Correction to: Some Oscillation and Asymptotic Properties for Linear Differential Equations

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The purpose of this note is to present some radical changes on the results of our paper mentioned above. These changes are necessary due to the fact that most of the results of our paper have been obtained following an erroneous step based on a similar argument by Singh [1]. The authors are indebted to Prof. H. Onose for his remarks on the subject. The above step could be easily modified in order that the results hold as exactly as they are formulated in the case of second order differential equations, while they hold at least for the bounded solutions in the case of differential equations of arbitrary order. This could be done by choosing the points $t_1, \tau_1, \tau_2, \dots, \tau_{n-1}$ (line 1 from below, p. 26 and line 10 from below, p. 28) so that $T < t_1 < \tau_1 < \tau_2 < \dots < \tau_{n-1}$.

We present here the corrected results and provide a new proof for the theorems which is a lot more simple.

We consider the n -th order ($n > 1$) general ordinary differential equation

$$(E) \quad (r_{n-1}(t)(r_{n-2}(t)(\dots(r_1(t)(r_0(t)x(t))')\dots)')' + a(t)x(t) = b(t), \quad t \geq t_0,$$

where the functions r_i ($i=0, 1, \dots, n-1$) are positive at least on the interval $[t_0, \infty)$. The continuity of the functions r_i ($i=0, 1, \dots, n-1$), a and b will be assumed without mention. Because of the linearity, all solutions of (E) are defined on the whole interval $[t_0, \infty)$. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$ is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

THEOREM 1. Consider the differential equation (E) subject to the conditions:

$$(R_0) \quad \liminf_{t \rightarrow \infty} r_0(t) > 0.$$

$$(A) \quad \int_{t_1}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |a(s)| ds ds_{n-1} \dots ds_2 ds_1 < \infty.$$

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$$(B) \quad \int_{r_1(s_1)}^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)| ds ds_{n-1} \cdots ds_2 ds_1 < \infty.$$

Then for every bounded oscillatory solution x of the equation (E),

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

Moreover, in the particular case $n=2$ all unbounded solutions of (E) are non-oscillatory.

THEOREM 2. Consider the differential equation

$$(E)_0 \quad (r_{n-1}(t)(r_{n-2}(t)(\cdots(r_1(t)(r_0(t)x(t))' \cdots))')' + a(t)x(t) = 0$$

subject to the conditions (R_0) and (A).

Then every nontrivial bounded solution of the equation $(E)_0$ is nonoscillatory.

PROOFS OF THEOREMS 1 AND 2. Let x be an oscillatory solution of (E). We introduce the notations $D_r^{(i)}x$ ($i=0, 1, \dots, n-1$) as follows:

$$D_r^{(0)}x = r_0x \quad \text{and} \quad D_r^{(i)}x = r_i(D_r^{(i-1)}x)' \quad (i=1, 2, \dots, n-1).$$

Obviously, all functions $D_r^{(i)}x$ ($i=0, 1, \dots, n-1$) are also oscillatory. We put $c = \inf\{r_0(t) : t \geq t_0\}$. Because of (R_0) , we have that $c > 0$.

Let x be bounded. For any fixed $t \geq t_0$ we choose τ_i ($i=0, 1, \dots, n-1$) with $t \leq \tau_0 \leq \tau_1 \leq \cdots \leq \tau_{n-1}$ and $(D_r^{(i)}x)(\tau_i) = 0$ ($i=0, 1, \dots, n-1$). Then on repeated integration from equation (E) we obtain

$$\begin{aligned} (-1)^n (D_r^{(0)}x)(t) + \int_t^{\tau_0} \frac{1}{r_1(s_1)} \int_{s_1}^{\tau_1} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\tau_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\tau_{n-1}} a(s)x(s) ds ds_{n-1} \cdots \\ \cdots ds_2 ds_1 \\ = \int_t^{\tau_0} \frac{1}{r_1(s_1)} \int_{s_1}^{\tau_1} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\tau_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\tau_{n-1}} b(s) ds ds_{n-1} \cdots \\ \cdots ds_2 ds_1. \end{aligned}$$

Therefore, for every $t \geq t_0$ we have

$$\begin{aligned} (*) \quad |x(t)| \leq \frac{S}{c} \int_t^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |a(s)| ds ds_{n-1} \cdots \\ \cdots ds_2 ds_1 \\ + \int_t^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)| ds ds_{n-1} \cdots \\ \cdots ds_2 ds_1, \end{aligned}$$

where $S = \sup\{|x(t)| : t \geq t_0\}$. Now, by conditions (A) and (B), inequality (*) gives

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

Next, we assume that $b=0$. If x is in addition nontrivial, then from (*) we get

$$0 < c \leq \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1} \cdots ds_2 ds_1,$$

which contradicts condition (A).

It remains to prove the second part of the conclusion of Theorem 1. This is the case of $n=2$. Because of (A) and (B), we choose a $T \geq t_0$ so that

$$\int_T^\infty \frac{1}{r_1(t)} \int_t^\infty |a(s)| ds dt \leq \frac{c}{2} \quad \text{and} \quad \int_T^\infty \frac{1}{r_1(t)} \int_t^\infty |b(s)| ds dt \leq \frac{c}{2}$$

and we consider a $\tau_0 \geq T$ with $x(\tau_0)=0$. Next, we suppose that x is unbounded and let $\hat{\tau}_0 > \tau_0$ be such that $|x(\hat{\tau}_0)| > 1$. Also, we choose a $\tau_1 \geq \hat{\tau}_0$ with $(D_r^{(1)}x)(\tau_1)=0$. Then from equation (E) we obtain

$$(D_r^{(0)}x)(t) - \int_{\tau_0}^t \frac{1}{r_1(u)} \int_u^{\tau_1} a(s)x(s) ds du = - \int_{\tau_0}^t \frac{1}{r_1(u)} \int_u^{\tau_1} b(s) ds du$$

and consequently for every $t \in [\tau_0, \tau_1]$ we get

$$\begin{aligned} |(D_r^{(0)}x)(t)| &\leq \int_{\tau_0}^t \frac{1}{r_1(u)} \int_u^{\tau_1} |a(s)| |x(s)| ds du + \int_{\tau_0}^t \frac{1}{r_1(u)} \int_u^{\tau_1} |b(s)| ds du \\ &\leq M \int_T^\infty \frac{1}{r_1(u)} \int_u^\infty |a(s)| ds du + \int_T^\infty \frac{1}{r_1(u)} \int_u^\infty |b(s)| ds du \\ &\leq M \frac{c}{2} + \frac{c}{2}, \end{aligned}$$

where $M = \max \{|x(t)| : t \in [\tau_0, \tau_1]\}$. Thus, $M \leq \frac{1}{2}M + \frac{1}{2}$, i.e. $M \leq 1$, which is a contradiction, since $M \equiv \max \{|x(t)| : t \in [\tau_0, \tau_1]\} \geq |x(\hat{\tau}_0)| > 1$.

References

- [1] Bhagat Singh, Forced oscillations in general ordinary differential equations, Tamkang J. Math. 6 (1975), 5–11.