A Note on Completeness of Real-Valued Functions

$$\{\phi^{n_p} : p=1, 2, \dots\}$$

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In this note we study a real-valued function ϕ defined on the real line $\mathbf{R}$ such that the sequence of functions $\{\phi^{n_p} : p=1, 2, \dots\}$ is complete on some interval $[\alpha, \beta]$, where $\{n_p : p=1, 2, \dots\}$ is a sequence of positive numbers which satisfies the Müntz-Szász condition:

$$\sum_{p=1}^{\infty} \frac{1}{n_p} = +\infty.$$

We now say that the sequence of functions $\{\phi_n : n=1, 2, \dots\}$ is complete on the interval $[\alpha, \beta]$ if for any summable function $f(t)$ on $[\alpha, \beta]$, the equalities

$$\int_{\alpha}^{\beta} f(t) \phi_n(t) d\mu(t) = 0, \quad n=1, 2, \dots,$$

imply that $f(t)=0$, a.e. on $[\alpha, \beta]$ (cf. [1]). Here μ denotes the Lebesgue measure on $\mathbf{R}$.

Throughout the remainder $\{n_p : p=1, 2, \dots\}$ will denote a sequence of positive numbers with $\lim_{p \rightarrow \infty} n_p = +\infty$ and ϕ will denote a real-valued function on $\mathbf{R}$ such that $\phi(\alpha_{\phi}) \geq 0$ and ϕ is strictly increasing on some interval $[\alpha_{\phi}, \alpha_{\phi} + \delta_{\phi}]$, where α_{ϕ} is a real number and δ_{ϕ} is a positive number.

In [3], the first author has shown that if ϕ is an absolutely continuous function on $[\alpha_{\phi}, \alpha_{\phi} + \delta_{\phi}]$ with $\phi'(t) \neq 0$, a.e. on $[\alpha_{\phi}, \alpha_{\phi} + \delta_{\phi}]$, and if $\sum_{p=1}^{\infty} 1/n_p = +\infty$, then $\{\phi^{n_p} : p=1, 2, \dots\}$ is complete on $[\alpha_{\phi}, \alpha_{\phi} + \delta_{\phi}]$ (see [3, Theorem 1 part (i)]).

The following theorem shows that the above result holds under a strictly weaker condition on ϕ .

THEOREM. *If ϕ^{-1} is absolutely continuous on $[\phi(\alpha_{\phi}), \phi(\alpha_{\phi} + \delta_{\phi})]$, and if $\sum_{p=1}^{\infty} 1/n_p = +\infty$, then $\{\phi^{n_p} : p=1, 2, \dots\}$ is complete on $[\alpha_{\phi}, \alpha_{\phi} + \delta_{\phi}]$.*

REMARK. If ϕ is absolutely continuous and $\phi'(t) \neq 0$, a.e. on $[\alpha_{\phi}, \alpha_{\phi} + \delta_{\phi}]$, then ϕ^{-1} is absolutely continuous on $[\phi(\alpha_{\phi}), \phi(\alpha_{\phi} + \delta_{\phi})]$ from Zarecki's theorem (see, for example, [2, p. 271] and [3, Lemma 4]).

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However, we can find a strictly increasing function ψ on some interval $[0, \delta_\psi]$ such that ψ^{-1} is absolutely continuous on $[0, \psi(\delta_\psi)]$ with $\psi(0)=0$ and $0 < \delta_\psi < 1$ but ψ is not absolutely continuous on $[0, \delta_\psi]$. Such a function has been indicated by M. Hayashi (see, for example, [3, Remark 3]).

PROOF OF THEOREM. From the proof of [3, Theorem 1 part (i)], we can assume that $\alpha_\phi=0$ and $\phi(\alpha_\phi)=0$. Now suppose that ϕ^{-1} is absolutely continuous on $[0, \phi(\delta_\phi)]$ and $\sum_{p=1}^{\infty} 1/n_p = +\infty$. To show the completeness of $\{\phi^{n_p}: p=1, 2, \dots\}$ on $[0, \delta_\phi]$, let f be any summable function on $[0, \delta_\phi]$ such that

$$\int_0^{\delta_\phi} f(t) \phi^{n_p}(t) d\mu(t) = 0$$

for all $p=1, 2, \dots$. Set

$$\psi(s) = \phi^{-1}(\phi(\delta_\phi)s)$$

for each $s \in [0, 1]$. Then ψ is a strictly increasing and absolutely continuous function on $[0, 1]$. Therefore we have, from [4, Theorem 2 p. 431], that

$$\begin{aligned} 0 &= \int_0^{\delta_\phi} f(t) \phi^{n_p}(t) d\mu(t) \\ &= \int_0^1 f(\psi(s)) \phi^{n_p}(\delta_\phi s) \psi'(s) d\mu(s) \\ &= \phi^{n_p}(\delta_\phi) \int_0^1 f(\psi(s)) \psi'(s) s^{n_p} d\mu(s) \end{aligned}$$

for all $p=1, 2, \dots$. We then obtain, from the well-known Müntz-Szasz theorem, that $f(\psi(s))\psi'(s)=0$, a.e. on $[0, 1]$. Set

$$E = \{s \in [0, 1]: \psi'(s) \neq 0\}.$$

Then there is a null set N in $[0, 1]$ such that

$$f(\psi(s))=0$$

for each $s \in E \cap ([0, 1] - N)$. It follows from [4, Lemma 1, p. 430] that

$$\begin{aligned} &\mu([0, 1] - \psi(E \cap ([0, 1] - N))) \\ &= \mu(\psi([0, 1] - E) \cup N) \\ &\leq \mu(\psi([0, 1] - E)) + \mu(\psi(N)) \\ &= \int_{[0, 1] - E} \psi'(s) d\mu(s) + \int_N \psi'(s) d\mu(s) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

In other words, $f(t)=0$, a.e. on $[0, \delta_\phi]$ and hence $\{\phi^{n_p}: p=1, 2, \dots\}$ is complete on $[0, \delta_\phi]$.

References

- [1] J. S. Hwang, C. S. Pan and L. S. Wu, An extension on Bernstein-Müntz-Szasz theorems, *Soochow J. Math.* **4** (1978), 121–126.
- [2] I. P. Natanson, *Theory of functions of a real variable*, Frederick Unger, Vol. 1, New York, 1955.
- [3] S. Takahasi, An extension of Müntz-Szasz theorem and Hwang-Pan-Wu theorem, *Soochow J. Math.* **6** (1980), 129–136.
- [4] J. F. Randolph, *Basic real and abstract analysis*, Academic Press, New York and London, 1968.