$\phi: \text{Pic}(R_0) \rightarrow \text{HPic}(R)$ is not necessarily surjective

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Let R be a commutative integral domain with quotient field K . We set $I^{-1} = \{x \in K; xI \subset R\}$ for a fractional ideal I of R . If $II^{-1} = R$, we say that I is an invertible ideal of R . The set $I(R)$ of invertible ideals of R forms a group under the usual product of ideals. The set $F(R)$ of nonzero principal fractional ideals of R is a subgroup of $I(R)$. We denote $I(R)/F(R)$ by $\text{Pic}(R)$, and call it the Picard group of R .

By a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, we mean an integral domain R graded by a torsionless grading monoid Γ . That is, Γ is a commutative cancellative monoid, written additively, and the quotient group $\langle \Gamma \rangle$ generated by Γ is a torsion-free abelian group. Let I be a fractional ideal of R . If sI is an integral homogeneous invertible ideal of R for some nonzero homogeneous element s of R , we say that I is a homogeneous invertible ideal of R . The set $HI(R)$ of homogeneous invertible ideals of R is a subgroup of $I(R)$. We set $HI(R) \cap F(R) = HF(R)$. We denote $HI(R)/HF(R)$ by $\text{HPic}(R)$, and call it the homogeneous Picard group of R . In the natural way, $\text{HPic}(R)$ is a subgroup of $\text{Pic}(R)$.

Anderson-Anderson studied divisorial ideals and invertible ideals of a graded integral domain in [1], and proved the following theorem among other things:

THEOREM ([1, Th. 5.4 and Th. 5.5]). *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with $R_0 \subset R$ an inert extension. Then the natural homomorphism $\phi: \text{Pic}(R_0) \rightarrow \text{HPic}(R)$ is a bijection. If the extension $R_0 \subset R$ is inert and if R is almost normal, then the natural homomorphism $\text{HPic}(R) \rightarrow \text{Pic}(R)$ is a bijection.*

They ask if the natural homomorphism $\phi: \text{Pic}(R_0) \rightarrow \text{HPic}(R)$ is necessarily surjective when $R_0 \subset R$ is not inert. In this paper we show that ϕ is not necessarily surjective when $R_0 \subset R$ is not inert.

Let D be an integral domain. We consider the polynomial ring $D[A, B, C]$ of three indeterminates A, B, C over D . We set $D[A, B, C]/(A^2 - A + BC) = R$. Let σ be the natural homomorphism of $D[A, B, C]$ to R . We set $\sigma(D) = E$. E is an integral domain isomorphic to D .

1) R is an integral domain, and E is a subdomain of R .

Set $\sigma(A) = a$, $\sigma(B) = b$ and $\sigma(C) = c$.

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2) R is an integral domain generated by a, b, c over E . That is $R = E[a, b, c]$.

We have $bc = a - a^2$. We define the R_d for integers d as follows:

$$R_d = \begin{cases} E[a]b^d & (\text{when } d > 0), \\ E[a] & (\text{when } d = 0), \\ E[a]c^{-d} & (\text{when } d < 0). \end{cases}$$

Each R_d is an additive subgroup of R .

3) We have $R_d R_e \subset R_{d+e}$ for $d, e \in \mathbb{Z}$.

PROOF. Suppose that $0 < e < -d$ for instance. Let $r \in R_d$ and $t \in R_e$. We have $r = fc^{-d}$ and $t = gb^e$ for $f, g \in E[a]$. Then $rt = fg \cdot (bc)^e c^{-d-e} = fg \cdot (a - a^2) \cdot c^{-d-e} \in R_{d+e}$. It follows $R_d R_e \subset R_{d+e}$. The other cases are similar.

Let X be an another indeterminate.

4) The sum $\sum_{d \in \mathbb{Z}} R_d$ of the additive subgroups R_d of R is a direct sum.

PROOF. Let $r_1 + r_2 + \cdots + r_n = 0$ for $r_i \in R_{d_i}$ ($i = 1, 2, \dots, n$), where $d_1 < d_2 < \cdots < d_n$. We may assume $d_1 > 0$ by 3). We have $r_i = f_i(a)b^{d_i}$ for $f_i(X) \in E[X]$ ($i = 1, 2, \dots, n$). It follows

$$f_1(A)B^{d_1} + f_2(A)B^{d_2} + \cdots + f_n(A)B^{d_n} = (A^2 - A + BC)f(A, B, C)$$

for some $f(A, B, C) \in D[A, B, C]$. Comparing degrees of C of both sides, we see that $f_1(A) = f_2(A) = \cdots = f_n(A) = 0$, and hence $r_1 = r_2 = \cdots = r_n = 0$.

Therefore $R = \bigoplus_{d \in \mathbb{Z}} R_d$ is a $\mathbb{Z}$ -graded integral domain. We denote the ideal (a, b) of R by I .

5) I is a homogeneous invertible ideal of R .

PROOF. Since I is generated by homogeneous elements a, b of R , it is a homogeneous ideal of R . We set $(1, c/a) = J$. We have $a \cdot 1 + b \cdot (c/a) = a + (1 - a) = 1$. It follows that $IJ = R$, and I is a homogeneous invertible ideal of R .

We have $I^{-1} = J$. We note that a is algebraically independent over E .

6) I is not a principal ideal of R .

PROOF. Suppose that I is a principal ideal of R . Then there exists a nonzero homogeneous element s of R such that $I = (s)$. We set $\deg(s) = d$. Suppose first that $d = 0$. We have $s \in R_0$. Since $b \in (s)$, it follows $b = sfb$ for some $f \in R_0$, hence $1 = sf$. Therefore $I = R$. We have $1 = ag + bhc$ for $g, h \in R_0$. It follows $1 = a(g + h(1 - a))$; which is a contradiction. The other cases are similar.

PROPOSITION. *The natural homomorphism $\phi: \text{Pic}(R_0) \rightarrow \text{HPic}(R)$ is not surjective.*

PROOF. Suppose that ϕ is surjective. There exists an integral invertible ideal I_0 of R_0 such that $\phi([I_0]) = [I]$. We have $I = I_0 \xi R$ for some $\xi \in K$ (K is the quotient field of R). Since I_0 is invertible, it is finitely generated: $I_0 = (f_1, f_2, \dots, f_n)R_0$. Since $I_0 \xi \subset R$, ξ is of the form r/f for $r \in R$ and $f \in R_0$. Since $a \in I_0 \xi R$, r is a homogeneous element of R . We set $\deg(r) = d$. At first, suppose $d > 0$. We have $r = gb^d$ for $g \in R_0$. $\xi = (g/f)b^d$. Since $b \in I_0 \xi R$, we have $b = \sum_i f_i \xi g_i c^{d-1}$ for $g_i \in R_0$ ($i = 1, 2, \dots, n$). $b = \sum_i f_i g g_i (a - a^2)^{d-1} b/f$. It follows $f = \sum_i f_i g g_i (a - a^2)^{d-1} \in (f_1 g, \dots, f_n g)R_0$. Since $f_i \xi \in I$, we have $f_i \xi = ah_i b^d + bk_i b^{d-1}$ for $h_i, k_i \in R_0$ ($i = 1, 2, \dots, n$). It follows $f_i g = f.(ah_i + k_i) \in fR_0$. Therefore $I_0 g = (f_1 g, \dots, f_n g)R_0 = fR_0$, and hence $I = b^d R$; which is a contradiction to 6). Next, suppose $d = 0$. Then $r \in R_0$. Since $b \in I_0 \xi R$, we have $b = \sum_i f_i \xi g_i b$ for $g_i \in R_0$ ($i = 1, 2, \dots, n$). It follows $f = \sum_i f_i r g_i \in (f_1 r, \dots, f_n r)R_0$. Since $f_i \xi \in I$, we have $f_i \xi = ah_i + bk_i c$ for $h_i, k_i \in R_0$ ($i = 1, 2, \dots, n$). It follows $f_i r = f.(ah_i + k_i(a - a^2)) \in fR_0$. Therefore $I_0 r = (f_1 r, \dots, f_n r)R_0 = fR_0$, and hence $I = R$; which is a contradiction to 6). The case of $d < 0$ is similar.

References

- [1] D. D. Anderson and D. F. Anderson, Divisorial ideals and invertible ideals in a graded integral domain, *Journ. Alg.* **76** (1982), 549–569.