Construction of Fundamental Solutions for Certain Degenerated Elliptic Operators

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§ 0. Introduction

The purpose of this memoire is to construct fundamental solutions for certain degenerated elliptic operators $A$ defined in a domain $\Omega$ of $\mathbb{R}^n$. Operators $A$ that we treat are of the same type as ones in Baouendi-Goulaouic [1] and in Goulaouic-Shimakura [2]. In other words, they have, roughly speaking, the following properties (see the hypotheses $[H-1] \sim [H-5]$ in §2):

(a) They are second-order linear differential operators with smooth coefficients.
(b) Elliptic in the interior of the domain.
(c) Degenerated in all directions at each point of the boundary surface (supposed to be smooth).

They are approximated, near the boundary, by the following simple operator $L_\alpha$ in the half-space $\mathbb{R}^{n+}$:

\begin{equation}
L_\alpha = -x_\alpha A + \alpha \partial_\alpha,
\end{equation}

where $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$ and $\alpha$ is a complex parameter with negative real part. We impose this hypothesis on $\alpha$ only for the technical reason. We note that with somewhat more elaboration we can develop the same argument, even if $\text{Re} \, \alpha > 0$ (see $[H-5]$ and its remark in §2).

In the interior of the domain $\Omega$, it is not difficult to construct fundamental solutions for $A$ by the aid of pseudo-differential operators, or more classically, by making use of a so-called geodesic distance decided by the principal part of $A$ (cf. Riesz [8]). Therefore our interest in this memoire is to explain how to construct a fundamental solution for $A$ near the boundary, where $A$ is degenerated. There are many authors who have studied various types of degenerated elliptic operators (see Oleinic-Radkevic [4], Shimakura [5], Graham [6] and Horiuchi [7]).

Our main plan of this paper is as follows. In §1, we explain our main idea in connection with two simple examples defined in $\mathbb{R}^n$ and in the ball $B \subset \mathbb{R}^n$. In §2, we describe the general setting of our operators $A$ on the domain $\Omega$, and state the main results. §§§3, 4 and 5 are devoted to the...
proofs of the above results. In §6 (Appendix), we establish Proposition 1 stated in §1.

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§1. Preliminaries

As C. Goulaouic and N. Shimakura [2] pointed out, $L_\sigma$ defined by (0.1) has the following fundamental solution $\sigma_\sigma(x, y)$ in $\mathbb{R}_\sigma^* = \{(x', y_n) | y_n > 0, x' \in \mathbb{R}^{n-1}\}$.

(1.1) \[ \sigma_\sigma(x, y) = \gamma(\alpha) y_n^{-a-1} (\tilde{\sigma} - 1) F(I, J, K, 1 - \Gamma/\tilde{\Gamma}) \]

\[ = \tilde{\gamma}(\alpha) y_n^{-a-1} \int_0^1 \{ (1 - \theta) + \tilde{\gamma}(\theta) \}^{-1} \{(1 - \theta)\}^{-1 - a/2} d\theta \]

where

\[ \Gamma = |x - y|^2, \quad \tilde{\Gamma} = |x' - y'|^2 + (x_n + y_n)^2, \]

\[ \{I, J, K\} = \{(n - 2 - a)/2, -a/2, -a\}, \]

\[ \gamma(\alpha) = 2^{-a-2} \pi^{-n/2} \Gamma(I) \Gamma(J)/\Gamma(K), \]

\[ \tilde{\gamma}(\alpha) = \gamma(\alpha) \Gamma(K)/\Gamma(J)^2 \]

and $F(I, J, K, \omega)$ is a hypergeometric function defined by

(1.2) \[ F(I, J, K, \omega) = \frac{\Gamma(K)}{\Gamma(I) \Gamma(J)} \sum_{p=0}^{\infty} \frac{\Gamma(I+p) \Gamma(J+p)}{\Gamma(K+p)(p!)} \omega^p. \]

Here, one should keep in mind that $\Gamma$ and $\tilde{\Gamma}$ are linked by the equality $\tilde{\Gamma} = \Gamma + 4x_n y_n$, and $\tilde{\Gamma}^{1/2}$ is a distance between $x$ and $y^* = (y', -y_n)$ (the symmetric point of $y$ with respect to the hyperplane $x_n = 0$). Since our operators $A$ are approximated, near the boundary, by $L_\sigma$, it may be natural to conceive that there exists a fundamental solution for $A$, near the boundary, of the form $\phi F$, where $\phi$ is a suitable function like $y_n^{a-1} \tilde{\Gamma}^{-1}$ and $F$ is a hypergeometric function. To confirm this conception, we consider the ball case.

Let us set, for $x \in B_\tau = \{x \in \mathbb{R}^n, |x| < \tau\}$,

(1.3) \[ \phi(x) = (\tau^2 - |x|^2)/(2\tau), \]

\[ L_b^a = -\phi(x) A + a(\sigma \phi, \sigma \cdot) + a(\alpha + 2 - n)/(2\tau), \]

\[ \sigma_\sigma(x, y) = \gamma(\alpha) \phi(y)^{-a-1} (\tilde{\sigma} - 1) F(I, J, K, 1 - \Gamma/\tilde{\Gamma}_b). \]

Where

\[ \Gamma = |x - y|^2, \quad \tilde{\Gamma}_b = \Gamma + 4\phi(x) \phi(y), \quad y \in B_\tau \ (x \equiv y), \]

$\gamma(\alpha)$ and $\{I, J, K\}$ are identical to the ones in (1.1), and $\alpha$ is a complex parameter with negative real part.
Then we obtain the following proposition, which will be established in §6 (Appendix).

\textbf{PROPOSITION 1.} \( \sigma_b^k \) is one of fundamental solutions for \( L_b^k \) above, more precisely;

1. If \( f \in C^0(\overline{B_r}) \), then \( \sigma_b^k f(x) \) is a solution of \( L_b^k u = f \).
2. If \( u \in C^2(\overline{B_r}) \), then we have \( \sigma_b^k (L_b^k u)(x) = u(x) \) in \( B_r \).

Here, note that \( \Gamma \) and \( \tilde{\Gamma}_b \) are linked by the equality \( \tilde{\Gamma}_b = \Gamma + 4 \phi(x) \phi(y) \) again, moreover \( \tilde{r}^{1/2} \) is almost equivalent to the distance between \( x \) and \( y^* = (r^2/|y|^2)y \) near the boundary, where \( y^* \) is the symmetric point of \( y \) with respect to the ball \( B_r \).

§ 2. Hypotheses and principal results

Let us return to the general case in which \( \Omega \) is a domain of \( \mathbb{R}^n \) and \( A \) is described as below.

Let \( \Omega \) be a domain of \( \mathbb{R}^n \) and \( \phi \) be a given non-negative smooth function defined on \( \overline{\Omega} \) and equivalent to a distance to the boundary (supposed to be smooth), that is to say;

\begin{equation}
\Omega = \{ x \in \mathbb{R}^n ; \phi(x) > 0 \} , \quad \partial \Omega = \{ x \in \mathbb{R}^n ; \phi(x) = 0 \} \quad \text{and} \quad d\phi \approx 0 \quad \text{on} \quad \partial \Omega \quad \text{(see also \([H-4]\))}.
\end{equation}

We consider a class of differential operators on \( \Omega \) of the form

\begin{equation}
A = -\phi(x) \sum_{j,k=1} a^{jk}(x) \partial_j \partial_k + \sum_{j=1} b^j(x) \partial_j + c(x) ,
\end{equation}

satisfying the following five hypotheses.

[H-1] \textbf{(Regularity of the coefficients)}

\( a^{jk} \) and \( b^j \) are smooth functions. Moreover we suppose that all \( a^{jk} \)'s are real valued and \( a^{jk} = a^{kj} \).

[H-2] \textbf{(Ellipticity in the interior)}

For any compact set \( W \subset \overline{\Omega} \), there is a constant \( C = C(W) > 0 \) such that, for any \( x \in W \) and \( \xi \in \mathbb{R}^n \), we have

\[ \sum_{j,k=1} a^{jk}(x) \xi_j \xi_k \geq C(W)|\xi|^2. \]

We assume without loss of generality

[H-3] \textbf{(A supplementary assumption)}
The behavior near the boundary of the solution of \( Au = f \) depends essentially on the values of the following function \( \alpha(x) \).

(2.3) \[
\alpha(x) = \sum_{j=1}^{n} b^j(x) \partial_j \phi, \quad \text{on } \partial \Omega.
\]

So we assume

\[ H-4 \]  
(Conditions on first order terms)

\[
b^j(x) - \alpha(x) \sum_{k=1}^{n} a^{jk}(x) \partial_k \phi = 0(\phi), \quad j = 1, \ldots, n.
\]

*Remark 2.1.* This is one of the sufficient conditions to reduce \( A \) to the model operator \( L_\phi \) (see (0.1)).

Finally we add

\[ H-5 \]  
(The entrance property of the boundary)

\[
\text{Re } \alpha(x) < 0, \quad \text{on } \partial \Omega.
\]

*Remark 2.2.* \[ H-2 \] implies that \( A \) has a Fuchsian principal part transversal to \( \partial \Omega \) and \( \alpha(x) + 1 \) is a characteristic root of this principal part, the other root being 0. Therefore \[ H-5 \] means that the real parts of these roots at every point \( x \in \partial \Omega \) are smaller than 1.

To treat our operator, we need more notations. We denote by \( \{a_{jk}(x)\} \) the inverse matrix of \( \{a^{jk}(x)\} \), and set

\[
g(x) = \det \{a_{jk}(x)\} \quad \text{and} \quad ds^2 = \sum_{j,k=1}^{n} a_{jk}(x) dx_j dx_k.
\]

Then by virtue of \[ H-2 \], we can regard the whole domain \( \Omega \) as a Riemannian manifold with a Riemannian metric induced by the metric tensor \( \{a_{jk}(x)\} \) and with a volume element \( dV = g(x)^{1/2} dx \). We also denote by \( r(x, y) \) the geodesic distance between \( x \) and \( y \) (supposing sufficiently close with each other).

Now we define \( \Gamma \) and \( \tilde{\Gamma} \) corresponding to the ones in (1.1) and (1.4).

(2.4) \[
\Gamma(x, y) = r(x, y)^2 \quad \text{and} \quad \tilde{\Gamma}(x, y) = \Gamma(x, y) + 4\phi(x)\phi(y).
\]

From this definition (2.4), we have obviously

(2.5) \[
a F \Gamma F \Gamma = 2n + O(r) \quad \text{and} \quad (a F \Gamma, F \Gamma) = 4 \Gamma.
\]

*Remark 2.3.* For the sake of simplicity, we denote \( A \) by

\[-\phi a F \Gamma + b F + c, \quad \text{or equivalently} \quad -\tilde{a} F \Gamma + b F + c.\]
We also note that $(\ ,\ )$ stands for the inner product.

Under these preparations, we now give a local fundamental solution for $A$ in the shape of $\Phi F + R$, where $\Phi F$ is described before and $R$ is a so-called compensating function. We fix an arbitrary point $x_0 \in \partial \Omega$, and fix a sufficiently small neighborhood $Q$ of $x_0$. Let us set, for any $(x, y) \in Q^o \times Q^o$ with $x \neq y$, and $Q^o = Q \cdot \partial Q$,

\begin{equation}
E_a(x, y) = \gamma(\beta)\phi(y)^{-\theta-1}\tilde{F}(I, J, K, 1 - \Gamma/\tilde{\Gamma}),
\end{equation}

where

\begin{equation}
\beta = \alpha(y), \quad \{I, J, K\} = \left\{ \frac{n - 2 - \alpha(x)}{2}, -\alpha(x)/2, -\alpha(x) \right\},
\end{equation}

$\Gamma$ and $\tilde{\Gamma}$ are defined by (2.4) and $\gamma(\cdot)$ is defined by (1.1).

Then, $E_a(x, y)$ has the following estimates.

**Theorem 1.** Under the above notations, we have

\begin{align}
|E_a(x, y)| &\leq C(Q)\phi(y)^{-\beta-1}\tilde{F}^{x/2}\log (1 + \Gamma/\tilde{\Gamma}), \quad \text{if} \quad n = 2; \\
|\partial^v_x E_a(x, y)| &\leq C(Q, v)\phi(y)^{-\beta-1}\tilde{F}^{x/2}\Gamma^{(2 - \alpha)/2} \\
& \times \{\Gamma^{-1/2} + |\log \Gamma| + |\log \phi(x)|\}^{v}, \quad \text{otherwise}.
\end{align}

Where $\alpha' = \Re \alpha(x)$, $\beta' = \Re \beta = \Re \alpha(y)$, $v$ is an arbitrary multi-index and $C(Q)$ (resp. $C(Q, v)$) is a positive constant depending only on $Q$ (resp. $Q$ and $v$).

As the special case of this theorem, we have immediately the following.

**Corollary 1.** In the case $\alpha' = \Re \alpha(x) \leq -1$ in $Q$, we have a simpler estimate as follows. We fix an arbitrary $\varepsilon_0 > 0$, then

\begin{equation}
|\partial^v_x E_a(x, y)| \leq C(Q, v)\Gamma^{(1 - n - \alpha - |v|)/2}
\end{equation}

holds for any $\varepsilon \geq \varepsilon_0$.

Moreover $E_a$ has the following properties with respect to the Riemannian metric.

**Theorem 2.** Suppose that $A$ satisfies the hypotheses $[H-1] \sim [H-5]$, and fix an arbitrary point $x_0 \in \partial \Omega$. Then, for a sufficiently small neighborhood $Q$ of $x_0$ in $\Omega$, $E_a$ is a local parametrix for $A$, more precisely, it holds that

\begin{equation}
AE_a(x, y) = \delta(x - y) + h(x, y),
\end{equation}

where $h(x, y)$ is a $C^\infty$ function defined for $(x, y) \in Q^o \times Q^o$ with $x \neq y$ satisfying the estimates

\begin{equation}
|h(x, y)| \leq C(Q)E_a(1 + \phi(x)/r) (1 + |\log \tilde{\Gamma}|).
\end{equation}
Especially when $\alpha' = \text{Re} \alpha(x) \leq -1$ in $Q$, we have, for any fixed $\varepsilon_0 > 0$,
\begin{equation}
|y(x, h)| \leq C(Q) \Gamma^{(1-n-\varepsilon)/2}, \quad \text{for any } \varepsilon \geq \varepsilon_0.
\end{equation}

From the above parametrix $E_a$, we can derive a correct solution of the equation
"$AE_a = \delta(x-y)$", by the method of E. E. Levi, in other words, by solving a suitable integral equation. To do so we use the measure $dV = g(x)^{1/2} dx$ instead of the ordinary Lebesgue measure. Thus, we can show the following.

**Theorem 3.** We fix an arbitrary point $x_0 \in \partial \Omega$. Suppose that the operator $A$ satisfies the hypotheses $[H-1] - [H-5]$. Moreover we suppose $\text{Re} \alpha(x) \leq -1$ in a neighborhood of $x_0$. Then, for a sufficiently small neighborhood $Q$ of $x_0$, $A$ has a fundamental solution $E_a$ of the form
\begin{equation}
E_a(x, y) = E_a(x, y) + R(x, y),
\end{equation}
where $R(x, y)$ is defined by
\begin{equation*}
R(x, y) = \int_Q E_a(x, z)S(z, y)g(z)^{1/2} dz,
\end{equation*}
using another function $S(x, y)$.

Here $S(x, y)$ and $R(x, y)$ are functions of $(x, y) \in Q^\circ \times Q^\circ$ with $x \neq y$, and satisfy the following estimates. For any fixed $\varepsilon_0 > 0$, we have
\begin{equation}
|S(x, y)| \leq C(Q) \Gamma^{1-n-\varepsilon}/2, \quad |R(x, y)| \leq C(Q) \Gamma^{(2-n-\varepsilon)/2}, \quad \text{for any } \varepsilon \geq \varepsilon_0 > 0.
\end{equation}

**Remark 2.4.** Also in the case $-1 < \text{Re} \alpha(x) < 0$, we can construct $E_a$, but its estimate is more complicated. We treat that case in a future publication.

**Remark 2.5.** In the definition of $E_a$, if we interchange $\alpha(x)$ (resp. $\alpha(y)$) with $\alpha(y)$ (resp. $\alpha(x)$), we can obtain the similar results. In other words, $E_a$ is a two-sided parametrix for $A$.

**Remark 2.6.** We know that, for such an operator as $A$ in this paper, one can apply $L^2$-Theory and the equation "$Au = f$" can be treated without boundary conditions (see [H-5] and its remark). Then, by using the above theorems, we can obtain the analogous results due to C. Goulaouic and N. Shimakura [2] (see also C. R. Graham [6]) for the uniqueness and the existence of the solutions of the equation "$Au = f$" without boundary conditions.

§3. **Proof of Theorem 1.**

For the sake of simplicity we begin with the case where $A = L_x$, $\Omega = R^*_+$, $E_a = \sigma_x$ (see (0.1) and (1.1)) and $x$ is a negative real number.
First we prepare the following elementary lemma (The proof is omitted.).

**Lemma 3.1.** Let us set, for $n \geq 2$, $z > 0$ and $\alpha < 0$,

\[ M(z) = \int_0^z (1 + \eta)^{(2-n+\alpha)/2} \eta^{-1-s/2} d\eta. \tag{3.1} \]

Then we have, for a suitable constant $C > 0$ independent of $z$ and $\alpha$,

\[ M(z) \leq C (z/(1+z))^{-s/2} \log (2+z). \tag{3.2} \]

Using this, we can show the following estimates for $\mathcal{E}_\alpha$.

**Lemma 3.2.** For any multi-index $\nu$, there is a constant $C(\nu) > 0$ such that we have, for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $x \neq y$,

\[ |\partial^n_\nu \mathcal{E}_\alpha(x, y)| \leq C(\nu) (\nu^{2-n-|\nu|})^{1/2} \log (1 + \frac{\Gamma}{\Gamma}). \tag{3.3} \]

**Remark 3.1.** In particular for $\alpha \leq -1$, we have from this lemma, for any fixed $\varepsilon_0 > 0$,

\[ |\partial^n_\nu \mathcal{E}_\alpha(x, y)| \leq C(\nu) \Gamma^{(2-n-1-|\nu|-\varepsilon)(s-1)/2}, \quad \text{uniformly for any} \quad \varepsilon \leq \varepsilon_0. \tag{3.4} \]

**Proof of Lemma 3.2.** Let us remember the definition of $\mathcal{E}_\alpha$, and then we have

\[ \gamma(x)^{-1} y_n^{2+n} \mathcal{E}_\alpha = \int_0^1 \{ (1 - \theta) + \tilde{\Gamma} \theta \}^{(s+2-n)/2} \theta (1 - \theta)^{-1-s/2} d\theta \]
\[ = \int_0^{1/2} + \int_{1/2}^1 \equiv R_1 + R_2. \]

For $R_1$ and $R_2$, we obtain

\[ R_1 \leq C \Gamma^{(2-n)/2} (x_n y_n)^{s/2} M(4x_n y_n/\Gamma) \quad \text{and} \]
\[ R_2 \leq R_1, \quad \text{(since} \quad \Gamma \leq \tilde{\Gamma}). \tag{3.5} \]

Therefore the assertion of Lemma 3.2 in the case $|\nu| = 0$ is obvious from (3.2) and (3.5). When $|\nu| > 0$, we can prove the assertion by a similar calculus. (The detailed proof of Lemma 3.2 is omitted, cf. Horiuchi [7].) Q.E.D.

To treat the general case, we prepare the following:

**Lemma 3.3.** Let $x_0 \in \partial \Omega$ and $Q$ be a sufficiently small neighborhood of $x_0$ in $\Omega$. Then we have the following inequality.

\[ \text{Max} \{ \phi(x)^2, \phi(x)\phi(y), \phi(y)^2, \Gamma(x, y) \} \leq C \tilde{\Gamma}(x, y), \tag{3.6} \]

where $C$ is a positive number depending only on $Q$.

If we admit this lemma, it is not difficult to show Theorem 1. Because the behavior of $E_\alpha$ near the boundary is locally equivalent to that of $\mathcal{E}_\alpha$ near the boundary.
**Proof of Lemma 3.3.** If suffices to show $\phi(x)^2 \leq C \tilde{r}(x, y)$. We use the following fact:

\[(3.7) \quad \phi(x) - \phi(y) - r(\mathcal{P}r, \mathcal{P}\phi) = 0(\Gamma).\]

Admitting this for a moment, we have

\[\tilde{r} \geq r + \phi(x)\phi(y) = r + \phi(x)^2 - \phi(x)r(\mathcal{P}r, \mathcal{P}\phi) = \phi(x)0(\Gamma).\]

Since $Q$ is sufficiently small, we may assume $|r(\mathcal{P}r, \mathcal{P}\phi)| \leq 3/2$ and $|\phi(x)0(\Gamma)| \leq \Gamma/4$, and so we obtain the desired estimate as follows,

\[\tilde{r} \geq r + \phi(x)^2 - (\phi(x)^2 + \Gamma)3/4 - \Gamma/4 = \phi(x)^2/4.\]

In order to verify (3.7), we need some well-known properties of the geodesic distance, that is to say:

\[(3.8) \quad \Gamma(x, y) = \sum_{j, k=1}^{n} a_{jk}(x_j - y_j)(x_k - y_k) + O(r^3) \quad \text{and} \quad (\mathcal{P}r, \mathcal{P}f) = \frac{d}{dr} f, \quad \text{for any} \quad f \in C^1(\mathcal{Q}).\]

Therefore, (3.7) is obvious. \(Q.E.D.\)

**Proof of Theorem 1.** By this lemma 3.3 and the definition of $E_{\alpha}$, we can see that $E_{\alpha}$ has the same singularity as $\mathcal{E}_{\alpha}$. Hence as is noted just after the statement of Lemma 3.3, the desired assertions follow by using an easy variant of Lemma 3.1. \(Q.E.D.\)

§ 4. Proof of Theorem 2.

Our main aim in this section is to show that $AE_{\alpha}$ with $x \neq y$ consists only of "small error terms".

Let us set,

\[(4.1) \quad E_{\alpha}(x, y) = \gamma(\beta)\phi(y)^{-d-1}\Phi_{\alpha}F(I, J, K, \omega),\]

where

\[\beta = \alpha(y), \{I, J, K\} = \left\{\frac{n-2-\alpha(x)}{2}, -\alpha(x)/2, -\alpha(x)\right\},\]

\[\omega = 1 - \Gamma/\tilde{r} \quad \text{and} \quad \Phi_{\alpha} = \tilde{r}^{-I}.\]

Since $F(I, J, K, \omega)$ is a hypergeometric function, it satisfies the following differential equation.

\[(4.2) \quad \omega(1-\omega)F_{\alpha\omega} + \{K-(I+J+1)\omega\}F_{\alpha} - IJF = 0, \quad \text{with} \quad F_{\alpha} = \partial_{\alpha}F.\]
Now we calculate $A(\Phi F)$ with $\Phi$ being an arbitrary function. For the sake of simplicity, we denote $A$ by $-\phi aP + bP + c$, or equivalently $-\Delta P + bP + c$, and we also denote $(aP^2, P^2)$ and $(aP^2, P^2)$ by $(P^2, P^2)$ and $|P|^2$ respectively (see Remark 2.3).

Then we have by the definition of $A$,

\begin{align}
A(\Phi F) &= -\phi(x)|P|^{2}\Phi F + \{(A_0 P - 2\phi(x)(P, P)\} F

&+ \{(A_0 P - 2\phi(x)(P, P)\} YF

&- 2\phi(x)(P, P)\Phi YF

&- \phi(x)|P|^2 \Phi Y^2 F,
\end{align}

where $2Y = \partial_x + \partial_y + 2\partial_s$, $F_o = \partial_o F$ and $A_0 = A - c$.

By comparing the coefficients of (4.2) with those of (4.3), we have the following three equations.

\begin{align}
\phi(x)|P|^2 &= \mu\omega(1 - \omega), \\
2\phi(x)(P, P) &= \Phi A_0 P + \mu\omega\{K - (I + J + 1)\omega\}, \\
A\Phi &= \mu IJ F.
\end{align}

Where $\mu$ is a function defined by (4.4) itself, and we regard terms including $YF$, $Y^2 F$ and $YF_o$ as "error terms". We denote these "error terms" by $P(x, y)$.

If the above three equations are satisfied by a suitable $\Phi$, then $\gamma(\beta)\Phi(y)^{\beta-1} \Phi F$ will turn out to be a parametrix for $A$ with an error term $P(x, y)$. But unfortunately, it is never elementary to solve them at all. Hence, we will try to show that $\Phi = \Phi_0$ defined by (4.1) is a good approximate solution of them.

From (4.4), we have, denoting $4\phi(x)\phi(y)$ by $f$,

\begin{align}
\mu &= \phi(x)\{4f - 2(P, P) + |Pf|^2/|f|\}.
\end{align}

And from (4.5), we have

\begin{align}
\{\phi(x)|\tilde{r}|^2\} X\Phi &= [A_0 \omega + \mu\{K - (I + J + 1)\omega\}] F,
\end{align}

where $X = 2(I/P, P) - 2f(P, P)$ and $f = 4\phi(x)\phi(y)$.

We note that from [H--4] $bP$ is of the form $a(P^2, P^2) + 0(\phi)T$ with $T$ being a first order differential operator. Hence, after a slightly long calculus we obtain

\begin{align}
X\Phi &= [-\{I/(2\tilde{r})\}] X\tilde{r} + \{(P, P) + |Pf|^2/2 - 2f\} n|\tilde{r}|
\end{align}
- \Gamma(a \nabla \nabla f + O(r \Gamma)) \Phi, \]

where we used \( a \nabla \nabla \Gamma = 2n + O(r) \) and \( |\Gamma|^2 = 4 \Gamma \) (see (2.5)).

Now we set \( \Phi_0 = \tilde{r}^{-1} \) with \( I = \{n - 2 - \alpha(x)\}/2 \), so that

(4.10) \[ X \Phi_0 = \left\{ -\{I/(2\tilde{r})\}X \tilde{r} + (1/2)(X \alpha) \log \tilde{r} \right\} \Phi_0. \]

\( \Phi_0 \) is never an exact solution of (4.4), (4.5) and (4.6) in general, but is very close to it in the following sense. We denote by \( G(\Phi) \) the right-hand side of (4.9). We have the following three estimates, denoting by \( C \) various constant depending only on \( Q \).

(4.11) \[ \{|\phi(x)/\tilde{r}^2\}{X \Phi_0 - G(\Phi_0)} F_{\phi} \| \leq (*), \]

where \( (*) = C|\Phi_0|/(r^{2-n}/2\{1 + \phi(x)/r\} (1 + |\log \tilde{r}|), \)

(4.12) \[ |(A \Phi_0) F - \mu I F_{\phi} F| \leq C|\Phi_0| F_{\phi} (1 + |\log \tilde{r}|), \]

(4.13) \[ |P(x, y)| \leq C|\Phi_0| F_{\phi} (1 + \phi(x)/r), \]

where \( P \) is an error term.

If we admit these estimates, we will have proved the next estimation of \( A E_{\phi} \).

\[ |A E_{\phi}| = |\gamma(\beta) \phi(y)^{-\beta-1} A(\Phi_0 F)| \]

\[ \leq C|E_{\phi}| F_{\phi}(1 + \phi(x)/r) (1 + |\log \tilde{r}|). \]

Obviously this assures Theorem 2, and therefore we have only to prove the above three estimates after this.

**PROOF OF (4.11).** We need two lemmas. First, by subtraction, we have

**LEMMA 4.1.**

(4.14) \[ X \Phi_0 = -G(\Phi_0) = \left\{ [8n \Gamma \phi(y)/\tilde{r}] N + (1/2)(X \alpha) \log \tilde{r} + O(r \tilde{r}) \right\} \Phi_0, \]

where \( N = \tilde{r} (a \nabla \nabla \phi)/(2n) - |\nabla \phi|^2 \phi(y) + \phi(x) - r F_{\phi(\phi)} \).

Moreover we have

(4.15) \[ N = O(\Gamma) \quad \text{and} \quad X \alpha = \phi(y) \Gamma \{1 + \phi(x)/r\}. \]

**PROOF OF LEMMA 4.1.** As for \( X \alpha \), the assertion is obvious. For \( N \), since \( |\nabla \phi| = 1 \) on \( \partial \Omega \), it suffices to remark the following,

\[ \phi(x) - \phi(y) - r F_{\phi(\phi)} = O(\Gamma) \quad \text{(see (3.7) in §3)} \]

Q. E. D.

As for \( F_{\phi} \), we prepare the next lemma which is easily verified, (for the proof, see [3]).
LEMMA 4.2. If \( i, j, k \) and \( i+j-k \) are positive, we have, for \( 0 < \eta \leq 1 \),

1. \( F(i, j, k, 1-\eta) = \eta^{k-i-j} F(k-i, k-j, k, 1-\eta) \),

2. \(|F(k-i, k-j, k, 1-\eta)| \leq |F(k-i, k-j, k, 1)|
   = \eta^{k-i-j} \frac{\Gamma(i+j-k)}{(i)\Gamma(j)} \).

For any \( \{i, j, k\} \) with \( k \neq 0 \), we have

3. \( F_{\infty}(i, j, k, \omega) = (ij/k) F(i+1, j+1, k+1, \omega) \).

By virtue of this lemma, we can show that

4. \( |F_{\infty}(I, J, K, 1-\frac{\eta}{\eta})| (\frac{\eta}{\eta}) \leq C(\eta/\eta)^{(2-n)/2} \).

Now (4.11) seems to be almost obvious. By the above two lemmas the desired estimate follows.

Q. E. D.

PROOF OF (4.12). We divide \( A\Phi_0 - IJ\Phi_0 \) into the following five terms, that is to say,

\[ (4.17) \quad A\Phi_0 - \mu IJ\Phi_0 = R_1 + R_2 \]

and

\[ R_2 = I\tilde{T}^{-1} = (R_3 + R_4 + R_5), \]

where

\[ R_1 = \{(A_0 I) \log \tilde{T} - \phi(x)|V|\|^2 (\log \tilde{T})^2 - 2\phi(x)(V \tilde{T}, V I)\tilde{T} \]

\[ - 2I \phi(x)(V \tilde{T}, V I)(\log \tilde{T})/I + c\} \Phi_0, \]

\[ R_2 = \{- (A_0 \tilde{T}) - (I + 1) \phi(x)|V|\|^2/\tilde{T} - \mu J\tilde{T}\} I\tilde{T}^{-1}, \]

\[ R_3 = \phi(x)aVf + \phi(x)b(r), \]

\[ R_4 = - (I + 1) \phi(x) \{- 4f + 2(V \tilde{T}, V f) + |V f|^2\}/I \]

and

\[ R_5 = a(x) \{2\phi(x) - (V \phi, V \tilde{T}) - (V \tilde{T}, V f) + 2\phi(x)f\}/I \]

\[ - \phi(x)(V \tilde{T}, V f)\tilde{T} + \Gamma|V f|^2 (2f\tilde{T}) \} \).

These five quantities are estimated as follows.

\[ (4.18) \quad |R_1| = 0 \{\tilde{T}^{-1} (1 + \log \tilde{T})\} \quad \text{and} \]

\[ |R_2| \leq |R_3| + |R_4| + |R_5| = 0(\tilde{T}). \]

PROOF. As for \( R_1 \) and \( R_3 \), there is no difficulty at all. From the definition of \( N \) and Lemma 4.1, we have

\[ (4.19) \quad -4f + 2(V \tilde{T}, V f) + |V f|^2 = 16\phi(y) \{\tilde{T}(aV V \phi)/(2n) - N\} \]

\[ = 0(\tilde{T} \phi(y)) \quad \text{and} \]

\[ -\alpha^{-1}R_2 = 2\phi(y)(1 - |V \phi|^2)(1 - f/\tilde{T}) \]

\[ = 0(\tilde{T} \phi(y)). \]
Thus we obtain the estimates for $R_4$ and $R_5$, which completes the proof of (4.12).

Q.E.D.

At last we estimate the error term $P(x, y)$ including $YF$, $Y^2F$ and $YF_\omega$ with $2Y=\delta_J + \delta_J + 2\delta_k$. We need the next lemma which is also one of the well-known properties of hypergeometric functions, for the proof, see [3].

**Lemma 4.3.** If $i$, $j$ and $k$ are positive, we have, for a suitable constant $C>0$,

\begin{equation}
|Y^mF(i, j, k, \omega)| \leq C|F(i, j, k, \omega)|,
\end{equation}

where $m=1$ or 2 and $0 \leq \omega \leq 1$.

**Remark 4.1.** Explicitly $YF$ can be written as follows,

\begin{equation}
2YF = \sum_{p=1}^{\infty} \frac{\Gamma(i+p)\Gamma(j+p)\Gamma(k)}{\Gamma(i)\Gamma(j)\Gamma(k+p)} \sum_{q=0}^{p-1} \left( \frac{1}{i+q} + \frac{1}{j+q} - \frac{2}{k+q} \right) \frac{\omega^p}{p!}
\end{equation}

and we note that $\left| \sum_{q=0}^{p-1} \right|$ is uniformly bounded for any $p$.

From Lemma 4.2, Lemma 4.3 and its easy variant, we can estimate $P(x, y)$ as we desire.

Q.E.D.

**§ 5. Proof of Theorem 3.**

We have already proved the equality "$AE_\omega = \delta(x-y) + h(x, y)$". Since $h(x, y)$ has only a weak singularity, it suffices to solve the next integral equation with respect to the kernel $S(x, y)$.

\begin{equation}
S(x, y) + h(x, y) + \int_\Omega h(x, z)S(z, y)g(z)^{1/2}dz = 0,
\end{equation}

for any $(x, y) \in \Omega \times \Omega$ with $x \neq y$.

If $S(x, y)$ is a solution kernel of (5.1), we immediately have

\begin{equation}
E_\omega(x, y) = \tilde{E}_\omega(x, y) + R(x, y),
\end{equation}

where

\begin{equation}
R(x, y) = \int_\Omega \tilde{E}_\omega(x, z)S(z, y)g(z)^{1/2}dz.
\end{equation}

And $S(x, y)$ itself is given by the following formal series,

\begin{equation}
S(x, y) = \sum_{j=1}^{\infty} (-1)^j h^{(j)}(x, y),
\end{equation}

where $h^{(1)} = h$ and $h^{(j+1)}(x, y) = \int_\Omega h(x, z)h^{(j)}(z, y)g(z)^{1/2}dz$. 

If \( \text{Re } \alpha(x) \leq -1 \), it holds from Theorem 2 that, for any fixed \( \varepsilon > 0 \),
\[
|h(x, y)| \leq C \varepsilon^{(1-\alpha-\gamma)/2} \leq C' |x - y|^{1-\alpha-\gamma}, \quad \text{for any } \varepsilon \geq \varepsilon_0.
\]
Therefore, it follows from the theory of Riesz potential that the series defined by (5.3) is absolutely convergent for a sufficiently small \( Q \), and we also obtain the estimates as follows. For a sufficiently small \( Q \) and any fixed \( \varepsilon > 0 \), we have
\[
|S(x, y)| \leq C(Q) |h(x, y)| \quad \text{and} \quad |R(x, y)| \leq C(Q) \varepsilon^{(2-\alpha+\gamma)/2}, \quad \text{for any } \varepsilon \geq \varepsilon_0.
\]
Q.E.D.

§ 6. Appendix (Proof of Proposition 1 in § 1)

We have postponed to this appendix the proof of Proposition 1 which is rather elementary but needs slightly long computations involving our kernel.

Let us recall the statement. We set again, for \((x, y) \in B \times B_e \) with \( x \neq y \),
\[
\phi(x) = (\tau^2 - |x|^2)/(2\tau),
\]
\[
L^b_\alpha = -\phi \Delta + \alpha(\phi \phi, \phi:) + \alpha(\alpha+2-n)/(2\tau),
\]
\[
\mathcal{E}^b_\alpha(x, y) = \gamma(\alpha) \phi(y)^{-\alpha-1} \tilde{\Gamma}_b \Gamma(I, J, K, 1-\Gamma/\tilde{\Gamma}_b)
\]
\[
= \tilde{\gamma}(\alpha) \phi(y)^{-\alpha-1} \int_0^1 \{\Gamma(1-\theta) + \tilde{\Gamma}_b \theta\}^{-1}(\theta(1-\theta))^{-1-\gamma/2} d\theta
\]
where \( \Gamma = |x - y|^2, \tilde{\Gamma}_b = \Gamma + 4\phi(x)\phi(y) \) and \( \{I, J, K\} = \{(n-2-\alpha)/2, -\alpha/2, -\alpha\} \).

**Remark 6.1.** If \( \alpha = 2 - n \ (n \geq 3) \), we can compute the right-hand side and obtain
\[
\mathcal{E}^b_\alpha(x, y) = \tilde{\gamma}(2-n) B(-\alpha/2, -\alpha/2) \phi(y)^{n-3}/(\Gamma \tilde{\Gamma}_b^{(n-2)/2}).
\]

Let us set
\[
A(x, y, \theta) = \Gamma(1-\theta) + \tilde{\Gamma}_b \theta,
\]
\[
F(x, y, \theta) = \tilde{\gamma}(\alpha) \phi(y)^{-\alpha-1} A(x, y, \theta)^{-1}\{\theta(1-\theta))^{-1-\gamma/2},
\]
\[
G(x, y, \theta) = 4\tilde{\gamma}(\alpha) \phi(y)^{-\alpha-1} A(x, y, \theta)^{-1}(\theta(1-\theta))^{-\gamma/2},
\]
so that
\[
\mathcal{E}^b_\alpha(x, y) = \int_0^1 F(x, y, \theta) d\theta.
\]

First we have
\[ \partial_\theta [\phi(y)G(x, y, \theta)] = L^b_x(x, \partial_\theta)F(x, y, \theta) \quad \text{and} \]
\[ \partial_\theta [\phi(x)G(x, y, \theta)] = \ell L^b_y(y, \partial_\theta)F(x, y, \theta), \]
where \( \ell L^b_x = -\phi A \alpha \phi (2\phi, \phi \cdot) + (\alpha + n)(\alpha + 2)/(2\tau) \) is the formal transpose of \( L^b_x \).

Next we set
\[ \mathcal{E}^b_{x,y}(x, y) = \int_0^1 F(x, y, \theta)d\theta. \]

Then
\[ L^b_x \mathcal{E}^b_{x,y}(x, y) = -\phi(y)G(x, y, \epsilon), \quad \text{since} \quad G(x, y, 1) = 0. \]

Considering the limit as \( \epsilon \to 0 \), \( -\phi(y)G(x, y, \epsilon) \) tends to zero uniformly on each compact set of \( y \subset B \), with \( x \neq y \). Moreover, on any compact set \( D \subset B \) and \( x \in \text{Interior of } D \), we can show
\[ \lim_{\epsilon \to 0} \int_D -\phi(y)G(x, y, \epsilon)dy = 1. \]

Hence \( L^b_x(\mathcal{E}^b_{x,y}f)(x) \) tends to \( f(x) \), for any \( f \in C^0(\overline{B}) \). Since the limit of \( \mathcal{E}^b_{x,y}f(x) \) is \( \mathcal{E}^b_{x,y}f(x) \), we have \( L^b_x(\mathcal{E}^b_{x,y}f)(x) = f(x) \). This proves the assertion (1) in Proposition 1.

Next we prove the assertion (2) in Proposition 1. We use the polar coordinate \((\rho, \sigma)\). Then \( L^b_x \) is of the form
\[ L^b_x = -\phi \left[ \partial_\rho + \{(n-1)/\rho\} \partial_\rho + (1/\rho^2)A_{\rho} \right] - \alpha \rho \partial_\rho + \alpha (\alpha + 2 - n)/(2\tau), \]
where \( A_{\rho} \) is the so-called Laplace-Beltrami operator on the unit sphere. Then we have
\[ vL^b_xu - \ell L^b_xv = -(\phi/\rho^2)(uA_{\rho}v - vA_{\rho}u) \]
\[ - \rho^{1-n} \partial_\rho (\phi \rho^{n-1}v \partial_\rho u - \phi^{-\sigma}u \rho^{n-1} \partial_\rho (\phi^{-1}v)). \]

Let us set \( v(y) = F(x, y, \theta) \) and \( u(y) \in C^2(\overline{B}) \).

Since \( \phi F \) and \( \phi^{-\sigma} \partial_\rho (\phi^{-1}v) \) vanish at \( \phi = 0 \) \((\rho = \tau)\), we have, using (A.8),
\[ \int_{\overline{B}} F(x, y, \theta) L^b_{x\epsilon}u(y)dy = \int_{\overline{B}} u(y)\ell L^b_{x\epsilon}F(x, y, \theta)dy. \]

Integrating the above equality with respect to \( \theta \) and using (A.4), we have
\[ \mathcal{E}^b_{x\epsilon}(L^b_{x\epsilon}u)(x) = \int_{\overline{B}} -\phi(x)G(x, y, \epsilon)u(y)dy. \]

Similarly to the proof of (1), we obtain \( \mathcal{E}^b_{x\epsilon}(L^b_{x\epsilon}u)(x) = u(x) \) in \( B \). Q. E. D.
References


