Euclidean rings

Mitsuo Kanemitsu* and Ken-ichi Yoshida**

The purpose of this paper is to give the structure theorem for a Euclidean ring in which zero-divisors can appear.

In this paper, all rings are commutative with identity. We give the following definition:

DEFINITION. Given a ring $R$, a Euclidean algorithm in $R$ is a map $\phi$ of $R$ into a well-ordered set $M$ such that

1. given $a, b \in R$, there exist $q$ and $r$ in $R$ such that
   
   \[ b = aq + r \quad \text{where} \quad r = a \quad \text{or} \quad \phi(r) < \phi(a) ; \]

2. $\phi(0) \leq \phi(a)$ for all $a \in R$.

We say that $R$ is Euclidean if it admits an algorithm $\phi$.

For an integral domain, the inequality "$\phi(ab) \geq \phi(a)$ for all $a, b \in R$" is also normally included in the definition. This inequality does not necessarily hold in general. For example, let $\phi: \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ be a Euclidean algorithm on $\mathbb{Z}$ defined by $\phi(n) = 2n$ for $n > 0$, $\phi(n) = -2n - 1$ for $n < 0$ and $\phi(0) = \infty$. But if we put $\bar{\phi}(a) = \min \{\phi(ua)\}$, where $u$ is taken over all units of $R$, then $\bar{\phi}$ is a Euclidean algorithm for $R$ which does satisfy the inequality.

First we give the following result.

LEMMA 1. If a ring $R$ is a Euclidean ring, then it is a principal ideal ring.

PROOF. If $\phi$ is a Euclidean algorithm on $R$, then, for each ideal $A$ of $R$, define

\[ \phi(a) = \min \{\phi(a)/a \in A\} . \]

For any $b \in A$, (2) of the above definition shows that

\[ b = aq + r \quad \text{where} \quad r = a \quad \text{or} \quad \phi(r) < \phi(a) . \]

In the case of $r = a, b \in Ra$. Otherwise, since $\phi(r) < \phi(a)$, we have $r \in A$. This is contradict to the minimality. Therefore $A$ is a principal ideal of $R$.

Received July 4, 1985.

* Department of Mathematics, Aichi University of Education, Igaya-cho, Kariya-shi, 448 Japan.

** Department of Applied Mathematics, Okayama University of Science, Ridai-cho, Okayama-shi, 700 Japan.
**COROLLARY 2.** If $R$ is a Euclidean ring, then it is a Noetherian ring with Krull dimension one at most.

As the proof of the next proposition is immediate, we omit its proof.

**PROPOSITION 3.** If $R$ is a Euclidean ring, then the homomorphic images of $R$ are also Euclidean.

**PROPOSITION 4.** Let $R$ be a Euclidean ring and $S$ be a multiplicatively closed set of $R$. Then $S^{-1}R$ is a Euclidean ring.

**PROOF.** We can assume that $S$ is a saturated multiplicatively closed set. So $S$ is generated by prime elements $\{p_\lambda\}$. Let $\phi: R \rightarrow M$ be a Euclidean algorithm, $\bar{\phi}: M \rightarrow M$ be a map such that $\phi(p_\lambda) \in \text{min } M$ for all $p_\lambda$ and $\bar{\phi}$ is their composition $\bar{\phi}\phi: R \rightarrow M$. We assume that $\phi$ is satisfying the condition $\phi(ab) \geq \phi(a)$ for $a \in R$. We can define $\bar{\psi}: S^{-1}R \rightarrow M$ by $\bar{\psi}$. Then $\bar{\psi}$ is a Euclidean algorithm on $S^{-1}R$. Therefore $S^{-1}R$ is Euclidean. q.e.d.

Next, we prove that there do not exist the embedded prime divisors of $(0)$ in Euclidean rings.

**PROPOSITION 5.** $R$ is a Euclidean ring, then there do not exist the embedded prime divisors of $(0)$.

**PROOF.** Let $(0)=q_1 \cap \cdots \cap q_t \cap Q_1 \cap \cdots \cap Q_s$ be an irredundant primary decomposition of $(0)$. Let the $Q_j$ be embedded primary components and $P_i=\text{rad } (Q_i)$ be the radical ideal of $Q_i$ for $i=1,\ldots, s$. Put $q_1 \cap \cdots \cap q_t=(a)$. We will prove that $(0): (a)\subset P_j$ for some $P_j$. We assume that $(0): (a)\subset P_i$ for $i=1,\ldots, s$. It is well known that there exists an element $y \in (0): (a)$ such that $y \notin P_1 \cap \cdots \cap P_s$. Since $ay=0$ and $Q_i$ is primary, we have $a \in Q_i$ for $i=1,\ldots, s$. Hence $a \in q_1 \cap \cdots \cap q_t \cap Q_1 \cap \cdots \cap Q_s=(0)$. This is a contradiction. Therefore we can assume that $(0): (a)\subset P_1=P$. Let $P=(p)$. Since $P$ is an embedded prime ideal of $(0)$, we have $(a)\subset q_i \subset P=(p)$ for some $q_i$. Write $a=pa'$ for some $a' \in R$. Since $p \notin \text{rad } (q_i)$ for $i=1,\ldots, r$, we have $a' \in q_1 \cap \cdots \cap q_t=(a)$, hence $a'=x a$ for some $x \in R$. Hence $(1-px)a=0$, so $1-px \in (0): (a)\subset P$. Therefore $1 \in P$, this is a contradiction.

**COROLLARY 6.** If $R$ is a Euclidean ring, then $R \cong R' \oplus A$ where $R'$ is a Euclidean ring such that the irreducible components of $\text{Spec}(R')$ equal all dimension one and $A$ is an Artinian Euclidean ring.

**PROOF.** Let $(0)=q_1 \cap q_2 \cap \cdots \cap q_t \cap Q_1 \cap \cdots \cap Q_s$ be an irredundant primary decomposition of $(0)$, where the $\text{rad } (Q_i)$ $(i=1,\ldots, s)$ are maximal ideals and the $\text{rad } (q_j)$ $(j=1,\ldots, r)$ are not maximal ideals. Put $q=q_1 \cap \cdots \cap q_t$ and $Q=Q_1 \cap \cdots \cap Q_s$. Then we have
Euclidean rings

\[ R \cong R/q \oplus R/Q \]

where \( R/q \) is an Artinian Euclidean ring and \( R/Q \) is a Euclidean ring such that each irreducible component of \( \text{Spec}(R/Q) \) has dimension 1. q.e.d.

To prove the structure theorem, we will prove the following proposition.

**Proposition 7.** If \( R \) is a Euclidean ring such that each irreducible component of \( \text{Spec}(R) \) is of dimension one, then

\[ R \cong R_1 \oplus \cdots \oplus R_t \]

where all \( R_i \) are Euclidean domains.

**Proof.** First, we will prove that \( R \) is a reduced ring. Suppose \( \text{rad}(0) \neq (0) \). Then \( \text{rad}(0) = (a) \) for some non-zero element \( a \in R \). Put \( A = (0):(a) \). Since \( A \neq R \), there exists a non-zero maximal ideal \( m = (p) \) containing \( A \). Hence \( (a, p) = (p) \). Therefore, we have \( a = pa_1 \) for \( a_1 \in R \). Since the irreducible components of \( \text{Spec}(R) \) are of dimension one, we have \( a_1 \in \text{rad}(0) = (a) \). Thus we have

\[ a = pa_1 = p^2a_2 = \cdots = p^ta_1 \quad (a_2, \ldots, a_t \in R). \]

Therefore we have a sequence of ideals:

\[ (a) \subset (a_1) \subset (a_2) \subset \cdots. \]

Since \( R \) is a Noetherian ring, we have \( (a_n) = (a_{n+1}) \) for some integer \( n \). Hence \( a_{n+1} = a_n x \) for some \( x \in R \). Thus we have

\[ a = p^{n+1}a_{n+1} = p^{n+1}a_n x = pxa, \quad \text{that is}, \quad a(1 - px) = 0. \]

Consequently, we have that

\[ 1 - px \in (0): (a) = A \subset m = (p). \]

Hence we have \( 1 \in (p) \), this is a contradiction. We have proved that \( R \) is reduced.

Next, let \( (0) = (p_1) \cap \cdots \cap (p_t) \) where each \( (p_i) \) is a prime ideal. If \( t = 1 \), then \( R \) is an integral domain. So nothing to do. Now, let \( t \geq 2 \). We will prove the proposition using induction on \( t \). Put \( p = p_1 \) and \( (p_2) \cap \cdots \cap (p_t) = (q) \). Then \( (0) = (p) \cap (q) \). We will prove that \( (p) + (q) = R \). Now, we assume that \( (p) + (q) \neq R \). Put \( A = (p) + (q) \) and \( A = (a) \) for some element \( a \in R \). If \( a \) is a zero divisor, then \( a \in (p_i) \) for some \( i \) \((1 \leq i \leq t)\). Suppose \( a \in (p_1) \), then \( A \subset (p_1) \). And so \( q \in (p_1) \), this is a contradiction. Next, we assume that \( a \in (p_j) \) for \( 2 \leq j \leq t \). Then we have \( p = p_1 \in (p_j) \). This is a contradiction. Hence we have proved that \( a \) is non-zero divisor. Since \( a = px + qy \) for some \( x, y \in R \), \( p = au \) for \( u \in R \) and \( q = av \) for \( v \in R \), we have that \( u \in (p) \), \( v \in (q) \) and \( a = px + qy = a(ux + vy) \). Hence \( 1 \in (ux + vy) \). This is a contradiction. Thus we have proved that \( (p) + (q) = R \). Therefore we have
\begin{align*}
R &= R/(p_1) \oplus R/(p_2) \cap \cdots \cap (p_i). 
\end{align*}

Now, we see that $R = R_1 \oplus \cdots \oplus R_n$, where each $R_i$ is a Euclidean domain, by the assumption of induction. This is complete the proof.

We summarize:

**Theorem 8** (Structure theorem for Euclidean rings). Let $R$ be a ring. Then $R$ is a Euclidean ring if and only if
\begin{align*}
R &= R_1 \oplus \cdots \oplus R_n \oplus A_1 \oplus \cdots \oplus A_s,
\end{align*}
where each $R_i$ is a Euclidean domain and each $A_j$ is an Artinian local Euclidean ring.

**Corollary 9.** Let $R$ be a ring and $\hat{R}$ be the completion of $R$. Then $R$ is a Euclidean ring if and only if $\hat{R}$ is so.

Now, we will consider the case of Artinian local rings.

**Proposition 10.** Let $R$ be an Artinian local ring with maximal ideal $M$ and let $D = R/M$. Then $R$ is Euclidean if and only if $R \cong D[X]/(X^s)$ for some integer $s$ or $R \cong V[X]/(f(X), X^s)$ for $f(X) \not\in V[X]$, where $V$ is a complete discrete valuation ring.

**Proof.** We can easily prove it by Proposition 3 and [2, Theorem 3.3].

We will study the relation between the Euclidean rings and the affine domains over a field.

**Lemma 11** (P. Samuel or Cunnea [1]). Let $k$ be an algebraically closed field, $R$ be an affine domain over the field $k$ and $K$ be the quotient field of $R$. Then $R$ is a principal ideal domain if and only if $K$ is of genus zero.

**Lemma 12.** Let $k$ be an algebraically closed field and $R$ be an affine domain over $k$. Then the following assertions are equivalent:

1. $R$ is a principal ideal domain.
2. $R \cong k[X][1/f(X)]$ where $X$ is an indeterminate and $f(X)$ is a polynomial of $k[X]$.
3. $R$ is a Euclidean ring.

**Proof.** The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are trivial. We will prove (1) $\Rightarrow$ (2). By Lemma 11, the quotient field $K$ of $R$ is of genus zero. Since $k[X] \subset R \subset k(X)$, it is well known that $R = k[X, 1/f(X)]$.

Now, for a ring $A$, we denote the units of $A$ by $A^\ast$.

**Proposition 13.** Let $R$ be a Euclidean domain containing a field $k$ with the
algorithm $\phi: R \to M$. Suppose that $R^* = k^*$ and $\phi(y + s) = \phi(y)$ for any $y \in R$ and any $s \in k$. Then there exists an element $x$ of $R$ such that $R = k[x]$.

**Proof.** We can assume that $\phi(ab) \geq \phi(a)$ for $a, b \in R$. It is easily proved that $R^* = \{a \in R/\phi(a) = \min M\}$. Put $\lambda = \min \{M - R^*\}$. Then $\lambda = \phi(x)$ for some $x \in R$. Since $k[x] \subset R$, we will prove $R \subset k[x]$. Let $a$ be any element of $R$. We will prove that $a \in R[x]$ using induction on $\phi(a)$. If $\phi(a)$ is the minimal element of $M$, then $a \in R^* = k^* \subset k[x]$. So, if $\phi(a) < n$ for $n \in M$, then we assume $a \in k[x]$. Let $\phi(a) = n$. Then there exist $b, c \in R$ such that $a = bx + c$ where $x = c$ or $\phi(c) < \phi(x)$. In the case of $x = c$, we have $a = x(b + 1)$, and so $\phi(a) \geq \phi(b + 1)$. Since $x$ is a non-unit in $R$, we see that $\phi(a) > \phi(b + 1)$. Hence $b + 1 \in k[x]$, whence $a = x(b + 1) \in k[x]$. In the case of $\phi(c) < \phi(x)$, since $\phi(c)$ is the minimal element of $M$, we have $c \in R^* = k^*$. Hence $\phi(a) = \phi(a - c) = \phi(bx) \geq \phi(b)$. Since $x$ is a non-unit, we see that $\phi(a) > \phi(b)$. Therefore $b \in k[x]$. Consequently, $a \in k[x]$.

q.e.d.

We denote the algebraic closure of a field $k$ by $\overline{k}$.

**Conjecture.** Let $k$ be a field, $R$ be a $k$-affine Euclidean domain and $R^* = k^*$. If $R \otimes_k k$ is a Euclidean ring, then $R$ is a field or $R \simeq k[X][1/f(X)]$ where $f(X)$ is a polynomial of $k[X]$.

**References**
