On the Submultiplicative classes

Dedicated to Professor Kentaro MURATA on his 60th birthday.

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ABSTRACT

In his paper [3], K. Oshima characterized a certain classes of submultiplicative matrix norms. It is intent of this paper to extend his work. The main result will be given in the diagram.

1. INTRODUCTION

Except when the appear as dimension and indices, or when otherwise indicated, lower Latin letter will denote column vectors, scalars, or variables. Context will allow one to distinguish among the three. Capital Latin letters will denote square matrices, and the set of square complex matrices will bedenoted by $M_n(\mathbb{C})$ and complex n -dimensional space by $\mathbb{C}^n$. The symbol $\|\cdot\|$ will be used to denote both vector and matrix norms. Context will allow one to distinguish between the two.

We will utilize the following symbols:

For a matrix $A = (a_{ij})_1^n$,

$$|A| = (|a_{ij}|)_1^n.$$

For $A, B \in M_n(\mathbb{C})$,

we write $|A| \leq |B|$, if $|a_{ij}| \leq |b_{ij}|$ for all i, j .

For $E_{ij} \in M_n(\mathbb{C})$,

we mean the matrix with 1 in i, j -component and zero otherwise.

DEFINTION 1.1. A submultiplicative matrix norm on $M_n(\mathbb{C})$ is an non-negative real valued function $\|\cdot\|$ defined on $M_n(\mathbb{C})$ satisfying:

- (a) $A=0$ if and only if $\|A\|=0$
- (b) $\|aA\| = |a| \cdot \|A\|$ for all $A \in M_n(\mathbb{C})$ and all scalars $a \in \mathbb{C}$
- (c) $\|A+B\| \leq \|A\| + \|B\|$ for all $A, B \in M_n(\mathbb{C})$
- (d) $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in M_n(\mathbb{C})$.

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For the sake of completeness, we will write the followings which were defined in [3].

DEFINITION 1.2.

$$\mathcal{D} = \{D \in M_n(\mathbb{C}); D = \text{diag}(d_1, d_2, \dots, d_n)\}$$

$$\mathcal{D}_1 = \{D \in M_n(\mathbb{C}); D = \text{diag}(d_1, d_2, \dots, d_n) \text{ where } \max_i |d_i| = 1\}$$

DEFINITION 1.3.

$$(\mathcal{M}): \text{ If } |A| \leq |B| \text{ then } \|A\| \leq \|B\|.$$

$$(\mathcal{F}): \|A\| = \| |A| \| \text{ for all } A \in M_n(\mathbb{C}).$$

$$(\mathcal{L}): \max_{A \neq 0} \|DA\|/\|A\| = 1 \text{ for all } D \in \mathcal{D}_1.$$

$$(\mathcal{R}): \max_{A \neq 0} \|AD\|/\|A\| = 1 \text{ for all } D \in \mathcal{D}_1.$$

$$(\mathcal{B}'): \max_{A \neq 0} \|DAD'\|/\|A\| = 1 \text{ for all } D, D' \in \mathcal{D}_1.$$

$$(\mathcal{B}'^*): \max_{A \neq 0} \|DAD'\|/\|A\| = 1 \text{ for } D, D' \in \mathcal{D}_1, \text{ and } D \text{ or } D' \text{ being non-singular.}$$

$$(\mathcal{S}): \|D\| = 1 \text{ for any } D \in \mathcal{D}_1.$$

Now the following class of submultiplicative matrix norms is defined:

DEFINITION 1.4.

$$(\mathcal{G}): \max_{A, B \neq 0} \|ADB\|/\|A\| \cdot \|B\| = 1 \text{ for all } D \in \mathcal{D}_1.$$

2. INCLUSIONS UNDER SOME CONDITIONS

In this section it will be investigated that if we impose some conditions on a matrix or a norm, then we may have more result, [3], on those classes which is defined in Definition 1.3 and Definition 1.4.

THEOREM 2.1. *Let $\|\cdot\| \in (\mathcal{L}) \cup (\mathcal{R})$ be a matrix norm such that $\|E_{ii}\| = 1$ for all $i = 1, 2, \dots, n$, then $\|\cdot\| \in (\mathcal{G})$.*

PROOF. For all $A, B \neq 0$ and $D \in \mathcal{D}_1$,
let

$$\|\cdot\| \in (\mathcal{R}) \quad [\text{or let } \|\cdot\| \in (\mathcal{L})].$$

Then we have

$$\|ADB\|/\|A\| \cdot \|B\| \leq \|AD\| \cdot \|B\|/\|A\| \cdot \|B\| = \|AD\|/\|A\|$$

$$[\text{or } \|ADB\|/\|A\| \cdot \|B\| \leq \|A\| \cdot \|DB\|/\|A\| \cdot \|B\| = \|DB\|/\|B\|].$$

Hence it follows that

$$\begin{aligned} \max_{A, B \neq 0} \|ADB\|/\|A\| \cdot \|B\| &\leq \max_{A \neq 0} \|AD\|/\|A\| = 1 \\ [\text{or } \max_{A, B \neq 0} \|ADB\|/\|A\| \cdot \|B\| &\leq \max_{A \neq 0} \|DB\|/\|B\| = 1]. \end{aligned}$$

Choose

$$A = B = E_{i_0 i_0} \quad \text{where} \quad |d_{i_0}| = 1 \quad \text{of} \quad D \in \mathcal{D}_1$$

then it follows from the fact

$$\|E_{i_0 i_0}\| = 1$$

that

$$\|E_{i_0 i_0} D E_{i_0 i_0}\| / \|E_{i_0 i_0}\|^2 = 1/1^2 = 1.$$

Therefore

$$(\mathcal{L}) \cup (\mathcal{R}) \subset (\mathcal{G}).$$

THEOREM 2.2. *For all non-negative matrices $A, B \in M_n(\mathcal{R}^+)$ and $D \in \mathcal{D}_1^+$, the class $(\mathcal{M}^+)$ is contained in the class $(\mathcal{G}^+)$, if $\|E_{ii}\| = 1$.*

Note. The symbol $^+$ means the sets or the classes which are defined on the set of all non-negative matrices.

PROOF. Since A, B and D are non-negative, it follows that

$$|ADB| \leq |AB|$$

so that

$$\|ADB\| \leq \|AB\| \leq \|A\| \cdot \|B\|.$$

Choose $A = B = E_{i_0 i_0}$ where $|d_{i_0}| = 1$ of $D \in \mathcal{D}_1$ then,

$$\max_{AB \neq 0} \|ADB\| / \|A\| \cdot \|B\| = \|E_{i_0 i_0} D E_{i_0 i_0}\| / \|E_{i_0 i_0}\|^2 = 1.$$

REMARK. The following example shows that we may not drop the non-negative.

Let

$$A = \begin{pmatrix} 1, & 1 \\ 1, & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1, & -2 \\ 1, & 2 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 1, & 0 \\ 0, & 1/2 \end{pmatrix} \quad \text{then,}$$

$$ADB = \begin{pmatrix} -1/2, & -1 \\ 0, & -4 \end{pmatrix}, \quad \text{but} \quad AB = \begin{pmatrix} 0, & 0 \\ -3, & 6 \end{pmatrix}.$$

We will state the following known Theorem [3] for our convenience.

THEOREM 2.3. *The following classes are equal:*

- (1) The class $(\mathcal{L}) \cap (\mathcal{R})$.
- (2) The class $(\mathcal{B}')$.
- (3) The class $(\mathcal{B}^{**})$ which has the property that $\max_{A \neq 0} \|DAD'\| / \|A\| = 1$ for all $D, D' \in \mathcal{D}_1$ with D or D' being nonsingular.

The foregoing may be summarized in the following diagram:

3. Certain classes of norms with $\|I\|=1$. In this section we study relation between the class $(\mathcal{G})$ with $\|I\|=1$ and all others.

LEMMA 3.1. Let $\|\cdot\| \in (\mathcal{L}) \cup (\mathcal{R})$. If $|D| \leq |D'|$ for $D, D' \in \mathcal{D}^*$, then

$$\|D\| \leq \|D'\|.$$

PROOF. If $|D| \leq |D'|$, then there exists a matrix D^* in $\mathcal{D}$ such that $\max_i \{|d_{*i}^i|\} \leq 1$ and $D = D^* D'$ ($D = D' D^*$).

Since $\|\cdot\| \in (\mathcal{L}) \cup (\mathcal{R})$ it follows that

$$\|D\|/\|D'\| = \|D^* D'\|/\|D'\| (= \|D' D^*\|/\|D'\|) \leq \max_i \{|d_{*i}^i|\} \leq 1.$$

Hence $\|D\| \leq \|D'\|$.

THEOREM 3.2. Let $\|\cdot\|$ be a matrix norm on $M_n(\mathbb{C})$ with $\|I\|=1$. Then the following conditions are equivalent

- (1) $\|D\|=1$ for any $D \in \mathcal{D}_1$ (i.e. $\|\cdot\| \in (\mathcal{L})$)
- (2) $\max_{A \neq 0} \|DA\|/\|A\| = 1$ for any $D \in \mathcal{D}_1$
(i.e. $\|\cdot\| \in (\mathcal{L})$)
- (3) $\max_{A \neq 0} \|AD\|/\|A\| = 1$ for any $D \in \mathcal{D}_1$
(i.e. $\|\cdot\| \in (\mathcal{R})$)
- (4) $\max_{A \neq 0} \|DAD'\|/\|A\| = 1$ for any $D, D' \in \mathcal{D}_1$
(i.e. $\|\cdot\| \in (\mathcal{B}')$)
- (5) $\max_{A \neq 0} \|DAD'\|/\|A\| = 1$ where $D, D' \in \mathcal{D}_1$ and D or D' is nonsingular
(i.e. $\|\cdot\| \in (\mathcal{B}'^*)$)
- (6) $\max_{A, B \neq 0} \|ADB\|/\|A\| \|B\| = 1$ for any $D \in \mathcal{D}_1$
(i.e. $\|\cdot\| \in (\mathcal{G})$).

PROOF. If $\|\cdot\| \in (\mathcal{L}) \cup (\mathcal{R})$, then by submultiplicativity and Lemma 3.1. we have $1 \leq \|E_{ii}\| \leq \|I\|=1$. thus $\|E_{ii}\|=1$. By using Theorem 3.2. and Lemma 3.1. we see that (5) $\rightarrow$ (4) $\rightarrow [(2) \cap (3)] \rightarrow$ (6). So we need only to show that (1) $\rightarrow$ (4) $\rightarrow$ (1) and (6) $\rightarrow$ (1).

Let $\|\cdot\| \in (\mathcal{S})$ and let $b = \max_{A \neq 0} \|DAD'\|/\|A\|$, where $D, D' \in \mathcal{D}_1$. By submultiplicativity we have $b \leq \|D\| \|D'\| = 1 \times 1 = 1$. Choose $A = E_{i_0 i_0}$ where $|d_{i_0}| = |d_{j_0}| = 1$, then $\|DE_{i_0 i_0}D'\|/\|E_{i_0 i_0}\| = 1 \leq b$. Thus $b = 1$, whence (1) implies (4).

Now let $\|\cdot\|$ satisfies condition (4) and let $D \in \mathcal{D}_1$.

Then

$$1 = \max \|D^{1/2}AD^{1/2}\| \geq \|D^{1/2}ID^{1/2}\|/\|I\| = \|D\|.$$

Since $1 = \max_i \{|d_i|\} \leq \|D\|$, we have $\|D\| = 1$, hence

$$\|\cdot\| \in (\mathcal{S}).$$

Finally we show that (6) implies (1).

If $\max_{A, B \neq 0} \|ADB\|/\|A\| \|B\| = 1$ for any $D \in \mathcal{D}_1$, then since $\|\cdot\|$ is submultiplicative, it follows that $\|D\| \geq 1$. Taking $A = B = I$ we have $\|D\| \leq 1$. Thus $\|D\| = 1$, which implies $\|\cdot\| \in (\mathcal{S})$, i.e. (6) implies (1).

Therefore if $\|I\| = 1$, then we have $(\mathcal{M}) = (\mathcal{F}) = (\mathcal{L}) = (\mathcal{R}) = (\mathcal{B}') = (\mathcal{B}'^*) = (\mathcal{S}) = (\mathcal{G})$, and $(\mathcal{M}^\dagger) = (\mathcal{G}^\dagger)$.

References

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