right. \right. \]

\[-\lambda \left. \left. \int_{-\infty}^{0} (Un)(u) \exp(izu) du \right\} t^{-1} \right| \]

\[+ i \lambda (2\pi)^{-1/2} \int_{0}^{\infty} (Un)(u) \exp(-\lambda tu + izu) du \]

\[+ \lambda (z - i) \lambda \int_{-\infty}^{0} \left| (2\pi)^{-1/2} \int_{-\infty}^{0} (Uk_u)(u) \exp(-\lambda tu + izu) du \right| \frac{dz}{z^2 + 1} \right]^{1/2} \]

\[+ \left[ \int_{-\infty}^{a} \left| (z - i) \left( (2\pi)^{-1/2} \int_{0}^{\infty} (Un)(u) \exp(-\lambda tu + izu) du \right. \right. \]

\[-\lambda \left. \left. \int_{0}^{\infty} (Un)(u) \exp(izu) du \right\} t^{-1} \right| \]

\[-i \lambda \left( (2\pi)^{-1/2} \int_{0}^{a} (Un)(u) \exp(izu) du \right) \]

\[+ \lambda (z - i) \lambda \int_{0}^{a} \left| (2\pi)^{-1/2} \int_{0}^{\infty} (Un)(u) \exp(-\lambda tu + izu) du \right| \frac{dz}{z^2 + 1} \right]^{1/2} \]

\[+ \left[ \int_{-\infty}^{0} \left| \left( \frac{2}{\pi} \right)^{1/2} i \lambda \left\{ \int_{0}^{\infty} (Uk_u)(u) \exp(\lambda t - 1) u du \right. \right. \]

\[-\int_{0}^{\infty} (Un)(u) \exp(-u) du \right\} \left| \frac{dz}{z^2 + 1} \right]^{1/2} \right]. \quad (2.9)

By the Minkowski inequality we have

\[\text{(the first term of (2.9))}\]
By the Parseval theorem it holds that
\[
[(Uf_{\lambda})(u) - (Un)(u)]r^{-1} = -i\lambda(2\pi)^{-1/2}\int_{-\infty}^{\infty} \frac{f(v)}{v - i + i\lambda}(v - i) \exp(-ivu)dv
\]
\[
= -i\lambda(2\pi)^{-1/2}\int_{-\infty}^{\infty} (Un)(v)Ur\left[\frac{1}{y - i + i\lambda}\right](u - v)dv
\]
\[
= \lambda\int_{-\infty}^{\infty} (Un)(v) \exp(1 - \lambda t)(u - v)dv.
\]
Hence by the Fubini theorem we see that
\[
(2\pi)^{-1/2}\int_{-\infty}^{0} \left[ (Uf_{\lambda})(u) - (Un)(u) \right]r^{-1} \exp(\lambda tu + izu)du
\]
\[
= \lambda(2\pi)^{-1/2}\int_{-\infty}^{0} (Un)(v) \exp(-1 + \lambda t)v dv \int_{-\infty}^{v} \exp(1 + iz)u du
\]
\[
= \lambda(2\pi)^{-1/2}\int_{-\infty}^{0} (Un)(v) \exp(-1 + \lambda t)v dv \int_{-\infty}^{v} \exp(1 + iz)u du
\]
\[
+ \lambda(2\pi)^{-1/2}\int_{0}^{\infty} (Un)(v) \exp(-1 + \lambda t)v dv \int_{-\infty}^{v} \exp(1 + iz)u du
\]
\[
= -\frac{i\lambda}{z - i}(2\pi)^{-1/2}\left[ \int_{-\infty}^{0} (Un)(u) \exp(\lambda t + iz)dv 
\right.
\]
\[
+ \int_{0}^{\infty} (Un)(v) \exp(-1 + \lambda t)v dv
\]
\]
From this equality it follows that
\[
\left[ \int_{-\infty}^{\infty} (2\pi)^{-1/2}\int_{-\infty}^{0} \left[ (Uf_{\lambda})(u) - (Un)(u) \right]r^{-1} \exp(\lambda tu + izu)du
\right.
\]
\[
+ \int_{0}^{\infty} (Un)(v) \exp(-1 + \lambda t)v dv
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\[
[ (2\pi)^{-1/2}\int_{-\infty}^{0} (Un)(v) \exp(-1 + \lambda t)v dv \int_{-\infty}^{v} \exp(1 + iz)u du
\]
\[
+ \int_{0}^{\infty} (Un)(v) \exp(-1 + \lambda t)v dv
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\[
+ \int_{0}^{\infty} (Un)(v) \exp(-1 + \lambda t)v dv
\]
Also by Plancherel's theorem we see that

\[
\left[ \int_{-\infty}^{\infty} \left\{ (2\pi)^{-1/2} \int_{-\infty}^{0} (Un)(u) \exp(\lambda t + iz) du \right\} \right]^{1/2}
\]

\[
= |\lambda| \left[ \int_{-\infty}^{\infty} \frac{1}{z^2 + 1} (2\pi)^{-1} \int_{-\infty}^{0} (Un)(u) \exp(\lambda t + iz) du \right]^{1/2}
\]

\[
+ \int_{0}^{\infty} (Un)(u) \exp(-u) du \left\{ \frac{2}{dz} \right\}^{1/2} \rightarrow 0 \quad \text{as} \quad t \rightarrow +0.
\]

Hence from (2.10) we obtain that (the first term of (2.9)) \( \rightarrow 0 \) as \( t \rightarrow +0 \). In the same manner we can prove that (the second term of (2.9)) \( \rightarrow 0 \) as \( t \rightarrow +0 \). It is easily seen that (the third term of (2.9)) \( \rightarrow 0 \) as \( t \rightarrow +0 \). From these facts it follows that \( A_{\lambda} f = B_{\lambda} f \). Q.E.D.

**Theorem 2.2.** The infinitesimal generator \( A_{-i\eta} \) of the group \( \{P_{-i\eta}(t): -\infty < t < \infty\} \) in Theorem 1.2 is given by the following form:

\[
D(A_{-i\eta}) = \{ f \in W^2 : |u|(Un)(u) \in L^2(R) \} \quad \text{and for } f \text{ in } D(A_{-i\eta})
\]

\[
(A_{-i\eta} f)(z) = q \int_{-\infty}^{\infty} (Un)(u) \text{sign}(u) \exp(izu) du
\]

\[
+ iq(z - i) \int_{-\infty}^{\infty} |u|(Un)(u) \exp(izu) du. \quad (2.11)
\]

**Proof.** In the same manner as the proof of Theorem 2.1 we obtain

\[
D(A_{-i\eta}) = \{ f \in W^2 : |u|(Un)(u) \in L^2(R) \}.
\]

If \( f \) is in \( D(A_{-i\eta}) \), then

\[
n(u) = (u - i)(Un)(u) \cdot \frac{1}{u - i} \in L^1(R).
\]

By the Parseval theorem we see that
Since

\[ t^{-1}(Uf_{-iqt} - Un)(u) = -(2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{f(v)}{(v-i)(v-i+qt)} \exp(-iuv) dv \]

\[ = -(2\pi)^{-1/2} \int_{-\infty}^{\infty} (Un)(w) U_v \left[ \frac{1}{v-i+qt} \right] (u-w) dw, \]

(2.12)

\[ t^{-1}(Uk_{-iqt} - Un)(u) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{f(v)}{(v-i)(v-i-qt)} \exp(-iuv) dv \]

\[ = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Un)(w) U_v \left[ \frac{1}{v-i-qt} \right] (u-w) dw. \]

(2.13)

Since

\[ (Un)(w), \ U_v \left[ \frac{1}{v-i+qt} \right] (w), \ U_v \left[ \frac{1}{v-i-qt} \right] (w) \]

are in \( L^1(R) \), (2.12) and (2.13) are functions in \( L^1(R) \) and in \( L^2(R) \). From these facts and from the same argument as the proof of Theorem 2.1 we obtain (2.11).

Q.E.D.

Note that if \( f \) is in \( D(A_{-iq}) \),

\[ \lim_{a \to \infty} (2\pi)^{-1/2} \int_{-a}^{a} (Un)(u) (i \text{ sign } u) \exp(izu) du \]

\[ = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Un)(u) (i \text{ sign } u) \exp(izu) du. \]

REMARK 2.1. In Theorem 2.1 and Theorem 2.1

1) \[ \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{n(u)}{u-z} du = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (Un)(u) (i \text{ sign } u) \exp(izu) du, \]

2) if \( n(u) \) is absolutely continuous on each bounded closed interval and \( n'(u) \) is absolutely integrable over \( R \),

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n'(u)}{u-z} du = -\lim_{a \to \infty} (2\pi)^{-1/2} \int_{-a}^{a} |u| (Un)(u) \exp(izu) du. \]

Here

\[ \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{n(u)}{u-z} du, \ \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{n'(u)}{u-z} du \]

denote the Hilbert transforms of \( n(u) \) and \( n'(u) \).

References

[2] R. R. Coifman and C. Fefferman Weighted norm inequalities for maximal functions and