Remarks on the Henstock Integral of Stieltjes Type

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ABSTRACT.

Let $f: [a, b] \rightarrow \mathbf{R}$ be bounded and $g: [a, b] \rightarrow \mathbf{R}$ be of bounded variation. It is shown that f is Henstock integrable with respect to g on $[a, b]$ if and only if f is Young refinement integrable with respect to g on $[a, b]$, and both integrals have the same value. Some relations to the mean Stieltjes integral will also be given.

Let $[a, b]$ be a compact interval on the real line $\mathbf{R}$. Given a positive function $\delta: [a, b] \rightarrow (0, \infty)$, we consider finite sequences of numbers $A = \{x_0, x_1, \dots, x_n; t_1, t_2, \dots, t_n\}$ such that

$$a = x_0 < x_1 < \dots < x_n = b, \quad x_{i-1} \leq t_i \leq x_i$$

$$[x_{i-1}, x_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i)) \quad (i = 1, 2, \dots, n).$$

Such a sequence A is called a δ -fine partitions of $[a, b]$.

If the second condition is replaced by

$$x_0 \leq t_1 < x_1, \quad x_{i-1} < t_i < x_i \quad (i = 2, 3, \dots, n-1), \quad x_{n-1} < t_n \leq x_n,$$

then A is called a δ^* -fine partition of $[a, b]$.

DEFINITION 1. A function $f: [a, b] \rightarrow \mathbf{R}$ is said to be H -integrable, (resp. H^* -integrable) with respect to $g: [a, b] \rightarrow \mathbf{R}$ on $[a, b]$ if there exists a number α such that to every $\epsilon > 0$ there is such a positive function $\delta(x)$ that

$$\left| \sum_{i=1}^n f(t_i)(g(x_i) - g(x_{i-1})) - \alpha \right| < \epsilon$$

for any δ -fine partition (resp. δ^* -fine partition) $A = \{x_0, x_1, \dots, x_n; t_1, t_2, \dots, t_n\}$. The number α is denoted by $(H) \int_a^b f dg$ ($(H^*) \int_a^b f dg$).

DEFINITION 2. Let $g: [a, b] \rightarrow \mathbf{R}$ be such that the limits $g(x+)$, $g(x-)$ exist

Received March 2, 1985. Research partially supported by the Grant-in-Aid for Scientific Research C-59540065 from the Ministry of Education.

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for all $x \in [a, b]$. A function $f: [a, b] \rightarrow \mathbf{R}$ is said to be Young refinement integrable with respect to g if there is number α such that to every $\varepsilon > 0$ there exists a partition P_0 of $[a, b]$ in the ordinary sense so that

$$\left| \sum_{i=1}^n [f(x_{i-1})(g(x_{i-1}+) - g(x_{i-1})) + f(t_i)(g(x_i-) - g(x_{i-1}+) + f(x_i)(g(x_i) - g(x_i-)) - \alpha] \right| < \varepsilon$$

holds for any ordinary partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ with $x_{i-1} < t_i < x_i$ ($i = 1, 2, \dots, n$) which is a refinement of P_0 . The number α is denoted by $(Y) \int_a^b f dg$.

In [1] and [2] Schwabik studied the relation between the H or H^* -integral and the Y -integral, and established the following theorem.

THEOREM 1. *Let $f: [a, b] \rightarrow \mathbf{R}$ and let $g: [a, b] \rightarrow \mathbf{R}$, be of bounded variation. The existence of $(Y) \int_a^b f dg$ does not in general imply the existence of $(H) \int_a^b f dg$. If $(Y) \int_a^b f dg$ exists then also $(H^*) \int_a^b f dg$ exists and both integrals are equal.*

In this note we want to show the following theorem.

THEOREM 2. *Let $f: [a, b] \rightarrow \mathbf{R}$ be bounded and $g: [a, b] \rightarrow \mathbf{R}$ be of bounded variation. Then $(Y) \int_a^b f dg$ exists if and only if $(H) \int_a^b f dg$ exists, and both integrals have the same value.*

To prove Theorem 2 we need some lemmas.

LEMMA 1. *If $f: [a, b] \rightarrow \mathbf{R}$ is bounded, $g: [a, b] \rightarrow \mathbf{R}$ is of bounded variation and $(Y) \int_a^b f dg$ exists then $(H) \int_a^b f dg$ exists and both integrals are equal.*

This lemma has been proved in Theorem 3.2 in [1].

LEMMA 2. *If $g: [a, b] \rightarrow \mathbf{R}$ is a pure break function of bounded variation and $f: [a, b] \rightarrow \mathbf{R}$ is bounded then $(Y) \int_a^b f dg$ and $(H) \int_a^b f dg$ both exist and are equal.*

For the proof, see Corollaries 1.2 and 1.2 in [1].

LEMMA 3. *If $f: [a, b] \rightarrow \mathbf{R}$ is bounded, $g: [a, b] \rightarrow \mathbf{R}$ is continuous and of*

bounded variation and if $(H)\int_a^b f dg$ exist then the Riemann-Stieltjes refinement integral $(R)\int_a^b f dg$ exists and both integrals are equal.

PROOF. For a given $\varepsilon > 0$, there is a function $\delta_1(x): [a, b] \rightarrow (0, \infty)$ such that

$$\left| \sum_{i=1}^n f(t_i)(g(x_i) - g(x_{i-1})) - (H)\int_a^b f dg \right| < \varepsilon$$

for any δ_1 -fine partition $\{x_0, x_1, \dots, x_n; t_1, t_2, \dots, t_n\}$ of $[a, b]$. Since g is continuous and of bounded variation on $[a, b]$, its total variation on $[a, x]$ $V_a^x g$ is continuous on $[a, b]$. Hence there is a positive number δ_2 such that $V_x^y g < \varepsilon$ for any $x, y \in [a, b]$ with $0 < y - x < \delta_2$. Let $\delta(x) = \min(\delta_1(x)/2, \delta_2)$. Let $\{y_0, y_1, \dots, y_m; t_1, t_2, \dots, t_m\}$ be the fixed $\delta(x)$ -fine partition of $[a, b]$ and put $P_0 = \{y_0, y_1, \dots, y_m\}$. If P is any refinement of P_0 and if $y_{j-1} = z_0 < z_1 < \dots < z_{l_j} = y_j$ is the corresponding part of P on $[y_{j-1}, y_j]$ ($j=1, 2, \dots, m$) then we have for $z_{k-1} \leq \xi_k \leq z_k$ ($k=1, 2, \dots, l_j$)

$$\begin{aligned} & \left| \sum_{k=1}^{l_j} f(\xi_k)(g(z_k) - g(z_{k-1})) - f(t_j)(g(y_j) - g(y_{j-1})) \right| \\ & \leq \sum_{k=1}^{l_j} |f(\xi_k) - f(t_j)| |g(z_k) - g(z_{k-1})| \leq 2M \sum_{k=1}^{l_j} |g(z_k) - g(z_{k-1})| \\ & \leq 2M V_{y_{j-1}}^{y_j} g < 2M\varepsilon, \end{aligned}$$

where $|f(x)| \leq M$ on $[a, b]$.

Let $S(P, f, g)$ denote the Riemann-Stieltjes sum of f with respect to g for P . Then we get

$$|S(P, f, g) - \sum_{j=1}^m f(t_j)(g(y_j) - g(y_{j-1}))| < 2M\varepsilon m$$

and therefore

$$|S(P, f, g) - (H)\int_a^b f dg| < 2M\varepsilon m + \varepsilon$$

which completes the proof.

PROOF OF THEOREM 2.

Necessary part is given in Lemma 1. To prove the sufficiency, we represent $g = g_b + g_c$ where g_b is the pure break part and g_c is the continuous part of g respectively. Then g_b, g_c are of bounded variation. It follows from Lemma 2 that $(H)\int_a^b f dg$ and $(Y)\int_a^b f dg$ exist and are equal. Hence $(H)\int_a^b f dg_c$ must exist since the H -integral is additive with respect to integrators. By Lemma 3,

$(R)\int_a^b f dg_c$ exists and equals $(H)\int_a^b f dg_c$. Since the Y -integral includes the R -integral in this case, we have

$$(H)\int_a^b f dg_c = (Y)\int_a^b f dg_c.$$

Hence $(Y)\int_a^b f dg$ exists and equals $(H)\int_a^b f dg$.

Finally we shall give some remarks to the mean Stieltjes refinement integral $(M)\int_a^b f dg$ (for the definition of the M -integral see [4]).

REMARK 1. The H -integral $(H)\int_a^b f dg$ is not compatible to the M -integral $(M)\int_a^b f dg$, that is, there exist functions $f, g: [a, b] \rightarrow \mathbf{R}$ such that f is both H and M -integrable with respect to g on $[a, b]$ but

$$(H)\int_a^b f dg \neq (M)\int_a^b f dg.$$

Let f, g be real valued functions defined on $[-1, 1]$ as follows; $f(x)=0$ except at 0 on $[-1, 1]$ and $f(0)=1$: $g(x)=0$ ($-1 \leq x < 0$), $g(0)=1/2$ and $g(x)=1$ ($0 < x \leq 1$). Then we have

$$\begin{aligned} (M)\int_{-1}^1 f dg &= 1/2(f(0-) + f(0))(g(0) - g(0-)) \\ &\quad + 1/2(f(0) + f(0+))(g(0+) - g(0)) \\ &= 1/2, \\ (H)\int_{-1}^1 f dg &= 0. \end{aligned}$$

REMARK 2. Suppose g is of bounded variation on $[a, b]$, f is bounded and $(M)\int_a^b f dg$ exists. Then $(H)\int_a^b f dg$ exists and equals

$$\begin{aligned} (M)\int_a^b f dg &- 1/2 \sum_{x \in (a, b)} (f(x+) - f(x))(g(x+) - g(x-)) \\ &+ 1/2 \sum_{x \in (a, b)} (f(x) - f(x-))(g(x) - g(x-)). \end{aligned}$$

It follows from Theorem 2.4 in [5] that $(Y)\int_a^b f dg$ exists and equals the above value. By Theorem 2, $(H)\int_a^b f dg$ exists and has the same value.

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