Notes on Noetherian Semigroup Rings

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Let A be a commutative ring with identity and let S be a torsion-free cancellative commutative semigroup with identity. We write the semigroup operation on S as addition and assume that $S \not\equiv \{0\}$. We consider the semigroup ring $A[X; S]$ of S over A . In [12] we determined conditions under which $A[X; S]$ is a Noetherian ring. We concern with $A[X; S]$ further as a Noetherian ring in this paper. Throughout this paper A denotes a commutative ring with identity. S denotes the above mentioned semigroup. The smallest group containing S is called quotient group of S and is denoted by $q(S)$. G denotes the quotient group of S .

1.

At first we list some Noetherian properties of a ring.

DEFINITION 1. The followings are some Noetherian properties of a ring and references definitions of which are due to:

Cohen-Macaulay ring	[13, (16. E)];
universally catenary	[13, p. 84];
excellent ring	[13, (34. A)];
Nagata ring	[13, (31. A)];
G -ring	[13, (33. A)];
integrally Noetherian ring	[2, chap. 5, §3, exerc. 6)];
J -0 ring	[13, (32. B)];
J -2 ring	[13, (32. B)].

THEOREM 2([8], [9], [10]). Let π be one of properties of a ring of Definition 1. If S is a group, $A[X; S]$ is a ring with property π if and only if A is a ring with property π and S is a finitely generated group.

THEOREM 3.

- (1) (a special case of [12, Appendix]) $A[X; S]$ is a Noetherian ring if and only if A is a Noetherian ring and S is a finitely generated semigroup;
- (2) $A[X; S]$ is an r -Noetherian ring if and only if $A[X; S]$ is a Noetherian ring.

Using the results of Connell (Canad. J. Math., 15(1963)) and of [9], the proof

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of (2) is similar with that of [12, Appendix]. We omit it.

Here, let Z_0 be the set of nonnegative integers. If there exists a finite subset $\{s_1, \dots, s_n\}$ of S such that $S = Z_0 s_1 + \dots + Z_0 s_n$, S is said to be a finitely generated semigroup. And, if each ideal of a ring A containing a nonzerodivisor is a finitely generated ideal, A is said to be an r -Noetherian ring ([7]).

As a corollary to Theorem 3, we can generalize a part of Theorem 2 for a semigroup;

THEOREM 4. *$A[X; S]$ is a ring with the respective property of universally catenary, excellent, Nagata, G-ring, integrally Noetherian and J-2, if and only if A is a ring with the respective property and S is a finitely generated semigroup.*

PROOF. Assume that $A[X; S]$ is a universally catenary ring. $A[X; G]$ is a quotient ring of $A[X; S]$. Therefore $A[X; G]$ is a universally catenary. By Theorem 2, A is universally catenary ring. Since $A[X; S]$ is a Noetherian ring by definition, S is a finitely generated semigroup by Theorem 3. Conversely assume that A is a universally catenary ring and S is a finitely generated semigroup. Then $A[X; S]$ is a finitely generated algebra over A . Therefore $A[X; S]$ is a universally catenary ring. Proofs for the respective property of excellent, Nagata, G-ring and integrally Noetherian are similar with that for universally catenary. Let A be a ring with the property J-2 and let S be a finitely generated semigroup. A finitely generated algebra of $A[X; S]$ is a finitely generated algebra over A . It follows that $A[X; S]$ is a ring with the property J-2. Conversely assume that $A[X; S]$ is a ring with the property of J-2. Let P be a prime ideal of A . We set $\varphi^{-1}(P) = \mathfrak{p}$ in the following Lemma 5. Since $A/P = A[X; S]/\mathfrak{p}$, A is a ring with the property of J-2 by [13, Theorem 73].

LEMMA 5. *There exists a canonical ring homomorphism φ of $A[X; S]$ onto A .*

PROOF. There exists a canonical ring homomorphism ψ of $A[X; G]$ onto A . Then the restriction φ of ψ to $A[X; S]$ is a desired homomorphism.

Let D be an integral domain. We note that Bouvier [3] asks to determine conditions under which $D[X; S]$ is a catenary ring and also asks to determine conditions under which $D[X; S]$ is a Cohen-Macaulay ring.

THEOREM 6. *$A[X; S]$ has property J-0 if and only if A has property J-0 and S is a finitely generated semigroup.*

PROOF. Sufficiency. Since G is a finitely generated group, $A[X; G]$ is a ring with property J-0 by Theorem 2. Therefore there exists an element $f \in A[X; G]$ such that $\emptyset \neq D'(f) \subset \text{Reg}(A[X; G])$. Here $D'(f)$ is the set of prime ideals P of $A[X; G]$ such that $f \notin P$ and $\text{Reg}(A[X; G])$ is the set of prime ideals P of $A[X; G]$

such that $A[X; G]_P$ is a regular local ring. Set $S = \sum_{i=1}^n Z_0 s_i$. Then the set $D(fX^{s_1} \dots X^{s_n})$ of prime ideals P of $A[X; S]$ such that $P \ni fX^{s_1} \dots X^{s_n}$ is not an empty set. Let $p \in D(fX^{s_1} \dots X^{s_n})$. We have $A[X; S]_p = A[X; G]_{pA[X; G]}$. $A[X; G]_{pA[X; G]}$ is a regular local ring, since $pA[X; G] \ni f$. We have proved that $A[X; S]$ is a ring with the property $J-0$. Necessity. There exists $f \in A[X; S]$ such that $\emptyset \neq D(f) \subset \text{Reg}(A[X; S])$. Set $f = a_1 X^{s_1} + \dots + a_n X^{s_n}$, where $0 \neq a_i \in A$ and $s_1 < s_2 < \dots < s_n$ ($G = q(S)$ becomes a totally ordered group). Since $\emptyset \neq D(f)$, there exists k such that a_k is not a nilpotent element. Then the set $D_0(a) = \{P; \text{prime ideal of } A \text{ such that } P \ni a\}$ is not an empty set. Let $P \in D_0(a)$. Then $PA[X; S] \in D(f) \subset \text{Reg}(A[X; S])$. Moreover $A[X; S]_{PA[X; S]} = A[X; G]_{PA[X; G]}$. By [9, (3.1)], A_P is a regular local ring. Therefore A has property $J-0$.

2.

PROPOSITION 1. *If $A[X; S]$ is a Cohen-Macaulay ring, then A is a Cohen-Macaulay ring and S is a finitely generated semigroup.*

PROOF. It follows that S is a finitely generated semigroup by Theorem 3 of section 1 and that $A[X; G]$ is a Cohen-Macaulay ring. Then A is a Cohen-Macaulay ring by Theorem 2 of section 1.

LEMMA 2([10, (6.5)]). *If A is a Noetherian ring, then we have $\dim A[X; S] = \dim A[X; G] = \dim A + \text{t.f.r.}(G)$.*

Here $\text{t.f.r.}(G)$ is the torsion-free rank of G .

LEMMA 3.

- (1) (e.g. [5, (31.16)]). *An integral domain which is a finitely generated algebra over a field satisfies the first chain condition (f.c.c.).*
- (2) ([1, chap. II, §1, Prop. 2]). *Let α be a subset of a ring A which is both additively and multiplicatively closed. Let $P_1, \dots, P_n$ be a finite number of prime ideals of A so that α is contained in the set-theoretical union $P_1 \cup \dots \cup P_n$. Then α is contained in some P_i .*
- (3) (e.g. [13, (16.D)]). *Let A be a Noetherian ring. Then A is a Cohen-Macaulay ring if and only if the unmixed theorem holds in A .*
- (4) (e.g. [13, Theorem 31]). *Let A be a Cohen-Macaulay ring and let $\{X_1, \dots, X_n\}$ be a system of indeterminates over A . Then a quotient ring of $A[X_1, \dots, X_n]$ is a Cohen-Macaulay ring.*

LEMMA 4. *Let k' be a field and let k be a subfield of k' . Let M' be a prime ideal of $k'[X; S]$ and let M be a prime ideal of $k[X; S]$ such that $M' \cap k[X; S] = M$. Let $m_1, \dots, m_s$ be a finite set of elements of M . Then $\{m_1, \dots, m_s\}$ is a regular sequence for $k'[X; S]_{M'}$ if and only if it is a regular sequence for $k[X; S]_M$.*

PROOF. Let $\{u_\lambda; \lambda \in A\}$ be a linear basis of k' over k . Then $k'[X; S]$ is a free module over $k[X; S]$ and $\{u_\lambda; \lambda \in A\}$ is a free basis of $k'[X; S]$ over $k[X; S]$. Assume $\{m_1, \dots, m_s\}$ is a regular sequence for $k'[X; S]_{M'}$. Let $1 \leq l \leq s$ and let $\xi \in k[X; S]_{M'}$. Assume $\xi m_l \in (m_1, \dots, m_{l-1})k[X; S]_{M'}$. There exist $f, g, f_1, \dots, f_{l-1} \in k[X; S]$ such that $g \in M$, $\xi = f/g$ and $fm_l = \sum_{i=1}^{l-1} m_i f_i$. We have $f = \sum_{i=1}^{l-1} m_i F_i / G$ for $F_1, \dots, F_{l-1}, G \in k'[X; S]$ such that $G \in M'$. Express $F_i = \sum f_{i\lambda} u_\lambda$ and $G = \sum g_\lambda u_\lambda$ where $f_{i\lambda}, g_\lambda \in k[X; S]$. Since $G \in M'$, there exists $\lambda_0 \in A$ such that $g_{\lambda_0} \in M$. We have $fg_{\lambda_0} = \sum_{i=1}^{l-1} m_i f_{i\lambda_0}$. Since $\xi = \sum m_i (f_{i\lambda_0}) / (gg_{\lambda_0})$, we have $\xi \in \sum_{i=1}^{l-1} m_i k[X; S]_{M'}$. Therefore $\{m_1, \dots, m_s\}$ is a regular sequence for $k[X; S]_{M'}$. Conversely assume that $\{m_1, \dots, m_s\}$ is a regular sequence for $k[X; S]_{M'}$. Let again $1 \leq l \leq s$ and let $\eta \in k'[X; S]_{M'}$ such that $\eta m_l \in (m_1, \dots, m_{l-1})k'[X; S]_{M'}$. There exist $F, G, F_1, \dots, F_{l-1} \in k'[X; S]$ such that $G \in M'$, $\eta = F/G$ and $Fm_l = \sum_{i=1}^{l-1} m_i F_i$. Express $F = \sum f_\lambda u_\lambda$ and $F_i = \sum f_{i\lambda} u_\lambda$ for $f_\lambda, f_{i\lambda} \in k[X; S]$. We have $f_\lambda m_l = \sum_{i=1}^{l-1} f_{i\lambda} m_i$. It follows $f_\lambda \in (m_1, \dots, m_{l-1})k[X; S]_{M'}$. It follows $\eta \in (m_1, \dots, m_{l-1})k'[X; S]_{M'}$. Therefore $\{m_1, \dots, m_s\}$ is a regular sequence for $k'[X; S]_{M'}$.

PROPOSITION 5. *Let k' be an algebraic extension field of a field k . Then if $k[X; S]$ is a Cohen-Macaulay ring, $k'[X; S]$ is a Cohen-Macaulay ring.*

PROOF. Let M' be a maximal ideal of $k'[X; S]$. Set $M' \cap k[X; S] = M$. Since $k'[X; S]$ is integral over $k[X; S]$, M is a maximal ideal of $k[X; S]$. We set $t.f.r.(G) = d$. We have $\dim k'[X; S] = \dim k[X; S] = d$ by Lemma 2. We have $ht(M') = ht(M) = d$ by Lemma 3(1) ('ht' means the height). There exists a finite set $m_1, \dots, m_d$ of d elements of M such that $\{m_1, \dots, m_d\}$ is a regular sequence for $k[X; S]_M$. $\{m_1, \dots, m_d\}$ is a regular sequence for $k'[X; S]_{M'}$ by Lemma 4. Since $k'[X; S]$ is a Noetherian ring by Theorem 3 of section 1, $k'[X; S]$ is a Cohen-Macaulay ring.

PROPOSITION 6. *Let k' be a purely-transcendental extension field of k . Then, if $k[X; S]$ is a Cohen-Macaulay ring $k'[X; S]$ is a Cohen-Macaulay ring.*

PROOF. Let $\{v_\sigma; \sigma \in \Sigma\}$ be a purely-transcendence basis of k' over k . Let M' be a maximal ideal of $k'[X; S]$. We have $ht(M') = d (= t.f.r.(G))$. Let $v_1, \dots, v_n$ be a finite number of elements of $\{v_\sigma; \sigma \in \Sigma\}$. We set $k_0 = k(v_1, \dots, v_n)$ and $k_0[X; S] \cap M' = M_0$. We can choose such elements $v_1, \dots, v_n$ that $ht(M_0) = d$. Then M_0 is a maximal ideal of $k_0[X; S]$. Since $k_0[X; S]$ is a quotient ring of $(k[X; S])[v_1, \dots, v_n]$, $k_0[X; S]$ is a Cohen-Macaulay ring by Lemma 3(4). There exists a finite number of elements $m_1, \dots, m_d$ of M_0 such that $\{m_1, \dots, m_d\}$ is a regular sequence for $k_0[X; S]_{M_0}$. Since $\{m_1, \dots, m_d\}$ is a regular sequence for $k'[X; S]_{M'}$ by Lemma 4, $k'[X; S]_{M'}$ is a Cohen-Macaulay ring.

LEMMA 7. *Let k' be an extension field of k . Let M be a maximal ideal of $k[X; S]$ and let M' be a maximal ideal of $k'[X; S]$ such that $M' \cap k[X; S] = M$. Then M' is a minimal prime ideal of $Mk'[X; S]$.*

PROOF. Let $\{v_\sigma; \sigma \in \Sigma\}$ be a transcendence basis of k' over k . Set $k_0 = k(v_\sigma; \sigma \in \Sigma)$ and set $M' \cap k_0[X; S] = M_0$. $k_0[X; S]$ is a quotient ring of a polynomial ring $(k[X; S])[v_\sigma; \sigma \in \Sigma]$ over $k[X; S]$. Hence $Mk_0[X; S]$ is a prime ideal of $k_0[X; S]$ of $ht \geq d$ ($=t.f.r.(G)$), and hence $M_0 = Mk_0[X; S]$. Since $k'[X; S]$ is integral over $k_0[X; S]$, each prime ideal of $k'[X; S]$ lying over M_0 is a maximal ideal of $k'[X; S]$. Therefore M' is a minimal prime ideal of $Mk'[X; S]$.

PROPOSITION 8. *Let k' be an extension field of a field k . Then if $k'[X; S]$ is a Cohen-Macaulay ring, $k[X; S]$ is a Cohen-Macaulay ring.*

PROOF. Let M be a maximal ideal of $k[X; S]$. Since $k'[X; S]$ is a free module over $k[X; S]$, there exists a maximal ideal M' such that $M' \cap k[X; S] = M$. Let $\{m_1, \dots, m_s\}$ be a maximal regular sequence for $k[X; S]_M$. We may assume $m_i \in M$ for $1 \leq i \leq s$. Suppose $s < d$ ($=t.f.r.(G)$). $\{m_1, \dots, m_s\}$ is a regular sequence for $k'[X; S]_{M'}$ by Lemma 4. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_k$ be the prime ideals belonging to $(m_1, \dots, m_s)k'[X; S]_{M'}$. We have $ht(\mathfrak{P}_i) = s$ for $1 \leq i \leq k$ by Lemma 3(3). By Lemma 7, M is not contained in any $\mathfrak{P}_i$. It follows $M \not\subset \mathfrak{P}_1 \cup \dots \cup \mathfrak{P}_k$ by Lemma 3(2). We take an element $m \in M$ such that $m \notin \mathfrak{P}_1 \cup \dots \cup \mathfrak{P}_k$. Then $\{m_1, \dots, m_s, m\}$ is a regular sequence for $k'[X; S]_{M'}$. $\{m_1, \dots, m_s, m\}$ is a regular sequence for $k[X; S]_M$ by Lemma 4, which is a contradiction. Therefore $k[X; S]_M$ is a Cohen-Macaulay ring.

Proposition 5, 6 and 8 implies the following result:

THEOREM 9.

- (1) *Let k' be an extension field of k . Then $k'[X; S]$ is a Cohen-Macaulay ring if and only if $k[X; S]$ is a Cohen-Macaulay ring.*
- (2) *Let k be a field and let k_0 be a prime field of k . Then $k[X; S]$ is a Cohen-Macaulay ring if and only if $k_0[X; S]$ is a Cohen-Macaulay ring.*

3.

Let n be a fixed positive integer throughout this section. If each finitely generated ideal of a ring A is generated by n elements of A , A is said to have n -generator property ([5, p. 320]).

PROPOSITION 1 ([9, (4.16)]). *Let D be a domain and assume that S is a group. Then $D[X; S]$ has n -generator property if and only if D is a field and S is isomorphic into Q .*

We denote the set of nonnegative rational numbers by Q_0 . Let S be a finitely generated (additive) subsemigroup of Q_0 . We have $q(S) = Ze$ for $e > 0$. Then the smallest natural number contained in $(1/e)S$ we will call the order of S . Next, if each ideal of a ring A is generated by n elements A is said to have rank n ([4]). We determined in [11, section 5] conditions under which $A[X; S]$ has rank n . Especially we have

LEMMA 2 ([11, (5.6)]). *Let D be a domain. Then $D[X; S]$ has rank n if and only if D is a field and S is either isomorphic with Z or isomorphic with a subsemigroup of Z_0 containing a natural number $m \leq n$.*

THEOREM 3. *Let D be a domain. Then $D[X; S]$ has n -generator property if and only if D is a field and S is either isomorphic with a subgroup of Q or isomorphic with a subsemigroup T of Q_0 such that if T' is a finitely generated subsemigroup of T , T' is contained in a finitely generated subsemigroup T'' of T of order $\leq n$.*

PROOF. Sufficiency. We may assume that S is a subsemigroup of Q_0 by Proposition 1. Let $\alpha = (f_1, \dots, f_m)$ be a finitely generated ideal of $D[X; S]$. There exists a finite subset $\{s_1, \dots, s_l\}$ of S such that $f_i \in D[X; \sum_{j=1}^l Z_0 s_j]$ for $1 \leq i \leq m$. $\sum_{j=1}^l Z_0 s_j$ is contained in a finitely generated subsemigroup T' of S of order $\leq n$. By Lemma 2, there exists n elements $g_1, \dots, g_n$ in $D[X; T']$ such that $(f_1, \dots, f_m)D[X; T'] = (g_1, \dots, g_n)D[X; T']$. Then α is an ideal of $D[X; S]$ generated by n elements. Therefore $D[X; S]$ has n -generator property. Necessity. It follows that $D[X; G]$ has n -generator property. Therefore D is a field and G is isomorphic into Q by Proposition 1. We may assume that S is a subsemigroup of Q_0 . If our assertion is false, there exists a finitely generated subsemigroup T of S such that the order m of T is greater than n and such that each finitely generated subsemigroup of S containing T has order $\geq m$. We may assume $q(T) = Z$. There exists a natural number h such that $\{h, h+1, h+2, \dots\} \subset T$. By Lemma 4, $(X^h, X^{h+1}, \dots, X^{h+m-1})$ is not generated by n elements, which is a contradiction.

LEMMA 4. *Let k be a field and S be a subsemigroup of Q_0 . Assume S has a finitely generated subsemigroup T such that order of $T = m$, $q(T) = Z$ and each finitely generated subsemigroup of S containing which has order $\geq m$. Assume $\{h, h+1, h+2, \dots\} \subset T$.*

(1) *Let $k \in Z$, $1 \leq k \leq m-1$. Let $g_1, \dots, g_k \in (X^h, X^{h+1}, \dots, X^{h+m-1})$ such that $o(g_1) = h$, $o(g_2) = h+1, \dots$, $o(g_k) = h+k-1$, and let $\varphi_1, \dots, \varphi_k \in k[X; S]$. Then we have $o(g_1\varphi_1 + \dots + g_k\varphi_k) \notin \{h+k, h+k+1, \dots, h+m-1\}$.*

(2) *$(X^h, X^{h+1}, \dots, X^{h+m-1})$ is not generated by $m-1$ elements of $k[X; S]$.*

PROOF. We note that $1, 2, \dots, m-1 \in S$. (1). We show (1) by induction

on k . Suppose that (1) is true for $k-1$. Suppose $o(g_1\varphi_1 + \dots + g_k\varphi_k) = \omega \in \{h+k, h+k+1, \dots, h+m-1\}$. By induction we may assume that each φ_i is either 0 or $o(\varphi_i) > 0$. ω appears as one of the powers of $g_i\varphi_i$ for some i . Each power in g_i is of the form $h+p+s$ where $Z \ni p, 0 \leq p \leq m-1, s \in S$. We have $h+p+s+s' = \omega$ for $0 < s' \in S$. It follows that $S \ni s+s' > 0, Z \ni s+s' \leq m-1$, which is a contradiction. (2). Suppose that $(X^h, X^{h+1}, \dots, X^{h+m-1})$ is generated by n elements $f_1, \dots, f_n$, where $n < m$. We take the minimal number n so that $(X^h, \dots, X^{h+m-1})$ is generated by n elements. We set $o(f_i) = d_i, 1 \leq i \leq n$. We may assume that $h = d_1 < d_2 < \dots < d_n$ and the coefficient of X^{d_i} in f_i is 1 for $1 \leq i \leq n$. Suppose $h+1 < d_2$. Since $X^{h+1} \in (f_1, \dots, f_n)$, we have $o(f_1\varphi_1) = h+1$ for some $\varphi_1 \in k[X; S]$, which contradicts to (1). Hence $h < d_2 \leq h+1$. Suppose $h < d_2 < h+1$. Since $f_2 \in (X^h, X^{h+1}, \dots, X^{h+m-1})$, it follows $d_2 - h \in S$. Replace f_2 by $f_2 - f_1X^{d_2-h}$. We note that there exists a discrete subgroup of Q containing Z and all powers of $f_1, \dots, f_n$. Thus we may assume $d_2 = h+1$. Suppose $h+2 < d_3$. Since $X^{h+2} \in (f_1, \dots, f_n)$ we have $o(f_1\varphi_1 + f_2\varphi_2) = h+2$ for $\varphi_1, \varphi_2 \in k[X; S]$, which contradicts to (1). Hence $h+1 < d_2 \leq h+2$. Suppose $h+1 < d_2 < h+2$. Since $f_3 \in (X^h, X^{h+1}, \dots, X^{h+m-1})$, we have $d_3 - h$ or $d_3 - h - 1 \in S$. Replace f_3 by $f_3 - f_1X^{d_3-h}$ or $f_3 - f_2X^{d_3-h-1}$. Thus we may assume $d_3 = h+2$. We continue the process. Thus we may assume $o(f_i) = h+i-1$ for each i . Then $X^{h+m-1} \in (f_1, \dots, f_n)$ by (1); a contradiction.

LEMMA 5([6, Cor. 3.1]). *Let S be a group. Then $A[X; S]$ is a Bezout ring (i.e. a ring with 1-generator property) if and only if A is a von Neumann regular ring and S is isomorphic into Q .*

LEMMA 6. *Let A be a von Neumann regular ring. If S is a subsemigroup of Z_0 containing n , then $A[X; S]$ has n -generator property.*

PROOF. Let $\alpha = (f_1, \dots, f_{n+1})$ be an ideal of $A[X; S]$ generated by $n+1$ elements. Set $d_i = \deg f_i$ for $1 \leq i \leq n+1$. We show that α is generated by n elements by the induction on $\sum_{i=1}^{n+1} d_i$. There exist $k \neq l$ such that $d_l > d_k$ and $d_l \equiv d_k \pmod{n}$. Set $d_l - d_k = nq$ and $f_k = a_0 + \dots + tX^{d_k}$. There exists an idempotent $e \in A$ such that $tA = eA$. Set $1 - e = e'$, $eA = A_1$ and $e'A = A_2$. We have $A[X; S] = A_1[X; S] \oplus A_2[X; S]$ and $\alpha = (ef_1, \dots, ef_{n+1})A_1[X; S] \oplus (e'f_1, \dots, e'f_{n+1})A_2[X; S]$. Since $\deg e'f_k < d_k$, $(e'f_1, \dots, e'f_{n+1})A_2[X; S]$ is generated by n elements of $A_2[X; S]$. Since et is a unit of A_1 , there exists $b \in A_1$ such that $\deg(ef_i - bef_kX^{nq}) < d_i$. Since $(ef_1, \dots, ef_{n+1})A_1[X; S] = (ef_1, \dots, ef_k, \dots, ef_i - bef_kX^{nq}, \dots, ef_{n+1})A_1[X; S]$, $(ef_1, \dots, ef_{n+1})A_1[X; S]$ is generated by n elements of $A_1[X; S]$. Therefore α is generated by n elements.

THEOREM 7. *Assume that A has the nilradical (0). Then $A[X; S]$ has n -generator property if and only if A is a von Neumann regular ring and S is*

either isomorphic with a subgroup of Q or isomorphic with a subsemigroup T of Q_0 such that if T' is a finitely generated subsemigroup of T , T' is contained in a finitely generated subsemigroup T'' of T of order $\leq n$.

PROOF. The necessity. Let P be a prime ideal of A . $(A/P)[X; S]$ has n -generator property. By Theorem 3, A/P is a field and S is a desired semigroup. Since A has dimension 0 and nilradical (0), A is a von Neumann regular ring. The sufficiency. By Lemma 5, we may assume that S is a subsemigroup of Q_0 . Let $\alpha = (f_1, \dots, f_{n+1})$ be an ideal of $A[X; S]$ generated by $n+1$ elements. There exists a finitely generated subsemigroup T' of S of order $\leq n$ such that $f_i \in A[X; T']$ for $1 \leq i \leq n+1$. By Lemma 6, $(f_1, \dots, f_{n+1})A[X; T']$ is generated by n elements of $A[X; T']$. Therefore α is generated by n elements of $A[X; S]$.

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