On good $\lambda$-inequalities

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1. Since 1970 (Burkholder and Gundy [2]), the so-called "good $\lambda$-inequality" has been used very effectively to investigate the continuities of many sort of operators such as singular integrals, the martingale square function and the martingale maximal function, etc.

The "good $\lambda$-inequality" denotes the following sort of inequalities:

\[(1.1)\quad m(\{x; |f(x)| > a\lambda, |g(x)| \leq b\lambda\}) \leq cm(\{x; |f(x)| > \lambda\})\]

for $\lambda > 0$, where $f$ and $g$ are $m$-measurable functions on a measure space $(X, m)$, and where $a$, $b$ and $c$ are constants satisfying $a > 1$, $b > 0$ and $0 < c < 1$.

If $m(X)$ is finite, then, as is shown in Burkholder [1],

\[(1.2)\quad \|f\|_{L^p(X, m)} \leq \frac{a}{b(1 - a^pc)^{1/p}} \|g\|_{L^p(X, m)},\]

provided $0 < p < \infty$ and $a^pc < 1$.

However, if $m(X)$ is infinite, one can not deduce inequality (1.2) from (1.1) without additional assumption such as $f \in L^p(X, m)$, (cf. [6, p. 3]).

In many cases, for singular integrals $K$ one obtains

\[(1.3)\quad |\{x \in \mathbb{R}^n; |f(x)| > 2\lambda, |g(x)| \leq \gamma \lambda\}| \leq C\gamma^\phi |\{x \in \mathbb{R}^n; |f(x)| > \lambda\}|\]

for $\lambda > 0$ and $0 \leq \gamma < 1$. Here $|E|$ denotes the Lebesgue measure of the subset of the euclidean space $\mathbb{R}^n$, $f(x) = K^*h(x)$ and $K^*$ is the maximal truncated operator of $K$, and $g$ is an appropriate Hardy-Littlewood maximal function of a function $h$ (see [3], [4], [5], [8] and [9], etc).

We shall give, in the next section, some sufficient conditions for a good $\lambda$-inequality to imply a norm inequality. In Section 3, we shall give some counterexamples which show that even such a good inequality as (1.3) does not generally imply a norm inequality. And in the last section, we shall give an application of the results in Section 2.

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2. Positive results

We give in this section some sufficient conditions for a good $\lambda$ inequality to imply a norm inequality.

**Proposition 1.** Let $a > 1$, $b > 0$ and let $\phi$ be a nonzero nondecreasing continuous function on $[0, \infty)$ satisfying $\phi(0) = 0$ and

$$
(2.1) \quad \phi(a\lambda) \leq c\phi(\lambda) \quad \text{for all} \quad \lambda > 0.
$$

Let $f$ and $g$ be nonnegative measurable functions on a measure space $(X, \mu)$. Suppose $b$ is a positive number less than $1/c$ and

$$
(2.2) \quad \mu(\{x \in X \mid f(x) > a\lambda, \ g(x) \leq \lambda\}) \leq bm(\{x \in X \mid f(x) < \lambda\})
$$

for all $\lambda > 0$. Finally suppose that

$$
(2.3) \quad \mu(\{x \in X \mid f(x) > \lambda\}) = O(1/\psi(\lambda)) \quad \text{as} \quad \lambda \to 0
$$

for some positive function $\psi$ satisfying

$$
(2.4) \quad \psi(a\lambda) \leq d\psi(\lambda)
$$

with some $d < 1/b$. Then

$$
\int_X \phi(f(x))d\mu(x) \leq \frac{c}{1-bc} \int_X \phi(g(x))d\mu(x).
$$

**Corollary 1.** Let $a > 1$, $b < 1$ and

$$
(2.5) \quad \mu(\{x \mid |f(x)| > a\lambda, \ |g(x)| \leq \lambda\}) \leq bm(\{x \mid |f(x)| > \lambda\}), \quad \lambda > 0.
$$

Suppose moreover for some $\alpha > 0$ with $a^\alpha b < 1$

$$
(2.5) \quad \mu(\{x \mid |f(x)| > \lambda\}) = O(\lambda^{-\alpha}) \quad \text{as} \quad \lambda \to 0.
$$

Then, if $a^p b < 1$, we have

$$
\|f\|_{L^p(\mu)} \leq \frac{a}{(1-a^p b)^{1/p}} \|g\|_{L^p(\mu)}.
$$

**Corollary 2.** Let $a > 1$ and

$$
\mu(\{x \mid |f(x)| > a\lambda, \ |g(x)| < \gamma\lambda\}) \leq C(\gamma)m(\{x \mid |f(x)| > \lambda\}), \quad \lambda > 0,
$$

where $C(\gamma) \to 0$ as $\gamma \to 0$. Suppose moreover for some $\alpha > 0$ the inequality (2.5) holds. Then for $0 < p < \infty$

$$
\|f\|_{L^p(\mu)} \leq C_p \|g\|_{L^p(\mu)},
$$

where $C_p$ is the constant in Corollary 1 with $b = C(\gamma)$ for $\gamma$ satisfying $C(\gamma)a^\alpha < 1$ and $C(\gamma)a^p < 1$. 
PROPOSITION 2. Let $a$, $\delta$, $\gamma_0$ and $p$ be positive numbers and suppose $a > 1$. Suppose $f$ and $g$ are nonnegative measurable functions on a measure space $(X, m)$ satisfying

$$m(\{x \in X; f(x) > a\lambda, g(x) \leq \gamma \lambda\}) \leq \gamma^p m(\{x \in X; f(x) > \lambda\})$$

for all $\lambda > 0$ and $0 < \gamma \leq \gamma_0$. Finally suppose that

$$m(\{x \in X; f(x) > \lambda\}) = O(\exp \lambda^{-s}) \quad \text{as} \quad \lambda \longrightarrow 0$$

for some positive number $\alpha$ satisfying $1 + \delta p^{-1} > a^\alpha$. Then

$$\int_X f(x)^p dm(x) \leq A^p \int_X g(x)^p dm(x),$$

where $A$ is a constant depending only on $a$, $\delta$, $\gamma_0$ and $p$. If we fix $a$, $\delta$, and $\gamma_0$, then $A = O(p^{-1/\delta})$ as $p \longrightarrow 0$ and $A = O(a^p/\delta)$ as $p \longrightarrow \infty$.

REMARK 1. The condition (2.5) is satisfied if $\min (1, |f(x)|) \leq L_{\gamma_0}(m)$.

REMARK 2. A merit of Corollaries 1 and 2 is that it suffices to check (2.5) only for one appropriate $\alpha$. We illustrate it for an example in Coifman and Fefferman [4]. Consider a singular integral operator $T: f \rightarrow K*f$ in $R^n$, with a convolution kernel $K$ satisfying the conditions: (a) $\|K\|_\infty \leq C$, (b) $|K(x)| \leq C|x|^{-n}$, and (c) $|K(x) - K(x-y)| \leq C|y||x|^{-n-1}$ for $2|y| < |x|$. And let $w(x)$ be a Muckenhoupt's $A_\infty$ weight function. Then there are constants $b$, $\delta > 0$ so that given any cube $Q$ and any measurable subset $E \subseteq Q$, $w(E)/w(Q) \leq b (|E|/|Q|)^b$. Let $T^*f(x) = \sup \int_{|x-y| > b} K(x-y)f(x)dy$ and $f^*$ be the Hardy-Littlewood maximal function of $f$. Then they showed the following good $\lambda$ inequality.

$$w(\{T^*f > 2\lambda, f^* \leq \gamma \lambda\}) \leq C\gamma^p w(\{T^*f > \lambda\}), \quad \text{for} \quad \lambda > 0.$$ 

Now for any bounded function $f$ with compact support one has clearly $T^*f(x) = O(|x|^{-n})$ as $|x| \rightarrow +\infty$. By Theorem V in [4], $w \in A_r$ for some $r \geq 1$. Hence, from an easy variant of Lemma 1 in [7], $\int_{R^n} w(x)(1 + |x|)^{-nr}dx < +\infty$. Hence $w(\{T^*f > \lambda\}) = O(\lambda^{-r})$. Therefore by Corollary 2 we have $\|T^*f\|_{L_p(wdx)} \leq C_p\|f^*\|_{L_p(wdx)} (0 < p < \infty)$, as is stated in Theorem III in [4].

We shall proceed to the proof of our results. Corollaries 1 and 2 follow directly from Proposition 1. In order to prove Propositions 1 and 2, we need the following lemma.

LEMMA 1. Let $a$, $c$ and $\phi$ be the same as in Proposition 1. Suppose that $f$ and $g$ are nonnegative measurable functions on a measure space $(X, m)$ and that they satisfy (2.2) with a constant $b < 1/c$. Finally suppose

$$m(\{x \in X; f(x) > \lambda\}) = O(1/\phi(\lambda)) \quad \text{as} \quad \lambda \longrightarrow 0.$$
Then
\[ \int_X \phi(f(x))dm(x) \leq \frac{c}{1 - bc} \int_X \phi(g(x))dm(x). \]

**Proof.** We use the notation \( \sigma_f(\lambda) = m(\{x \in X ; f(x) > \lambda\}) \). Recall the identity
\[ \int_X \phi(f(x))dm(x) = \int_0^\infty \sigma_f(\lambda)d\phi(\lambda), \]
which can be shown by the use of Fubini's theorem.

We may assume \( \int_X \phi(g)dm < \infty \). From (2.2), we have
\[ \sigma_f(a\lambda) \leq \sigma_f(\lambda) + b\sigma_f(\lambda). \]
Integrating this inequality with respect to \( d\phi(\lambda) \) on the interval \( (\varepsilon, A) \), \( 0 < \varepsilon < A < \infty \), we obtain
\[ (2.9) \quad \int_\varepsilon^A \sigma_f(\lambda)d\phi(\lambda/a) \leq \int_\varepsilon^A \sigma_\phi(\lambda)d\phi(\lambda) + b\int_\varepsilon^A \sigma_f(\lambda)d\phi(\lambda). \]
By integration by parts, the left hand side of this inequality can be rewritten as
\[ \sigma_f(aA)\phi(A) - \sigma_f(a\varepsilon)\phi(\varepsilon) - \int_\varepsilon^A \phi(\lambda/a)d\sigma_f(\lambda). \]
Hence we use the inequalities \( \phi(A) \geq \phi(aA)/c \), \( \phi(\lambda/a) \geq \phi(\lambda)/c \) and integration by parts again to obtain
\[ (2.10) \quad \int_\varepsilon^A \sigma_f(\lambda)d\phi(\lambda/a) \geq -\sigma_f(a\varepsilon)\phi(\varepsilon) + \frac{1}{c} \phi(a\varepsilon)\sigma_f(a\varepsilon) + \frac{1}{c} \int_\varepsilon^A \sigma_f(\lambda)d\phi(\lambda). \]
From (2.9) and (2.10), we can deduce the following inequality:
\[ (2.11) \quad \left(\frac{1}{c} - b\right) \int_\varepsilon^A \sigma_f(\lambda)d\phi(\lambda) \leq \sigma_f(a\varepsilon)\phi(\varepsilon) + \frac{1}{c} \int_\varepsilon^A \sigma_f(\lambda)d\phi(\lambda) + \int_\varepsilon^A \sigma_\phi(\lambda)d\phi(\lambda). \]
By the assumptions (2.1) and (2.8), the first and the second terms on the right hand side of (2.11) are bounded as \( \varepsilon \) tends to zero. The last term on the right hand side of (2.11) is also bounded since it is majorized by \( \int_0^\infty \sigma_\phi(\lambda)d\phi(\lambda) = \int_X \phi(g)dm < \infty \). Hence, letting \( \varepsilon \to 0 \) and \( A \to \infty \) in (2.11), we see that \( \int_0^\infty \sigma_f(\lambda)d\phi(\lambda) < \infty \). Hence it turns out that the first and the second terms on the right hand side of (2.11) tends to zero as \( \varepsilon \) tends to zero. Thus, finally, we let \( \varepsilon \to 0 \) and \( A \to \infty \) in (2.11) to obtain
\[ \left(\frac{1}{c} - b\right) \int_0^\infty \sigma_f(\lambda)d\phi(\lambda) \leq \int_0^\infty \sigma_\phi(\lambda)d\phi(\lambda), \]
which is the desired inequality. This completes the proof of Lemma 1.
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**Proof of Proposition 1.** We can assume $\int_X \phi(g)dm = B < \infty$. We use the notation $\sigma_f(\lambda) = m(\{x \in X; f(x) > \lambda\})$. By Lemma 1, it is sufficient to prove that $\sigma_f(\lambda) = O(1/\phi(\lambda))$ as $\lambda \to 0$. The inequality (2.2) implies that

$$\sigma_f(\lambda) \leq \sigma_g(\lambda/a) + b \sigma_f(\lambda/a).$$

By induction, we have, for every positive integer $N$,

$$\sigma_f(\lambda) \leq \sum_{k=0}^{N-1} b^k \sigma_g(\lambda/a^{k+1}) + b^n \sigma_f(\lambda/a^n).$$

Since the function $\phi$ is nondecreasing, we have $\sigma_g(\lambda) \leq B/\phi(\lambda)$. Using this inequality and the inequalities (2.1), (2.3) and (2.4), we obtain

$$\sigma_f(\lambda) \leq \sum_{k=0}^{N-1} Bc(bc)^k/\phi(\lambda) + B'(bd)^n/\psi(\lambda),$$

where $B'$ is some constant. Since $bc < 1$ and $bd < 1$, we obtain the desired estimate by letting $N \to \infty$. This completes the proof of Proposition 1.

**Proof of Proposition 2.** We may assume $\int_X g^p dm < \infty$. We shall use the notation $\sigma_f(\lambda) = m(\{x \in X; f(x) > \lambda\})$. We shall prove that $\sigma_f(\lambda) = O(\lambda^{-p})$ as $\lambda \to 0$. Once this is proved, Lemma 1 shows that the conclusion holds with

$$A = \inf \{a^p (1 - \gamma^a)^{-1/p}; 0 < \gamma \leq \gamma_0, \gamma^a \alpha < 1\}.$$

Simple calculation, which will be omitted, gives the asymptotic behavior of $A$ as $p \to 0$ or $p \to \infty$.

By (2.6), we have

$$\sigma_f(\lambda) \leq \sigma_g(\lambda^a) + \gamma^a \sigma_f(\lambda^a).$$

Repeated use of this inequality gives

$$\sigma_f(\lambda) \leq \sum_{k=0}^{N-1} (\gamma_1 \cdots \gamma_k)^a \sigma_g(\lambda^a) + (\gamma_1 \cdots \gamma_k)^a \sigma_f(\lambda^a),$$

where $(\gamma_k)$ is any sequence with $0 < \gamma_k \leq \gamma_0$. By the estimate $\sigma_g(\lambda) = O(\lambda^{-p})$ (since $\int g^p dm < \infty$) and (2.7), we have

$$\sigma_f(\lambda) \leq C \sum_{k=0}^{N-1} (\gamma_1 \cdots \gamma_k)^a (\gamma_{k+1})^a \lambda^{-p} + C(\gamma_1 \cdots \gamma_N)^a \exp(a^{N^a} \lambda^{-s}).$$

Thus it is sufficient to show that we can take a sequence $(\gamma_k)$ so that we have

$$0 < \gamma_k \leq \gamma_0, \quad k = 1, 2, 3, \ldots,$$

$$\sum_{k=1}^{\infty} (\gamma_1 \cdots \gamma_k)^a (\gamma_{k+1})^p < \infty,$$

$$\lim_{N \to \infty} (\gamma_1 \cdots \gamma_N)^a \exp(a^{N^a} \lambda^{-s}) = 0 \quad \text{for all } \lambda > 0.$$
These conditions are satisfied if we take a sufficiently large number $B$ and define $\gamma_k$ by
\[
\log \gamma_k = -B \left( 1 + \frac{\delta}{p} \right)^{k-1} - \frac{p}{\delta} \log a - \frac{1}{\delta}.
\]
In fact, for this $(\gamma_k)$, the ratio of the $(k+1)$-th term to the $k$-th term in the series in (2.13) is $1/e$ and hence the series is convergent. The estimate (2.14) follows from
\[
\log \left[ (\gamma_1 \cdots \gamma_n)^{\lambda} \exp(a^{N\lambda - n}) \right]
= C_1 + C_2 N - pB \left( 1 + \frac{\delta}{p} \right)^{N} + a^{N\lambda - n} \rightarrow -\infty \quad \text{as} \quad N \rightarrow \infty
\]
(this is seen from $1 + \frac{\delta}{p} > a^\lambda$). Finally (2.12) is assured if we take $B$ sufficiently large. This completes the proof of Proposition 2.

3. Negative Results

In this section we shall give two counter examples. The first one will show that a good $\lambda$ inequality such as the one in Corollary 2, however strong it is, cannot imply any norm inequality without a priori estimate of the function $f$. The second counter example will show that Proposition 2 is sharp.

In order to construct the examples, we shall use the following lemma.

LEMMA 2. Let $(X, m)$ be a measure space and $C(\gamma)$ be a positive nondecreasing function defined for $0<\gamma<\infty$. Suppose $(\gamma_n)_{n \geq 1}$ is a sequence of positive numbers, $(E_n)_{n \geq 0}$ is a disjoint sequence of measurable sets of $X$ and $C(\gamma_n) m(E_n) \geq m(E_{n-1})$ for all $n \geq 1$. Take a number $a>1$ and define measurable functions $f$ and $g$ on $X$ as follows: $f=a^{-n}$ and $g=a^{-n-1}\gamma_{n+1}$ on $E_n$ ($n=0, 1, \ldots$) and $f=g=0$ outside $\bigcup E_n$. Then the following inequality holds for all $\gamma>0$ and all $\lambda>0$:

$$m(\{x \in X; f(x)>a\lambda, g(x)\leq \gamma \lambda\}) \leq C(\gamma)m(\{x \in X; f(x)>\lambda\}).$$

PROOF. Observe that the following inequality holds for all $n \geq 0$ and all $\gamma>0$:

$$m(E_n \cap \{g \leq a^{-n-1} \gamma\}) \leq C(\gamma)m(E_{n+1}).$$

If $a^{-N-1} \leq \lambda< a^{-N}$ with a positive integer $N$, then $\{f>a\lambda\} = \bigcup_{n=0}^{N-1} E_n \{f>\lambda\} = \bigcup_{n=0}^{N} E_n$ and $\{g \leq \gamma \lambda\} \subset \{g \leq a^{-n-1} \gamma\}$ for $n=0, 1, \ldots, N-1$, and hence

$$m(\{f>a\lambda, g \leq \gamma \lambda\}) \leq \sum_{n=0}^{N-1} m(E_n \cap \{g \leq a^{-n-1} \gamma\}) \leq \sum_{n=0}^{N-1} C(\gamma)m(E_{n+1}) \leq C(\gamma)m(\{f>\lambda\})$$
If $\lambda \geq a^{-1}$, then the set $\{f > a\lambda\}$ is empty and hence the inequality obviously holds. This completes the proof.

**Counter Example 1.** Suppose $a > 1$, $C(\gamma)$ is a positive nondecreasing function defined for $0 < \gamma < \infty$ such that $C(\gamma) \to 0$ as $\gamma \to 0$, $\phi(\lambda)$ is a nonnegative nondecreasing function defined for $0 \leq \lambda < \infty$ such that $\phi(\lambda) \to 0$ as $\lambda \to 0$, and $\theta(\lambda)$ is a positive nonincreasing function defined for $0 < \lambda < 1$ such that $\theta(\lambda) \to \infty$ as $\lambda \to 0$. Also suppose $(X, m)$ is a $\sigma$-finite measure space with $m(X) = \infty$. Then there exist nonnegative measurable functions $f$ and $g$ on $X$ with the following properties:

$$m({x \in X : f(x) > a\lambda, g(x) \leq \gamma \lambda}) \leq C(\gamma)m({x \in X : f(x) > \lambda})$$

for all $\gamma > 0$ and all $\lambda > 0$;

$$\int_X \phi(g(x))dm(x) < \infty;$$

$$\theta(\lambda) \leq m({x \in X : f(x) > \lambda}) < \infty \quad \text{for} \quad 0 < \lambda < 1.$$  

**Proof.** Choose, by induction, a sequence $(E_n)_{n \geq 0}$ of measurable subsets and a sequence $(\gamma_n)_{n \geq 1}$ of positive numbers as follows: choose $E_0$ with $\theta(a^{-1}) \leq m(E_0) < \infty$; if $E_0, \ldots, E_{n-1}$ and $\gamma_1, \ldots, \gamma_{n-1}$ are chosen, choose $\gamma_n$ such that $\phi(a^{-n} \gamma_n)m(E_{n-1}) \leq 2^{-n}$ and $m(E_{n-1})/C(\gamma_n) \geq \theta(a^{-n-1})$ and choose $E_n$ so that $E_n$ does not intersect with $E_j$ for $j \leq n-1$ and $m(E_{n-1})/C(\gamma_n) \leq m(E_n) < \infty$. Define functions $f$ and $g$ in the same way as in Lemma 2. Then they have the desired properties.

**Counter Example 2.** Suppose $a > 1$, $\delta > 0$, $p > 0$, $\alpha > 0$ and $1 + \frac{\delta}{p} \leq a^{\alpha}$. Also suppose $(X, m)$ is a non atomic measure space with $m(X) = \infty$. Then there exist nonnegative measurable functions $f$ and $g$ on $X$ with the following properties:

$$m({x \in X : f(x) > a\lambda, g(x) \leq \gamma \lambda}) \leq \gamma^p m({x \in X : f(x) > \lambda})$$

for all $\gamma > 0$ and all $\lambda > 0$;

$$\int_X g(x)^pdm(x) < \infty;$$

$$m({x \in X : f(x) > \lambda}) = O(\exp \lambda^{-\alpha}) \quad \text{as} \quad \lambda \to 0;$$

$$\int_X f(x)^pdm(x) = \infty.$$  

**Proof.** Choose a disjoint sequence $(E_n)_{n \geq 0}$ of measurable subsets of $X$ with $m(E_n) = \exp (a^{\alpha n})$ and define $\gamma_n$ ($n = 1, 2, \ldots$) by $\gamma_n^p m(E_n) = m(E_{n-1})$. Define functions $f$ and $g$ in the same way as in Lemma 2. Then they have the desired properties. In fact, the first property follows from Lemma 2. As for the second property, observe that
the ratio of the \((n+1)\)-th term to the \(n\)-th term in the above series is
\[
a^{-p} \exp \left[ \frac{\rho}{\delta} \left( 1 + \frac{\delta}{\rho} - a^2 \right) (a^a - 1) a^{a^a} \right] \leq a^{-p} < 1
\]
and hence the series is convergent and \(\int_X g^a dm < \infty\). The third and the fourth properties can be checked easily. This completes the proof.

REMARK. The second counter-example can be constructed on some atomic measure spaces, but not on all infinite \(\sigma\)-finite measure spaces. In fact, on some pathological atomic measure space, we can improve the results in Section 2. We give an example of such a space. Suppose \(X\) is the set of positive integers and \(m\) is a measure on \(X\) such that \(m(\{n\}) \to \infty\) as \(n \to \infty\) and
\[
(3.1) \quad (\log \log m(\{n+1\}))/\exp m(\{n\}) \to \infty \quad \text{as} \quad n \to \infty.
\]
For this measure space \((X, m)\), Corollary 1 holds with the assumption (2.5) replaced by
\[
(3.2) \quad m(\{x \in X; |f(x)| > a\}) = O(\exp \exp a) \quad \text{as} \quad a \to 0.
\]

PROOF. We may assume that the functions \(f\) and \(g\) are nonnegative. It is sufficient to show that, if \(f\) satisfies (3.2) and (2.2) with \(g \in L^p(X, m)\), then \(m(\{f > \lambda\}) = O(\exp \lambda^{-1})\) as \(\lambda \to 0\).

Using (3.1) and (3.2), we see that \(f(k) < f(n)/2a\) for all \(k > n\). Hence (2.2) implies that
\[
(3.3) \quad m(\{n\} \cap \{x \in X; g(x) \leq f(n)/2a\}) \leq bm(\{1, \ldots, n\}).
\]
From (3.1), it follows that \(m(\{1, \ldots, j\})/m(\{j\}) \to 1\) as \(j \to \infty\) and hence \(bm(\{1, \ldots, n\}) < m(\{n\})\) since \(b < 1\). Thus (3.3) implies \(g(n) > f(n)/2a\) and hence \(g(n)^p m(\{n\}) > (2a)^{-p}\). But this inequality can hold for only a finite number of \(n\)'s since \(g \in L^p(X, m)\). Thus \(m(\{n\}) \leq f(n)^{-p}\) except for a finite number of \(n\)'s. This, together with the fact that \(m(\{1, \ldots, j\})/m(\{j\}) \to 1\), implies that \(m(\{f > \lambda\}) = O(\lambda^{-p})\) as \(\lambda \to 0\). This completes the proof.

4. Applications

One more application of Corollary 2 is an improvement of a theorem of Journé [6, p. 41].

PROPOSITION 3. Let \(w \in A_\infty\) (Muckenhoupt's), \(0 < p < \infty\) and \(f \in L_{loc}(R^n)\). If \(w(\{f^* > \lambda\}) = O(\lambda^{-p})\) as \(\lambda \to 0\) for some \(\alpha \geq 0\), then

\[
\int_X g(x)^a dm(x) = \sum_{n=0}^{\infty} (a^{-a-1}g_{n+1})^a \exp (a^{a^a});
\]
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In the above, $f^*$ is the Hardy-Littlewood maximal function and $f^*$ is the $\#$-function, introduced by Fefferman and Stein, i.e., $f^*(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q| dy; Q: \text{cube and } f_Q = |Q|^{-1} \int_Q f(x) dx$.

**PROOF.** The proof is analogous to the one in Journe [6, pp. 41-42]. Note that in order to prove the good $\lambda$ inequality one has only to use the assumption $w\{f^* > \lambda\} < \infty$ for each $\lambda > 0$. In order to deduce the $L^p$ inequality from the good $\lambda$ inequality, we use our Corollary 2 instead of the corresponding lemma in Journe [6, p. 3].

Using the above proposition, we have:

**PROPOSITION 4.** Let $w \in A_\infty$ and $1 \leq p < \infty$. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f^* \in L^p(wdx)$. Then there exists a complex number $a$ such that

$$\int_{\mathbb{R}^n} (f^* - a)^p w(x) dx \leq C(p, w) \int_{\mathbb{R}^n} (f^*)^p w(x) dx. \tag{4.1}$$

**PROOF.** When $w(x) \equiv 1$ and $1 < p < \infty$, this is known as the one part of the Fefferman-Stein-Strömberg theorem, [10] or [11]. One can prove this by modifying the usual one a little. In fact, as is known, $w \in A_q$ for some $1 \leq q < \infty$. Hence, if $f \in L^p(wdx) \cap L^q(wdx)$, $f^* \in \text{weak}-L^q(wdx)$ and so $w\{\{f^* > \lambda\}\} = O(\lambda^{-q})$. Hence applying the above proposition, we have the desired inequality for $f \in L^p(wdx) \cap L^q(wdx)$. Therefore, we have only to find a number $a$ and an appropriate limit process. A way is to modify the proof by Uchiyama [11]: let $t_k = 2^k/3$ for even $k$ and $t_k = 2^{k+1}/3$ for odd $k$ and consider the translated dyadic cubes

$$I = \prod_{j=1}^n (t_k + l_j 2^k, t_k + (l_j + 1) 2^k),$$

$l_j \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then these translated dyadic cubes have the same properties as the usual dyadic cubes: If $I_1 \cap I_2 \neq \emptyset$, then either $I_1 \subset I_2$ or $I_2 \subset I_1$. An important property for these translated dyadic cubes is the following: there exists a sequence $(J_j)$ such that $|J_j| = 2^{jn}$, $J_1 \subset J_2 \subset \cdots$ and $\bigcup_{j=1}^\infty J_j = \mathbb{R}^n$. Now $f^*_d$ will denote the translated dyadic $\#$-function of a function $f$. This means that in the definition of the $\#$-function cubes move only over the translated dyadic cubes. Then Proposition 3 also holds for this $\#$-function and the translated dyadic maximal function $f^*_d$.

We have
where $C$, $C'$ and $C(w)$ are positive constants, and $C(w) > 1$ (see [6, p. 41]). Hence $(f_j)$ tends to a number $a$ as $j \to \infty$. Let $h(x) = (f - f_j)x_j$. Then one can easily see that $h_\#_p(x) \leq 2 f_\#_p(x)$. Let $h_k(x) = h(x)$ for $|h(x)| \leq k$ and $h_k(x) = k \text{sgn } h(x)$ for $|h(x)| > k$. Then $h_k(x) \in L^p(wdx)$ and one can see that $(h_k)_\#_p(x) \leq 2h_\#_p(x)$. Hence, since $f_\#_p \in L^p(wdx)$ by assumption, we get $h_\#_p \in L^p(wdx)$ and hence $(h_k)_\#_p \in L^p(wdx)$. Thus, since $h_k \in L^p(wdx) \cap L^q(wdx)$, we have by Proposition 3

$$\|h_k\|_{L^p(wdx)} \leq C f_\#_p \|_{L^p(wdx)}.$$

Since $(h_k)_\#_p$ tends to $h_\#_p$ monotonically, we have by the monotone convergence theorem

$$\lim_{k \to \infty} \|h_k\|_{L^p(wdx)} = \|h_\#_p\|_{L^p(wdx)} \leq C f_\#_p \|_{L^p(wdx)}.$$

Finally, since one can easily see that $h_\#_p$ tends to $(f-a)_\#_p$ pointwise, by Fatou's lemma we have

$$\langle (f-a)_\#_p \rangle \leq \liminf_{j \to \infty} \int h_\#_p w(x) dx \leq C \int \langle f_\#_p \rangle w(x) dx.$$

From (4.2) we obtain the desired inequality (4.1) in the same way as in Fefferman-Stein [12, p. 112]. In fact, for $h \in \mathbb{R}^n$ and $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define

$$g_{*,i}(x) = \sup_{x+I \cap h} |I|^{-1} \int_{x+I \cap h} |g(y)| dy,$$

where $I$'s move only over the translated dyadic cubes containing $x-h$. Then, by making an elementary observation we have for any integer $k$

$$2^{-3n} \sup_{l(Q) \leq 2^k} \|Q\|^{-1} \int_{Q} |g(y)| dy \leq 2^{-n(k+1)} \int_{|h| < 2^k} g_{*,i}(x) dh,$$

where $l(Q)$ is the side length of the cube $Q$. Combining this with (4.2) and letting $k \to \infty$, we obtain (4.1). This completes the proof of Proposition 4.

As a direct consequence we have the following:

**Corollary 3.** Let $w \in A_\infty$ and $1 \leq p < \infty$. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w(\{|f| > \lambda\}) < \infty$ for each $\lambda > 0$. Then
On good \( \lambda \)-inequalities

\[
\int_{\mathbb{R}^n} (f^*)^p w(x) dx \leq C(p, w) \int_{\mathbb{R}^n} (f^*)^p w(x) dx.
\]

**Proof.** If the right hand side of the above inequality is finite, then by Proposition 4 there exists a number \( a \) with \( f - a \in L^p(w dx) \). However, the assumption \( w(\{|f| > \lambda\}) < \infty \) for each \( \lambda > 0 \) implies that the number \( a \) must be zero. q.e.d.

**Remark.** If \( \inf (1, |f|) \in L^\alpha(w dx) \) for some \( \alpha > 0 \), then the above condition is satisfied.

**References**