Torsion-free Abelian Semigroup Rings VI

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If Γ is a nonempty set which is associative under an operation on Γ , we say that Γ is an associative set. We call a torsion-free cancellative commutative associative set $S \cong \{0\}$ a semigroup. We call a commutative ring A with the identity 1 a ring. Let A be a ring, S a semigroup. The semigroup ring $A[X; S]$ of S over A is the ring of elements $a_1 X^{\alpha_1} + \cdots + a_n X^{\alpha_n}$, where $a_i \in A$ and $\alpha_i \in S$ for each i . A general reference on semigroup rings is [6]. The aim of this paper is to continue [12].

1. On the essentiality. Let A be a ring. We denote the total quotient ring of A by $q(A)$. Let D be an integral domain with the quotient field K . Let v be a valuation on K with the valuation ring V . If V is of the form D_P for some prime ideal P of D , v (resp. V) is called essential for D . Let $\{V_\lambda; A\}$ be the set of essential valuation rings for D . If $D = \bigcap_\lambda V_\lambda$, we say that D is an essential ring. We get a condition for $D[X; S]$ to be essential in this section.

LEMMA (1.1). *If $D[X; S]$ is an essential ring, then D is an essential ring.*

PROOF. Let $\{W_\lambda; A\}$ be the set of essential valuation rings for $D[X; S]$. Let K be the quotient field of D and $V_\lambda = W_\lambda|K$ for each λ . Then V_λ is an essential valuation ring for D for each λ . We have $D = \bigcap_\lambda V_\lambda$. Therefore D is an essential ring.

Let S be a semigroup. We denote the quotient group of S by $q(S)$. We set $G = q(S)$. Let V be a valuation semigroup on G . If V is of the form S_P for some prime ideal P of S , we say that V is an essential valuation semigroup for S . Let $\{V_\lambda; A\}$ be the set of essential valuation semigroups for S . If $S = \bigcap_\lambda V_\lambda$, we say that S is an essential semigroup. A similar proof to Lemma (1.1) shows the following.

LEMMA (1.2). *If $D[X; S]$ is an essential ring, then S is an essential semigroup.*

LEMMA (1.3). *Let a group G be a semigroup, F a free subgroup of G such that G/F is a torsion group. If $\{\alpha_1, \dots, \alpha_n\}$ is a finite subset of G , then $F + \sum_{i=1}^n \mathbb{Z}\alpha_i$*

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is a free subgroup of G .

PROOF. We may suppose that $n=1$. Let $\alpha \in G$, and set $H=F+Z\alpha$. Let $\{u_\lambda; \Lambda\}$ be a free basis of F . We have $\alpha = \sum_1^m l_i u_i$ for a finite subset $\{1, \dots, m\}$ of Λ , where $\{l, l_1, \dots, l_m\} \subset Z$. Set $\Sigma = \Lambda - \{1, \dots, m\}$. We have $H = (\sum_1^m Z u_i + Z\alpha) \oplus \sum_\Sigma Z u_j$. Since $\sum_1^m Z u_i + Z\alpha$ is a free subgroup of G , it follows that H is a free subgroup of G .

LEMMA (1.4) ([15, Lemma 9]). (1) Let v be an essential valuation for D . Then the canonical extension v^* of v to $q(D[X; S])$ is an essential valuation for $D[X; S]$.

(2) Let u be an essential valuation for S . Then the canonical extension u^* of u to $q(D[X; S])$ is an essential valuation for $D[X; S]$.

We denote a group which is a semigroup by G .

LEMMA (1.5). Let D be a domain with the quotient field K and S a semigroup with the quotient group G . Let F be a free subgroup of G such that G/F is a torsion group. Let f be an irreducible element of $K[X; F]$, w the f -adic valuation on $q(K[X; F])$ and t an extension of w to $q(K[X; G])$. Then t is an essential valuation for $D[X; S]$.

PROOF. Assume that $t(\varphi) \geq 0$ for $0 \neq \varphi \in q(K[X; G])$. There exists a free subgroup $F' \supset F$ of G such that $\varphi \in q(K[X; F'])$ by Lemma (1.3). Let $\mathfrak{p}$ be the center of t on $K[X; F]$ and $\mathfrak{p}'$ the center of t on $K[X; F']$. $\mathfrak{p}$ is a prime ideal of $K[X; F]$ of height 1. It follows that $\mathfrak{p}'$ is a prime ideal of $K[X; F']$ of height 1. Therefore $K[X; F']_{\mathfrak{p}'}$ is the valuation ring for $t|_{q(K[X; F'])}$. Since $\varphi \in K[X; F']_{\mathfrak{p}'}$, t is an essential valuation for $K[X; G]$. Since t is trivial both on K and on G , t is an essential valuation for $D[X; S]$.

LEMMA (1.6). If D is an essential ring and S is an essential semigroup, then $D[X; S]$ is an essential ring.

PROOF. Set $G=q(S)$ and $K=q(D)$. Let $\{v_i; I\}$ be the set of essential valuations for D and $\{u_j; J\}$ the set of essential valuations for S . Let F be a free subgroup of G such that G/F is a torsion group, $\{f_\sigma; \Sigma\}$ a complete representative system of irreducible elements of $K[X; F]$, w_σ the f_σ -adic valuation on $q(K[X; F])$ and $\{t_\tau; T\}$ the set of valuations on $q(K[X; G])$ which are extensions of members of $\{w_\sigma; \Sigma\}$. $\{v_i^*, u_j^*, t_\tau; I, J, T\}$ is a family of essential valuations for $D[X; S]$ by Lemmas (1.4) and (1.5). Let V_i^* (resp. U_j^*, T_τ) be the valuation ring for v_i^* (resp. u_j^*, t_τ) for each $i \in I$ (resp. $j \in J, \tau \in T$). Let $0 \neq \varphi \in \bigcap_i V_i^* \cap \bigcap_j U_j^* \cap T_\tau$. There exists a free subgroup $F' \supset F$ such that $\varphi \in q(K[X; F'])$ by Lemma (1.3). Let $(q_1 \cdots q_m)/(p_1 \cdots p_n)$ be the irreducible decomposition of φ in $K[X; F']$. If $n \geq 1$, let w' be the

p_1 -adic valuation on $q(K[X; F'])$. $w'|q(K[X; F])$ is equivalent to some w_σ . Therefore w' is equivalent to $t_i|q(K[X; F'])$ for some $t \in T$. It follows that $t_i(\varphi) < 0$, which is a contradiction. Therefore φ is of the form $\sum_1^m a_i X^{\alpha_i}$, where $0 \neq a_i \in K$ and $\alpha_i \in G$ for each i . Since $\varphi \in \bigcap_i V_i^*$, we have $a_i \in D$ for each i . Since $\varphi \in \bigcap_j U_j^*$, we have $\alpha_i \in S$ for each i . Hence we have $\varphi \in D[X; S]$. It follows that $D[X; S] = \bigcap_i V_i^* \bigcap_j U_j^* \bigcap_\tau T_\tau$. Therefore $D[X; S]$ is an essential ring.

Lemmas (1.1), (1.2) and (1.6) imply the following,

THEOREM (1.7). *$D[X; S]$ is an essential ring if and only if D is an essential ring and S is an essential semigroup.*

2. On the FQR and the QQR properties. Let A be a ring. A ring of the form A_M is called a quotient ring of A , where M is a multiplicative subset of A . A ring B such that $A \subset B \subset q(A)$ is called an overring of A . Let D be a domain with the quotient field K . If each overring of D is a quotient ring of D , D is said to have the QR property. If each overring of D is a finite intersection of quotient rings of D , D is said to have the FQR property. If each overring of D is an intersection of quotient rings of D , D is said to have the QQR property. Let $\mathcal{Q}$ (resp. $\mathcal{Q}_0$) be the additive semigroup of rational numbers (resp. nonnegative rational numbers). We know the following results.

LEMMA (2.1) (A part of [10, §14]). *Let D be a domain and G a group which is a semigroup. The following four conditions are equivalent.*

- (1) $D[X; G]$ has the QR property.
- (2) $D[X; G]$ has the FQR property.
- (3) $D[X; G]$ has the QQR property.
- (4) D is a field and G is isomorphic to a subgroup of $\mathcal{Q}$.

LEMMA (2.2) ([11, Proposition (8.9)]). *$D[X; S]$ has the QR property if and only if D is a field and S is isomorphic either to a subgroup of $\mathcal{Q}$ or to a Prüfer subsemigroup of $\mathcal{Q}_0$.*

We get a condition for a semigroup ring to have the FQR property and a condition to have the QQR property in this section. Let $0 \neq f \in A[X; S]$. The ideal of S generated by $\text{Supp}(f)$ is denoted by $e(f)$. $q(S)$ is regarded as a subset of $q(A[X; S])$ in a canonical way.

LEMMA (2.3). *Let k be a field, S a subsemigroup of $\mathcal{Q}_0$. If $k[X; S]$ has the QQR property, then S is a Prüfer subsemigroup of $\mathcal{Q}_0$.*

PROOF. Set $G = q(S)$ and $G_0 = G \cap \mathcal{Q}_0$. Suppose that $S \not\subseteq G_0$. We have $k[X; G_0] = \bigcap_I k[X; S]_{M_i}$ for a family $\{M_i; I\}$ of multiplicative subsets of $k[X; S]$. We have $k[X; S]_{M_i} \cap G \supset G_0$ for each i . Suppose that $k[X; S]_{M_i} \cap G = G_0$ for some

i. We take $s_0 \in G_0$ such that $s_0 \bar{\in} S$. We have $X^{s_0}m = f$ for some $m \in M_i$ and some $f \in k[X; S]$. We may assume that the subgroup of G generated by $\{s_0\} \cup e(m) \cup e(f)$ is Z . The order $o(m)$ of m is a positive integer. There exists $0 < h \in Z_0$ such that $\{h, h+1, h+2, \dots\} \subset S$. We have $X^{-1}m^{h+1} \in k[X; S]$, and hence $X^{-1} = (X^{-1}m^{h+1})/m^{h+1} \in k[X; S]_{M_i} \cap G$. It follows that $-1 \in G_0$, which is a contradiction. Therefore $k[X; S]_{M_i} \cap G \not\supseteq G_0$ for each i . Then $k[X; S]_{M_i} \cap G$ is a subsemigroup of $\mathcal{Q}$ which contains a negative rational number for each i . Therefore $k[X; S]_{M_i} \cap G = G$ for each i . It follows that $G_0 = \bigcap_i (k[X; S]_{M_i} \cap G) = G$; which is a contradiction. We have proved that $S = G_0$. Therefore S is a Prüfer subsemigroup of $\mathcal{Q}_0$.

THEOREM (2.4). *The following four conditions are equivalent.*

- (1) $D[X; S]$ has the QR property.
- (2) $D[X; S]$ has the FQR property.
- (3) $D[X; S]$ has the QQR property.
- (4) D is a field and S is isomorphic either to a subgroup of $\mathcal{Q}$ or to a Prüfer subsemigroup of $\mathcal{Q}_0$.

PROOF. Lemma (2.2) shows that (4) implies (1). Suppose that $D[X; S]$ has the QQR property. Set $G = q(S)$. $D[X; G]$ has the QQR property. Then D is a field and G is isomorphic to a subgroup of $\mathcal{Q}$ by Lemma (2.1). Lemma (2.3) shows that S is isomorphic either to a subgroup of $\mathcal{Q}$ or to a Prüfer subsemigroup of $\mathcal{Q}_0$.

3. On the $n(1/2)$ -generator property. Let A be a ring and $0 < n \in Z_0$. If each finitely generated ideal a of A is generated by n elements, A is said to have the n -generator property. If $A/(a)$ has the n -generator property for each nonzero nonunit $a \in A$, A is said to have the $n(1/2)$ -generator property. A condition for a semigroup ring to have the n -generator property was obtained in [3]. We will get a condition for a semigroup ring to have the $n(1/2)$ -generator property in this section after [3]. We denote the nilradical of A by N .

LEMMA (3.1). *Let S be a subsemigroup of $\mathcal{Q}_0$ with $o(S) = k < \infty$. If $A[X; S]$ has the $n(1/2)$ -generator property, then $A[X; S]$ has the n -generator property.*

PROOF. We can modify the proof “(1) $\Rightarrow$ (2)” of [3, Theorem 8]. Namely, if $A[X; S]$ has the $n(1/2)$ -generator property, then it has the $(n+1)$ -generator property. It follows that $\dim A = 0$. Let $\{a_1, \dots, a_m\} \subset N$ with $(m+1)k > n$. Set $S_1 = S \cap Z_0$. We may assume that $o(S_1) = k$ and $q(S_1) = Z$. We have $\{g+i; i \in Z_0\} \subset S_1$ for some $0 < g \in Z_0$. We set $\beta = \{a_j X^{g+jk-i}; 1 \leq j \leq m+1, 1 \leq i \leq k\}$, $I = (\beta, X^*)A[X; S]$ and $\bar{A} = A[X; S]/(X^*)$, where $a_{m+1} = 1$ and $s \geq g + (m+1)k$. Then $\bar{A}$ has dimension 0 and has the n -generator property. Therefore there exists

a decomposition $\bar{A} = \bar{A}e_1 \oplus \cdots \oplus \bar{A}e_n$ such that $\bar{I}e_j$ is generated by n elements from $\bar{\beta}e_j$ for each j . We have $e_j \in A$ for each j . We now take $A = Ae_j$. We have $a_i X^i \in (\beta - \{a_i X^i\}, X^s)A[X; S]$ for some $a_i X^i \in \beta$. It follows that $a_i = 0$ if $i = 1$ and $a_i \in (a_1, \dots, a_{i-1})$ if $i > 1$. [3, Theorem 8] shows that $A[X; S]$ has the n -generator property.

LEMMA (3.2). *Let G be a subgroup of $\mathcal{Q}$. If $A[X; G]$ has the $n(1/2)$ -generator property, then $A[X; G]$ has the n -generator property.*

PROOF. We may assume that $G \supset Z$. Since $A[X; G]$ has the $(n+1)$ -generator property, the dimension of A is zero. Let $\{a_1, \dots, a_n\} \subset N - \{0\}$ and $I = (a_1, \dots, a_n)$. Take $0 \neq a \in I$ such that $aI = (0)$. We set $Y = 1 - X$ and $\beta = \{a_i Y^i; 1 \leq i \leq n+1\}$, where $a_{n+1} = 1$. There exists $\{f_1, \dots, f_n\} \subset A[X; G]$ such that $\beta A[X; G] = (f_1, \dots, f_n, aY^{n+2})A[X; G]$. f_i is of the form $Y^{n+1}F_i + H_i$ and Y^{n+1} is of the form $\sum_{i=1}^n f_i G_i + H$, where $H_i \in IA[X; G]$ and $H \in IA[X; G]$ for each i . We have $1 = \sum f_i G_i + H'$ for some $H' \in IA[X; G]$. It follows that

$$\begin{aligned} aY^{n+2} &= \sum aY^{n+2}F_i G_i + aY^{n+2}H' \\ &= \sum aYf_i G_i - \sum aYH_i G_i + aY^{n+2}H' \\ &= \sum aYf_i G_i \in (f_1, \dots, f_n)A[X; G]. \end{aligned}$$

Therefore $\beta A[X; G]$ is generated by n elements $f_1, \dots, f_n$. There exists a subgroup $H \supset Z$ of G such that $\beta A[X; H]$ has n generators over $A[X; H]$. We may assume that $\beta A[X; Z]$ has n generators, where $\beta = \{a_i Y^i; 1 \leq i \leq n+1\}$ and $Y = 1 - X^k$. If $T = 1 - X$, we have $Y^i = [1 - (1 - T)^k]^i = f_i(T)$ for $i \geq 1$, where $f_i(0) = 0$. We may assume that each coefficient of $f_i(T)$ is either a unit or a nilpotent of A . There exists $0 < m \in \mathbb{Z}_0$ such that $(T^m): (Y^i) \subset (T)$ for $1 \leq i \leq n+1$. If we set $\bar{A} = A[X; Z]/(T^m)$, then we have a decomposition $\bar{A} = \bar{A}e_1 \oplus \cdots \oplus \bar{A}e_n$ such that $\bar{\beta}\bar{A}e_j$ can be generated by n elements from $\bar{\beta}e_j$. Since $\bar{A} = A[X]/(T^m)$, e_i belongs to A for each i . We now take $A = Ae_j$. There exists i such that $a_i Y^i \in (\beta - \{a_i Y^i\}, T^m)A[T]$. If we set $A' = A/(a_1, \dots, a_{i-1})$ and $a' = a + (a_1, \dots, a_{i-1})$ for each $a \in A$, then $a'_i Y^i$ is of the form $\sum_{j=i+1}^{n+1} a'_j Y^j h_j(T) + h(T)T^m$. Then $a'_i - \sum_{j=i+1}^{n+1} a'_j Y^{j-i} h_j(T)$ belongs to $(T^m): (Y^i)$, hence to (T) . It follows that $a'_i = 0$. Therefore $a_i \in (a_1, \dots, a_{i-1})$. [3, Theorem 11] shows that $A[X; G]$ has the n -generator property.

[3, Corollary 12] and Lemmas (3.1) and (3.2) imply the following,

THEOREM (3.3). *$A[X; S]$ has the $n(1/2)$ -generator property if and only if $A[X; S]$ has the n -generator property.*

4. On the rank n and the rank $n(1/2)$. Let A be a ring and $0 < n \in \mathbb{Z}$. If each ideal of A is generated by n elements, we say that A has the rank n . If $A/(a)$

has the rank n for each nonzero nonunit $a \in A$, we say that A has the rank $n(1/2)$. Let $\mathfrak{a}$ be a finitely generated ideal of A . We denote the number of elements in a minimal generating set for $\mathfrak{a}$ by $d(\mathfrak{a})$. (We set $d((0))=0$.) If I is a finitely generated nil ideal of A , we set $\nu(I) = \sum_1^{\infty} d(I^i/I^{i+1})$. We know the following result.

LEMMA (4.1) ([12, Section 5]). $A[X; S]$ has the rank n if and only if either (1) or (2) of the following holds.

- (1) $S \cong \mathbb{Z}$ and there exists a decomposition $A = A_1 \oplus \cdots \oplus A_h$ such that (A_i, N_i) is a Noetherian local ring with $\nu(N_i) < n$, where N_i is the nilradical of A_i for each i .
- (2) S is isomorphic to a subsemigroup of $\mathbb{Z}_0$ and there exists a decomposition $A = A_1 \oplus \cdots \oplus A_h$ such that (A_i, N_i) is a Noetherian local ring with $(\nu(N_i) + 1)o(S) \leq n$ for each i .

We will get another characterization for a semigroup ring to have the rank n and a condition for a semigroup ring to have the rank $n(1/2)$.

THEOREM (4.2). $A[X; S]$ has the rank n if and only if either (1) or (2) of the following holds.

- (1) $S \cong \mathbb{Z}$ and there exists a decomposition $A = A_1 \oplus \cdots \oplus A_h$ such that (A_i, N_i) is a Noetherian local ring such that if $\{a_1, \dots, a_n\} \subset N_i$, then $a_1 = 0$ or $a_j \in (a_1, \dots, a_{j-1})$ for some $j > 1$ for each i .
- (2) S is isomorphic to a subsemigroup of $\mathbb{Z}_0$ and there exists a decomposition $A = A_1 \oplus \cdots \oplus A_h$ such that (A_i, N_i) is a Noetherian local ring such that if $\{a_1, \dots, a_m\} \subset N_i$ with $(m+1)o(S) > n$, then $a_1 = 0$ or $a_j \in (a_1, \dots, a_{j-1})$ for some $j > 1$ for each i .

PROOF. If $A[X; S]$ has the rank n , then S is isomorphic either to $\mathbb{Z}$ or to a subsemigroup of $\mathbb{Z}_0$ by Lemma (4.1). There exists a primary decomposition $(0) = Q_1 \cap \cdots \cap Q_h$ in A such that $A = A/Q_1 \oplus \cdots \oplus A/Q_h$. $(A/Q_i)[X; S]$ has the n -generator property for each i . Then (1) or (2) holds by [3, Corollary 12]. The converse follows also from [3, Corollary 12].

THEOREM (4.3). $A[X; S]$ has the rank $n(1/2)$ if and only if $A[X; S]$ has the rank n .

PROOF. If $A[X; S]$ has the rank $n(1/2)$, then $A[X; S]$ is a Noetherian ring with the $n(1/2)$ -generator property. Theorem (3.3) shows that $A[X; S]$ is a Noetherian ring with the n -generator property. Therefore $A[X; S]$ has the rank n . The converse is obvious.

5. On the M-ring. Let D be a domain. If (1) and (2) of the following properties hold for D , we say that D is an M-ring.

- (1) If $\{\mathfrak{a}_i; i = 1, 2, \dots\}$ is a family of divisorial ideals of D such that $\mathfrak{a}_i \subset \mathfrak{a}_{i+1} \subset D$ for each i , then there exists n such that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$.

(2) (a, b) is a divisorial ideal of D for each two elements $a, b \in D$.

We know the following result.

LEMMA (5.1). (1) ([10, §14, Proposition 27]). $D[X; G]$ is an M-ring if and only if D is a field and $G \cong \mathbb{Z}$.

(2) ([11, Proposition (2.10)]). If $D[X; S]$ is an M-ring, then D is a field and S is isomorphic either to $\mathbb{Z}$ or to a subsemigroup of $\mathbb{Z}_0$.

If (1) and (2) of the following properties hold for S , we say that S is an M-semigroup.

(1) If $\{a_i; i=1, 2, \dots\}$ is a family of divisorial ideals of S such that $a_i \subset a_{i+1} \subset S$ for each i , then there exists n such that $a_n = a_{n+1} = \dots$.

(2) (a, b) is a divisorial ideal of S for each two elements $a, b \in S$.

LEMMA (5.2). Let a be a fractional ideal of S . Then we have $(aD[X; S])^v \cap G = a^v$, where $G = q(S)$.

PROOF. Set $b = aD[X; S]$. Let $s \in a^v$ and $b \subset \xi D[X; S]$ for $\xi \in q(D[X; S])$. We have $\xi = X^t/f$ for some $t \in a$ and $f \in D[X; S]$. If we set $f = \sum_1^n a_i X^{s_i}$ with $0 \neq a_i \in D$ for $1 \leq i \leq n$, we have $a \subset t - s_i + S$ for each i . Therefore $X^s/\xi \in D[X; S]$, and hence $X^s \in b^v$. It follows that $a^v \subset b^v \cap G$. Conversely let $s \in b^v \cap G$ and $a \subset (\alpha)$ for $\alpha \in G$. Since $b \subset X^\alpha D[X; S]$, we have $X^s \in X^\alpha D[X; S]$. Therefore $s \in (\alpha)$, and hence $s \in a^v$. It follows $b^v \cap G \subset a^v$, and hence $b^v \cap G = a^v$.

PROPOSITION (5.3). If $D[X; S]$ is an M-ring, then S is an M-semigroup.

PROOF. Let $a_i \subset S$ be a divisorial ideal of S such that $a_1 \subset a_2 \subset \dots$. There exists n such that $(a_i D[X; S])^v = (a_{i+1} D[X; S])^v$ for each $i \geq n$. Then $a_i = a_{i+1}$ for each $i \geq n$ by Lemma (5.2). Let $s_1, s_2 \in S$ and $a = (s_1, s_2)$. By Lemma (5.2) we have $a^v = (aD[X; S])^v \cap q(S) = aD[X; S] \cap q(S) = a$, and hence $a^v = a$. Therefore S is an M-semigroup.

Let k be a field and let $S = \{0, 2, 3, 4, \dots\}$. [12, Example (6.6)] shows that $k[X; S]$ is an M-ring. Proposition (5.3) shows that S is an M-semigroup. Moreover we have the following,

PROPOSITION (5.4). If S is a subsemigroup of $\mathbb{Z}_0$ with $O(S) = 2$, then S is an M-semigroup.

PROOF. We may assume that $q(S) = \mathbb{Z}$. We take the minimal number h such that $\{h+i; i \in \mathbb{Z}_0\} \subset S$. h is of the form $2m$ for $m \in \mathbb{Z}_0$. Note that (α, β) is divisorial if and only if $(\alpha+\gamma, \beta+\gamma)$ is divisorial for each $\alpha, \beta, \gamma \in \mathbb{Z}$. We must show that the ideal $a = (s, s')$ is a divisorial ideal of S for each $s, s' \in S$. We may assume that $s < s'$. If $s' - s$ is an even number, then $(0, s' - s)$ is a divisorial ideal of S . Assume that $s' - s$ is an odd number. If $s' - s \geq h$, then $(0, s' - s)$ is a di-

visorial ideal of S , and hence α is a divisorial ideal of S . Assume that $s' - s < h$. We will show that $(s + h + 1 - s', h + 1)$ is a divisorial ideal of S . Hence we may assume that $s' = h + 1$. We have $s' = 2m + 1$ and $s = 2l$ for an $l \in \mathbb{Z}_0$. We have

$$S = \{0, 2.1, 2.2, \dots, 2.l, \dots, 2m, 2m + 1, 2m + 2, \dots\}.$$

If we set

$T = \{2l, \dots, 2m, 2m + 1, 2m + 2, \dots\}$, we have $\alpha = T$. We have $\{\alpha \in \mathbb{Z}; \alpha \subset (\alpha)\} = \{s - t; t \in T\}$. Therefore we have

$$\alpha^v = (0) \cap (-2) \cap (-4) \cap \dots \cap (-2m + 2l) \cap (-2m + 2l - 1) \cap (-2m + 2l - 2) \cap \dots.$$

We have

$$\begin{aligned} 2(l - 1) \bar{\in} (-2m + 2l - 1), 2(l - 2) \bar{\in} (-2m + 2l - 3), \dots, 2 \bar{\in} (-2m + 3), \\ 0 \bar{\in} (-2m + 1). \end{aligned}$$

It follows that $\alpha^v = \alpha$, and hence S is an M -semigroup.

6. On the N-2 property. Let D be a domain with the quotient field K . If the integral closure of D in each extension field L of K of finite degree is a finitely generated D -module, D is said to have the N-2 property.

LEMMA (6.1). *If $D[X; S]$ has the N-2 property, then D has the N-2 property.*

PROOF. Let K be the quotient field of D , L an extension field of K of finite degree, $\bar{D}$ the integral closure of D in L and $\bar{S}$ the integral closure of S . Then $q(L[X; S])$ is an extension field of $q(D[X; S])$ of finite degree and $\bar{D}[X; \bar{S}]$ is the integral closure of $D[X; S]$ in $q(L[X; S])$. We have $\bar{D}[X; \bar{S}] = \sum_1^n D[X; S]F_i$, where $F_i \in \bar{D}[X; \bar{S}]$ for each i . If we set $F_i = \sum x_{ij} X^{a_{ij}}$ for each i , then $\bar{D} = \sum_{i,j} D x_{ij}$. Therefore D has the N-2 property.

Let S be a semigroup and T a nonempty set. Assume that $s + t \in T$ is determined for each $s \in S$ and $t \in T$. If $0 + t = t$ and $(s + s') + t = s + (s' + t)$ for each $t \in T$ and $s, s' \in S$, then we say that T is an S -module. Let T be an S -module. If $T = \bigcup_1^n (S + t_i)$ for some finite subset $\{t_1, \dots, t_n\}$ of T , we say that T is a finitely generated S -module. Set $G = q(S)$. If the integral closure of S in each extension semigroup group G' of G which is a finitely generated module over G is a finitely generated S -module, we say that S has the N-2 property.

LEMMA (6.2). *If $D[X; S]$ has the N-2 property, then S has the N-2 property.*

PROOF. Let $G = q(S)$ and let G' be an extension semigroup group of G which is a finitely generated G -module. Then $q(D[X; G'])$ is an extension field of $q(D[X; S])$ of finite degree. If D' is the integral closure of D and S' the integral closure of S in G' , then $D'[X; S']$ is the integral closure of $D[X; S]$ in $q(D[X; G'])$.

We have $D'[X; S'] = \sum_1^n D[X; S]F_i$ for a finite subset $\{F_1, \dots, F_n\}$ of $D'[X; S']$. If we set $F_i = \sum b_{ij}X^{\alpha_{ij}}$ for each i , then $S' = \bigcup_{i,j} (S + \alpha_{ij})$. Therefore S has the N-2 property.

THEOREM (6.3). *Assume that $D[X; S]$ is a Noetherian ring. Then $D[X; S]$ has the N-2 property if and only if D has the N-2 property and S has the N-2 property.*

PROOF. The necessity follows from Lemmas (6.1) and (6.2). Conversely assume that D has the N-2 property and S has the N-2 property. $D[X; S]$ has the N-1 property by Theorem (8.4) of Section 8. If D has the characteristic zero, it follows that $D[X; S]$ has the N-2 property. Assume that D has the characteristic $p > 0$. We set $K = q(D)$, $G = q(S)$, $G = Z\alpha_1 \oplus \dots \oplus Z\alpha_n$, $H = Q\alpha_1 \oplus \dots \oplus Q\alpha_n$. Let $\bar{K}$ be the algebraic closure of K and Ω the algebraic closure of $q(\bar{K}[X; H])$. If L is a radical extension of $q(K[X; G])$ of finite degree, we may assume that $L \subset \Omega$ and $L^{p^e} \subset q(K[X; G])$ for some e . We have $L = q(K[X; G])(x_1, \dots, x_m)$, where x_i is integral over $D[X; S]$ for each i . If $\bar{D}$ is the integral closure of D and $\bar{S}$ the integral closure of S , we have $x_i^{p^e} \in \bar{D}[X; \bar{S}]$ for each i . Hence x_i is of the form $\sum b_{ij}X^{\beta_{ij}}$ for $b_{ij} \in \bar{K}$ and $\beta_{ij} \in H$ for each i, j . If we set $K' = K(b_{ij}; i, j)$ and $G' = G + \sum_{i,j} Z\beta_{ij}$, then K' is an extension field of K of finite degree and G' is a finitely generated G -module. If D' is the integral closure of D in K' and S' the integral closure of S in G' , then $D'[X; S']$ is a finitely generated $D[X; S]$ -module. The integral closure I of $D[X; S]$ in L is a subring of $D'[X; S']$. Therefore I is a finitely generated $D[X; S]$ -module. Therefore $D[X; S]$ has the N-2 property by [16, Section 31, Corollary 2].

7. On the J-1 property. Let A be a ring. We set $\text{Reg}(A) = \{P \in \text{Spec}(A); A_P \text{ is a regular local ring}\}$. If A is a Noetherian ring and $\text{Reg}(A)$ is an open set of $\text{Spec}(A)$, we say that A has the J-1 property. We know the following results.

LEMMA (7.1) ([11, Theorem (2.13)]). *$A[X; G]$ has the J-1 property if and only if A has the J-1 property and G is a finitely generated group.*

If a ring A has a unique maximal ideal, A is called a local ring.

LEMMA (7.2) ([10, §3, Proposition 1]). *Let A be a local ring with the unique maximal ideal M . If $A[X; G]_{MA[X; G]}$ is a regular local ring, then A is a regular local ring.*

If $\mathfrak{a}$ is an ideal of A , we denote $\{P \in \text{Spec}(A); P \supset \mathfrak{a}\}$ by $D(\mathfrak{a})$.

PROPOSITION (7.3). *If $\text{Reg}(A) = D(\mathfrak{a})$ for an ideal $\mathfrak{a}$ of A , then $\text{Reg}(A[X; S]) \subset D(\mathfrak{a}A[X; S])$.*

PROOF. Let $\mathfrak{p} \in \text{Reg}(A[X; S])$, $P = \mathfrak{p} \cap A$, $G = q(S)$, $B = A_P[X; S]$, $C =$

$A_P[X; G]$, $E = B_{pB}$ and $F = A[X; S]_p$. Then we have $C_{PC} = B_{PB} = E_{PE} = F_{PF}$. Since F is a regular local ring, F_{PF} is a regular local ring, and hence C_{PC} is a regular local ring. Then A_P is a regular local ring by Lemma (7.2). Therefore $P \supsetneq \mathfrak{a}$, and hence $p \supsetneq \mathfrak{a}A[X; S]$. Therefore $\text{Reg}(A[X; S]) \subset D(\mathfrak{a}A[X; S])$.

LEMMA (7.4). *Let A be a multiplicative subset of A . If A has the J-1 property, then A_M has the J-1 property.*

PROPOSITION (7.5). *$A[X_1, \dots, X_n]$ has the J-1 property if and only if A has the J-1 property.*

PROOF. If $A[X_1, \dots, X_n]$ has the J-1 property, then $A[X; \overbrace{Z \oplus \dots \oplus Z}^n]$ has the J-1 property by Lemma (7.4). Hence A has the J-1 property by Lemma (7.1). Conversely assume that A has the J-1 property. We have $\text{Reg}(A) = D(\mathfrak{a})$ for an ideal $\mathfrak{a}$ of A . If $p \in D(\mathfrak{a}A[X_1, \dots, X_n])$ and $P = p \cap A$, then $P \in D(\mathfrak{a})$. Therefore A_P is a regular local ring. If we set $B = A_P[X_1, \dots, X_n]$, then B_{pB} is a regular local ring. Therefore $A[X_1, \dots, X_n]_p$ is a regular local ring. It follows that $D(\mathfrak{a}A[X_1, \dots, X_n]) \subset \text{Reg}(A[X_1, \dots, X_n])$, and hence $D(\mathfrak{a}A[X_1, \dots, X_n]) = \text{Reg}(A[X_1, \dots, X_n])$ by Proposition (7.3). Therefore $A[X_1, \dots, X_n]$ has the J-1 property.

8. On the N-1 property. Let D be a domain. If the integral closure of D is a finitely generated D -module, we say that D has the N-1 property. We know the following result.

LEMMA (8.1) ([9, §6, Proposition 11]). *$D[X; G]$ has the N-1 property if and only if D has the N-1 property.*

If the integral closure of a semigroup S is a finitely generated S -module, S is said to have the N-1 property.

LEMMA (8.2). *If $D[X; S]$ has the N-1 property, then D has the N-1 property.*

PROOF. Set $G = q(S)$. Since $D[X; G]$ is a quotient ring of $D[X; S]$, $D[X; G]$ has the N-1 property. By Lemma (8.1), D has the N-1 property.

LEMMA (8.3). *If $D[X; S]$ has the N-1 property, then S has the N-1 property.*

PROOF. If $\bar{D}$ is the integral closure of D and $\bar{S}$ the integral closure of S , then $\bar{D}[X; \bar{S}]$ is the integral closure of $D[X; S]$. There exists a finite subset $\{F_1, \dots, F_n\}$ of $\bar{D}[X; \bar{S}]$ such that $\bar{D}[X; \bar{S}] = \sum_{i=1}^n D[X; S]F_i$. If we set $F_i = \sum x_{ij}X^{\alpha_{ij}}$ for each i , then $\bar{S} = \bigcup_{i,j} (S + \alpha_{ij})$. Therefore $\bar{S}$ has the N-1 property.

THEOREM (8.4). *$D[X; S]$ has the N-1 property if and only if D has the N-1 property and S has the N-1 property,*

PROOF. The necessity follows from Lemmas (8.2) and (8.3). Conversely assume that D has the N-1 property and S has the N-1 property. If $\bar{D}$ is the integral closure of D and $\bar{S}$ is the integral closure of S , we have $\bar{D} = \sum_{i=1}^m Dx_i$ and $\bar{S} = \bigcup_{j=1}^n (S + \alpha_j)$. Then $\bar{D}[X; \bar{S}] = \sum_{i,j} D[X; S]x_iX^{\alpha_j}$. Therefore $D[X; S]$ has the N-1 property.

9. On the a.c.c.p. property. Let D be a domain. Assume that if $\{a_i; i=1, 2, \dots\} \subset D$ such that $(a_1) \subset (a_2) \subset \dots$, then there exists n such that $(a_n) = (a_{n+1}) = \dots$. Then we say that D has the a.c.c.p. property. We know the following result.

LEMMA (9.1) ([10, §14, Proposition 25]). $D[X; G]$ has the a.c.c.p. if and only if D has the a.c.c.p. and G has type $(0, 0, 0, \dots)$.

LEMMA (9.2). If $D[X; S]$ has the a.c.c.p., then D has the a.c.c.p.

Assume that if $\{s_i; i=1, 2, \dots\} \subset S$ such that $(s_1) \subset (s_2) \subset \dots$, then there exists n such that $(s_n) = (s_{n+1}) = \dots$. Then we say that S has the a.c.c.p. property.

LEMMA (9.3). If $D[X; S]$ has the a.c.c.p., then S has the a.c.c.p.

Let G be the quotient group of S and $\{u_\lambda; \lambda\}$ an independent basis of G . Let Σ be the family of a nonempty finite subset σ of λ such that $(\sum_{\sigma} Qu_i) \cap S \neq \{0\}$.

LEMMA (9.4) ([13, Section 2, Lemma 6]). Let G be a group, H a subgroup of G and $0 \neq f \in D[X; H]$ with $O(f) = 0$. If F, G are elements of $D[X; G]$ with $O(F) = 0$ such that $f = FG$, then we have $F, G \in D[X; H^*]$, where H^* is the integral closure of H in G .

PROPOSITION (9.5). $D[X; S]$ has the a.c.c.p. if and only if S has the a.c.c.p. and $D[X; (\sum_{\sigma} Qu_i) \cap S]$ has the a.c.c.p. for each $\sigma \in \Sigma$.

PROOF. Assume that $D[X; S]$ has the a.c.c.p. Set $T = (\sum_{\sigma} Qu_i) \cap S$ for $\sigma \in \Sigma$. If $\{f_1, f_2, \dots\} \subset D[X; T]$ and $f_1D[X; T] \subset f_2D[X; T] \subset \dots$, then $f_1D[X; S] \subset f_2D[X; S] \subset \dots$. There exists n such that $f_i/f_{i+1} = a_iX^{s_i}$, where a_i is a unit of D and s_i is a unit of S for each $i \geq n$. Since $f_i, f_{i+1} \in D[X; \sum_{\sigma} Qu_i]$, we have $s_i \in \sum_{\sigma} Qu_i$. Therefore s_i is a unit of T for each $i \geq n$. Hence $f_nD[X; T] = f_{n+1}D[X; T] = \dots$. Therefore $D[X; T]$ has the a.c.c.p. Conversely, let $\{f_1, f_2, \dots\} \subset D[X; S]$ and $f_1D[X; S] \subset f_2D[X; S] \subset \dots$. We may assume that $f_1 \neq 0$. There exists $\sigma \in \Sigma$ such that $f_1 \in D[X; (\sum_{\sigma} Qu_i) \cap S]$. Set $(\sum_{\sigma} Qu_i) \cap S = T$ and $O(f_i) = s_i$. We have $s_1 + S \subset s_2 + S \subset \dots$. There exists n such that $s_n + S = s_{n+1} + S = \dots$. Hence we may assume that $s_1 = s_2 = \dots = s$. If we set $f_i = f_{i+1}g_{i+1}$ for each i , then $X^{-s}f_1 = (X^{-s}f_2)g_2$. Then $f_2, g_2 \in D[X; T]$ by Lemma (9.4), and $f_1D[X; T] \subset f_2D[X; T]$. Similarly $f_iD[X; T] \subset f_{i+1}D[X; T]$ for each i . There exists n such that

$f_n D[X; T] = f_{n+1} D[X; T] = \dots$. g_i is of the form $a_i X^{t_i}$, where a_i is a unit of D and t_i is a unit of T for each $i \geq n$. Since t_i is a unit of S , we have $f_n D[X; S] = f_{n+1} D[X; S] = \dots$. Therefore $D[X; S]$ has the a.c.c.p.

Proposition (9.5) shows that conditions for $D[X; S]$ to have the a.c.c.p. reduce to the case where the torsion-free rank of S is finite.

10. *-operations. If $F(S)$ is the set of fractional ideals of our semigroup S , a mapping $\alpha \mapsto \alpha^*$ of $F(S)$ into $F(S)$ is called a *-operation on S if the following conditions hold for each $\alpha \in q(S)$ and all α, β in $F(S)$:

- (1) $(\alpha)^* = (\alpha)$; $(\alpha + \alpha)^* = \alpha + \alpha^*$.
- (2) $\alpha \subset \alpha^*$; if $\alpha \subset \beta$, then $\alpha^* \subset \beta^*$.
- (3) $(\alpha^*)^* = \alpha^*$.

Let $*$ be a *-operation on S . A fractional ideal α is called a *-ideal if $\alpha^* = \alpha$. A *-ideal α is said to be *-finite if there exists a finitely generated element β of $F(S)$ such that $\alpha = \beta^*$. $*$ is said to be a. b. if for all $\alpha, \beta, \gamma \in F(S)$ such that α^* is *-finite, the containment $(\beta + \alpha)^* \subset (\gamma + \alpha)^*$ implies that $\beta^* \subset \gamma^*$; $*$ is e.a.b. if the same statement is valid for all $\alpha, \beta, \gamma \in F(S)$, each of α^*, β^* and γ^* is *-finite. If S admits an e.a.b. *-operation, then S is integrally closed. If S (resp. D) admits an a.b. *-operation $*$, then S (resp. D) is called a *-regularly integrally closed semigroup (resp. *-regularly integrally closed ring). If $\alpha \in F(S)$, we denote the fractional ideal $\{\alpha \in q(S); \alpha + \alpha \subset s\}$ by α^{-1} , and denote $(\alpha^{-1})^{-1}$ by α^v . Then v is a *-operation on S . A v -regularly integrally closed semigroup is called a regularly integrally closed semigroup. If v is an e.a.b. *-operation on S , then S is regularly integrally closed. Two *-operations $*_1$ and $*_2$ on S are said to be equivalent if $\alpha^{*_1} = \alpha^{*_2}$ for each finitely generated fractional ideal α of S . We call a subsemigroup T of $q(S)$ containing S an oversemigroup of S .

PROPERTY (10.1). Assume that $\{S_\lambda; \lambda \in A\}$ is a family of oversemigroups of S such that $S = \bigcap_{\lambda} S_\lambda$. We define α^* to be $\bigcap_{\lambda} (\alpha + S_\lambda)$ for each $\alpha \in F(S)$. Then the mapping $\alpha \mapsto \alpha^*$ is a *-operation on S and $\alpha + S_\lambda = \alpha^* + S_\lambda$ for each $\alpha \in F(S)$ and for each λ . If each S_λ is a valuation semigroup, then this *-operation is a.b.

If $\{V_\lambda; \lambda \in A\}$ is a family of valuation oversemigroups of an integrally closed semigroup S such that $S = \bigcap_{\lambda} V_\lambda$, then the *-operation $\alpha \mapsto \bigcap_{\lambda} (\alpha + V_\lambda)$ is called a w-operation induced by the family $\{V_\lambda; \lambda \in A\}$. If $\{V_\lambda; \lambda \in A\}$ is the set of all valuation oversemigroups of S , the w-operation induced by the family $\{V_\lambda; \lambda \in A\}$ is called the b-operation on S . If α is a nonzero ideal of $D[X; S]$, the ideal of S generated by $\{e(f); 0 \neq f \in \alpha\}$ is denoted by $e(\alpha)$.

LEMMA (10.2) (Dedekind-Mertens Lemma for semigroups, cf. [7, Proposition 6.2]). If $f, g \in D[X; S] - \{0\}$, then there exists a positive integer m such that

$(m+1)e(f) + e(g) = me(f) + e(fg)$. In fact m can be chosen to be $|\text{Supp}(g)|$.

Lemma (10.2) implies the following:

LEMMA (10.3). *Let $*$ be an e.a.b. $*$ -operation on S . Then we have $e(fg)^* = (e(f) + e(g))^*$ for all $f, g \in D[X; S] - \{0\}$.*

Lemma (10.3) implies the following:

PROPOSITION (10.4). *Let D be a domain and let $*$ be an e.a.b. $*$ -operation on a semigroup S with quotient group G .*

- (1) *If $f, g, f', g' \in D[X; S] - \{0\}$ such that $f/g = f'/g'$, then $e(f)^* \subset e(g)^*$ if and only if $e(f')^* \subset e(g')^*$.*
- (2) *Let $S_*^D = \{0\} \cup \{f/g; f, g \in D[X; S] - \{0\}\}$ and $e(f)^* \subset e(g)^*$. Then S_*^D is an overring of $D[X; S]$ such that $S_*^D \cap G = S$ and $S_*^D = S_*^{q(D)}$.*
- (3) *S_*^D is a Bezout ring.*
- (4) *If α is a finitely generated ideal of S , then $\alpha S_*^D \cap G = \alpha^*$ and $\alpha^* S_*^D = \alpha S_*^D$.*

S_*^D is called the Kronecker function ring of S with respect to the operation $*$ and the domain D . The Kronecker function ring S_b^D with respect to the b -operation is called the Kronecker function ring of S .

THEOREM (10.5). *Let S be an integrally closed semigroup with quotient group G and let $*$ be an e.a.b. $*$ -operation on S . If v^* is a valuation on $q(D[X; S])$ which is nonnegative on S_*^D , then v^* is the natural extension of $G|v^*$ to $q(D[X; S])$.*

PROPOSITION (10.6). *Let S be an integrally closed semigroup with quotient group G and let $\{V_\lambda; \lambda \in \Lambda\}$ be a family of valuation oversemigroups of S such that $S = \bigcap_\lambda V_\lambda$. Let w be the w -operation on S induced by the family $\{V_\lambda; \lambda \in \Lambda\}$, and for each $\lambda \in \Lambda$, let V^* be the natural extension of V_λ to $q(D[X; S])$. Then $S_*^D = \bigcap_\lambda V_\lambda^*$.*

PROPERTY (10.7). *Each e.a.b. $*$ -operation on S is equivalent to a w -operation on S , and hence to an a.b. operation. The Kronecker function ring S_*^D with respect to $*$ contains S_b^D . If S is regularly integrally closed, then S_*^D is contained in S_b^D .*

THEOREM (10.8). *Let S be an integrally closed semigroup with quotient group G .*

- (1) *If T is an integrally closed oversemigroup of S , then each Kronecker function ring T_*^D of T is an overring of S_b^D .*
- (2) *If R is an overring of S_b^D , then R is a Kronecker function ring of $R \cap G$.*

Theorems (10.5) and (10.8) imply that there is a one-to-one correspondence between valuation oversemigroups of an integrally closed semigroup S and valuation overrings of S_b^D . And because S_b^D is a Bezout ring, the set of valuation

overrings of S_P^b is in one-to-one correspondence with the set of proper prime ideals of S_P^b . If each finitely generated ideal of S is invertible, then each $*$ -operation on S is a.b., and any two $*$ -operations on S are equivalent. And the mapping $T \mapsto T_P^b$ is a one-to-one mapping from the set of oversemigroups of S onto the set of overrings of S_P^b .

Let $*$ be a $*$ -operation on S . We set $d^*(a) = \{b \in F(S); b^* = a^*\}$ for each $a \in F(S)$, set $D^*(S) = \{d^*(a); a \in F(S)\}$ and $D_f^*(S) = \{d^*(a); a \text{ is } * \text{-finite}\}$. $D^*(S)$ and $D_f^*(S)$ are additive semigroups under the $*$ -multiplication. We define $D(S)$ (resp. $D_f(S)$) to be $D^v(S)$ (resp. $D_f^v(S)$). If $*$ is a $*$ -operation on a domain D , then we define $D^*(D)$ and $D_f^*(D)$ similarly. If $D_f^*(S)$ (resp. $D_f^*(D)$) is a group, then S (resp. D) is called a Prüfer $*$ -multiplication semigroup (resp. Prüfer $*$ -multiplication ring).

THEOREM (10.9). (1) *Let S be a Prüfer $*$ -multiplication semigroup. Then $D_f^*(S)$ is canonically isomorphic onto the group of divisibility of S_P^* for each domain D .*

(2) *Let $*$ be an e.a.b. $*$ -operation on S . Then S is a Prüfer $*$ -multiplication semigroup if and only if S_P^b is a quotient ring of $D[X; S]$.*

11. Prüfer v -multiplication semigroup rings and regularly integrally closed semigroup rings.

PROPERTY (11.1). (1) *If S is a regularly integrally closed semigroup, then each element of $D_f(S)$ is invertible in $D(S)$.*

(2) *If $D^*(S)$ (resp. $D^*(D)$) is a group, then $D(S)$ (resp. $D(D)$) is a group.*

(3) *If S (resp. D) is a Prüfer $*$ -multiplication semigroup (resp. Prüfer $*$ -multiplication ring), then S (resp. D) is a Prüfer v -multiplication semigroup (resp. Prüfer v -multiplication ring).*

Let $*$ be a rule which assigns a $*$ -operation $\ast_s$ (resp. $\ast_D$) on S (resp. D) for each semigroup S (resp. domain D). We denote $a^{\ast_s}$ (resp. $a^{\ast_D}$) simply by a^* for each fractional ideal a of each semigroup S (resp. domain D). $*$ is called an allied $*$ -operation if the following conditions hold for each semigroup S and for each domain D :

- (1) $(a^* D[X; S])^* = (a D[X; S])^*$ for each $a \in F(S)$.
- (1)' $(a^* D[X; S])^* = (a D[X; S])^*$ for each $a \in F(D)$.
- (2) If a is a nonzero ideal of $D[X; S]$, then $e(a^*)^* = e(a)^*$ and $c(a^*)^* = c(a)^*$.
- (3) If a is a finitely generated nonzero ideal of $D[X; S]$, then $(a^* K[X; G])^* = (a K[X; G])^*$, where $G = q(S)$ and $K = q(D)$.

For example the v -operation is an allied $*$ -operation.

PROPOSITION (11.2). *Let $*$ be an allied $*$ -operation.*

- (1) *If $D[X; S]$ is a $*$ -regularly integrally closed ring, then S (resp. D) is a $*$ -regularly integrally closed semigroup (resp. $*$ -regularly integrally closed ring).*

- (2) If $D^*(D[X; S])$ is a group, then $D^*(S)$ and $D^*(D)$ are groups.
 (3) If $D[X; S]$ is a Prüfer $*$ -multiplication ring, then S is a Prüfer $*$ -multiplication semigroup and D and $K[X; G]$ are Prüfer $*$ -multiplication rings.

LEMMA (11.3). Let S be a $*$ -regularly integrally closed semigroup and let D be a $*$ -regularly integrally closed domain, where $*$ is an allied operation.

- (1) There exist natural homomorphisms $\varphi_S^*: D_f^*(D[X; S]) \rightarrow q(D_f^*(S))$, $\varphi_D^*: D_f^*(D[X; S]) \rightarrow q(D_f^*(D))$ and $\varphi_{G,K}^*: D_f^*(D[X; S]) \rightarrow D_f^*(K[X; G])$, where $G=q(S)$ and $K=q(D)$.
 (2) Assume that the homomorphism $\varphi_{G,K}^* \times \varphi_S^* \times \varphi_D^*$ of $D_f^*(D[X; S])$ to $D_f^*(K[X; G]) \times q(D_f^*(S)) \times q(D_f^*(D))$ is a monomorphism. If S is a Prüfer $*$ -multiplication semigroup and if D and $K[X; G]$ are Prüfer $*$ -multiplication rings, then $D[X; S]$ is a Prüfer $*$ -multiplication ring.
 (3) If S is a regularly integrally closed semigroup and D is a regularly integrally closed ring, then $\varphi_{G,K}^* \times \varphi_S^* \times \varphi_D^*$ is a monomorphism and $D[X; S]$ is a regularly integrally closed ring.

For the proof of the latter half of (3) let α be a finitely generated ideal of $D[X; S]$. If an element ξ of $q(D[X; S])$ satisfies $\alpha\alpha^{-1} \subset \xi D[X; S]$, then $\xi^{-1}\alpha^v \subset \alpha^v$, and hence $(\xi^{-1}, 1)\alpha^v = \alpha^v$. Since $K[X; G]$ is regularly integrally closed, $\varphi_{G,K}^v(\xi^{-1}, 1) = 0$. It follows that $\xi^{-1} \in K[X; G]$. $\varphi_S^v(\xi^{-1}, 1) = 0$ implies that $\xi^{-1} \in K[X; S]$. $\varphi_D^v(\xi^{-1}, 1) = 0$ implies that $\xi^{-1} \in D[X; S]$. Therefore $(\alpha\alpha^{-1})^v = D[X; S]$.

Proposition (11.2) and Lemma (11.3) imply the following:

THEOREM (11.4) (1) ([1, Proposition 6.5]). $D[X; S]$ is a Prüfer v -multiplication ring if and only if S is a Prüfer v -multiplication semigroup and D is a Prüfer v -multiplication ring.

(2) $D[X; S]$ is a regularly integrally closed ring if and only if S is a regularly integrally closed semigroup and D is a regularly integrally closed ring.

12. On pseudo-principal semigroup rings. In [14] we obtained among other things a condition for the semigroup ring $D[X; S]$ of S over D to be a pseudo-principal ring when S is a group. The purpose of this paper is to obtain a necessary and sufficient condition for $D[X; S]$ to be a pseudo-principal ring for a semigroup S .

LEMMA (12.1). If $D[X; S]$ is a pseudo-principal ring ([4, §1, Exercice 21]), then D is a pseudo-principal ring.

We set $G=q(S)$. Let H be the set of units of S . If $(\alpha) \subset (\beta)$, we set $\alpha + H \geq \beta + H$ for $\alpha, \beta \in G$, where $(\alpha) = \alpha + S$ and $(\beta) = \beta + S$. Then G/H is a partially ordered group. If G/H is a complete lattice (cf. [4, §1, Exercice 21]), we call S a pseudo-principal semigroup.

LEMMA (12.2). *S is a pseudo-principal semigroup if and only if each divisorial ideal of S is a principal ideal of S .*

LEMMA (12.3). *If $D[X; S]$ is a pseudo-principal ring, then S is a pseudo-principal semigroup.*

PROOF. Let α be a divisorial ideal of S , and let $\{\alpha_\lambda; \lambda \in \Lambda\}$ be the set of elements $\alpha \in S$ such that $(\alpha) \supset \alpha$. We have $\alpha = \bigcap_{\lambda} (\alpha_\lambda)$. We set $\bigcap_{\lambda} X^{\alpha_\lambda} D[X; S] = \alpha'$. Since $D[X; S]$ is a pseudo-principal ring, we have $\alpha' = f D[X; S]$ for some $f \in D[X; S]$. f is of the form aX^s for some $a \in D$ and $s \in S$. It follows that $\alpha = (s)$. Therefore S is a pseudo-principal semigroup by Lemma (12.2).

Our semigroup S is a totally ordered semigroup ([6, Theorem 3.1]).

LEMMA (12.4). *Let D be an integrally closed domain with quotient field K . For elements $f, g \in K[X; S]$ we have the following properties.*

(1) *Let $f = \sum a_\alpha X^\alpha$, $g = \sum b_\beta X^\beta$ and $c \in K$. If $c(fg) \subset (c)$, then $a_\alpha b_\beta \in (c)$ for each α, β .*

(2) *Assume that f, g are monic elements of $K[X; S]$. If $fg \in D[X; S]$, then $f, g \in D[X; S]$.*

(3) *We have $\text{div } c(f) + \text{div } c(g) = \text{div } c(fg)$.*

(4) *If $c(fg)$ is a divisorial fractional ideal of D , then $c(f)c(g) = c(fg)$.*

(5) *Assume that D is a pseudo-Bezout ring ([4, §1, Exercice 21]). Then $\text{GCD } c(f) \cdot \text{GCD } c(g) = \text{GCD } c(fg)$, where $\text{GCD } c(f)$ is the greatest common divisor of $c(f)$ over D .*

Lemma (12.4) is a generalization of [9, §6, Proposition 1] for semigroups. And the proof is similar with its proof.

LEMMA (12.5). *Let D be a pseudo-Bezout ring with quotient field K , and let $f, g \in K[X; S]$. Then f divides g in $D[X; S]$ if and only if f divides g in $K[X; S]$ and $\text{GCD } c(f)$ divides $\text{GCD } c(g)$ in D .*

PROOF. If f divides g in $D[X; S]$, we have $g = fh$ for some $h \in D[X; S]$. Then clearly f divides g in $K[X; S]$. We have $\text{GCD } c(g) = \text{GCD } c(f) \cdot \text{GCD } c(h)$ by Lemma (12.4), (5). Therefore $\text{GCD } c(f)$ divides $\text{GCD } c(g)$ in D . The converse follows also from Lemma (12.4), (5).

Lemma (12.5) is a generalization of [9, §2, Proposition 1] for semigroups.

LEMMA (12.6). *If $D[X; S]$ is a pseudo-principal ring, then $K[X; S]$ is a pseudo-principal ring, where $K = q(D)$.*

PROOF. Let α' be a nonzero divisorial ideal of $K[X; S]$. Let $\{\varphi_\lambda; \lambda \in \Lambda\}$ be the set of elements $\varphi \in q(K[X; S])$ such that $\varphi K[X; S] \supset \alpha'$. We have $\alpha' = \bigcap_{\lambda} \varphi_\lambda K[X; S]$. D is a pseudo-principal ring by Lemma (12.1). We may suppose

that there exist $f, g_\lambda \in D[X; S]$ such that $\varphi_\lambda = f/g_\lambda$ and $\text{GCD } c(f) = \text{GCD } c(g_\lambda) = 1$ for each λ . We set $\alpha = \bigcap_\lambda (f/g_\lambda)D[X; S]$. Since $D[X; S]$ is a pseudo-principal ring, we have $\alpha = f_0 D[X; S]$ for some $f_0 \in D[X; S]$. Let $h \in \alpha'$. We must show that $h \in f_0 K[X; S]$. We may suppose that $\text{GCD } c(h) = 1$. We have $h = (f/g_\lambda)f_\lambda$ for some $f_\lambda \in K[X; S]$ and for each λ . Then $f_\lambda \in D[X; S]$ by Lemma (12.5). It follows that $\alpha' = f_0 K[X; S]$. Therefore $K[X; S]$ is a pseudo-principal ring.

Let $f \in D[X; S]$ and let $\text{Supp}(f) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, where $\alpha_1 < \alpha_2 < \dots < \alpha_n$. If $\alpha_n = 0$, we will call f an e-monic element of $D[X; S]$.

LEMMA (12.7). *Let D be an integral domain with quotient field K . Assume that S is integrally closed. For $f, g \in K[X; G]$ we have the following properties, where $G = q(S)$.*

- (1) *If $e(fg) \subset (\gamma)$, then $\alpha + \beta \in (\gamma)$ for each $\alpha \in \text{Supp}(f)$, $\beta \in \text{Supp}(g)$.*
- (2) *Assume that f, g are e-monic elements of $D[X; G]$. If $fg \in D[X; S]$, then $f, g \in D[X; S]$.*
- (3) *We have $\text{div } e(f) + \text{div } e(g) = \text{div } e(fg)$.*
- (4) *If $e(fg)$ is a divisorial fractional ideal of S , then $e(f) + e(g) = e(fg)$.*
- (5) *Assume that S is a pseudo-Bezout semigroup (i.e. GCD-semigroup), then $\text{GCD } e(f) + \text{GCD } e(g) = \text{GCD } e(fg)$.*

PROOF. Let $\{v_\lambda; \lambda \in \Lambda\}$ be the set of valuations of G which is nonnegative on S . We have $S = \bigcap_\lambda V_\lambda$, where V_λ is the valuation semigroup associated with v_λ for each λ ([6, Theorem 12.8]). Each v_λ has the canonical extension w_λ on $q(K[X; G])$ ([2, Section 1]). Let $\lambda \in \Lambda$. We have

$$v_\lambda(\alpha + \beta) \geq w_\lambda(f) + w_\lambda(g) = \inf \{v_\lambda(\alpha'); \alpha' \in \text{Supp}(fg)\} \geq v_\lambda(\gamma)$$

for each $\alpha \in \text{Supp}(f)$ and $\beta \in \text{Supp}(g)$. It follows that $\alpha + \beta - \gamma \in v_\lambda$ for each λ . Therefore $\alpha + \beta \in (\gamma)$. (2) We set $\gamma = 0$ in (1). (3) follows from (1). (4) We have $\text{div } e(fg) = \text{div}(e(f) + e(g))$ by (3). Since $e(fg)$ is divisorial, it follows that $e(f) + e(g) = e(fg)$. (5) follows from (1).

Lemma (12.7) is a semigroup version of Lemma (12.4).

LEMMA (12.8). *Assume that S is a pseudo-Bezout semigroup. Let $f, g \in K[X; G]$, where $G = q(S)$. Then f divides g in $D[X; S]$ if and only if f divides g in $D[X; G]$ and $\text{GCD } e(f)$ divides $\text{GCD } e(g)$ in S .*

PROOF. If f divides g in $D[X; S]$, we have $g = fh$ for some $h \in D[X; S]$. Then clearly f divides g in $D[X; G]$. We have $\text{GCD } e(g) = \text{GCD } e(f) + \text{GCD } e(h)$ by Lemma (12.7), (5). Therefore $\text{GCD } e(f)$ divides $\text{GCD } e(g)$ in S . The converse follows also from Lemma (12.7), (5).

LEMMA (12.9). *Let K be a field. If $K[X; S]$ is a pseudo-principal ring, then*

$K[X; G]$ is a pseudo-principal ring, where $G=q(S)$.

PROOF. Let α' be a nonzero divisorial ideal of $K[X; G]$. Let $\{\varphi_\lambda; \lambda \in A\}$ be the set of elements $\varphi \in q(K[X; G])$ such that $\varphi K[X; G] \supset \alpha'$. We have $\alpha' = \bigcap_\lambda \varphi_\lambda K[X; G]$. S is a pseudo-principal semigroup by Lemma (12.3). We may suppose that there exist $f, g_\lambda \in K[X; S]$ such that $\varphi_\lambda = f/g_\lambda$ and $\text{GCD } e(f) = \text{GCD } e(g_\lambda) = 0$ for each λ . We set $\alpha = \bigcap_\lambda (f/g_\lambda)K[X; S]$. Since $K[X; S]$ is a pseudo-principal ring, we have $\alpha = f_0 K[X; S]$ for some $f_0 \in K[X; S]$. Let $h \in \alpha'$. We must show that $h \in f_0 K[X; G]$. We may suppose that $\text{GCD } e(h) = 0$. We have $h = (f/g_\lambda)f_\lambda$ for some $f_\lambda \in K[X; G]$ and for each λ . Then $f_\lambda \in K[X; S]$ by Lemma (12.8). It follows that $\alpha' = f_0 K[X; G]$. Therefore $K[X; G]$ is a pseudo-principal ring.

LEMMA (12.10) ([14, §2, Theorem 8]). Assume that $S=G$ is a group. Then $D[X; G]$ is a pseudo-principal ring if and only if D is a pseudo-principal ring and each nonzero element of G is of type $(0, 0, 0, \dots)$.

LEMMA (12.11). If $D[X; S]$ is a pseudo-principal ring, then $D[X; G]$ is a pseudo-principal ring and each nonzero element of G is of type $(0, 0, 0, \dots)$, where $G=q(S)$.

PROOF. $K[X; S]$ is a pseudo-principal ring by Lemma (12.6), where $K=q(D)$. It follows that $K[X; G]$ is a pseudo-principal ring by Lemma (12.9). Hence each nonzero element of G is of type $(0, 0, 0, \dots)$ by Lemma (12.10). D is a pseudo-principal ring by Lemma (12.1). Therefore $D[X; G]$ is a pseudo-principal ring by Lemma (12.10).

LEMMA (12.12). Let D be an integral domain, N a multiplicative system of D . Let T be the subset of elements $t \in D$ that is prime to N ([6, §14]). Assume that the following two conditions are satisfied.

- (1) Let $\{a_\lambda; \lambda \in A\}$ be a subset of K of elements of the form n/m for $n, m \in N$, where $K=q(D)$. Then $\bigcap_\lambda a_\lambda D \cap D$ is a principal ideal of D .
- (2) Each element of D can be expressed in the form nt for some $n \in N, t \in T$. If D_N is a pseudo-principal ring, then D is a pseudo-principal ring.

PROOF. Since the elements of T are prime to N , it follows that $(t/t')D_N \cap D = (t/t')D \cap D$ for each $t, t' \in T$. Let α be a nonzero divisorial ideal of D . Let $\{a_\lambda; \lambda \in A\}$ be the set of elements $a \in K$ such that $(a) \supset \alpha$. We have $\alpha = \bigcap_\lambda (a_\lambda)$. There exist $b, c_\lambda \in D$ such that $a_\lambda = b/c_\lambda$ for each λ . We have $b = nt, c_\lambda = n_\lambda t_\lambda$ for some $n, n_\lambda \in N$ and $t, t_\lambda \in T$ for each λ . We have $\bigcap_\lambda (n/n_\lambda)D \cap D = n_0 D$ for some $n_0 \in D$ by (1). We set $\bigcap_\lambda a_\lambda D_N = \alpha'$. Since D_N is a pseudo-principal ring, we have $\alpha' = t_0 D_N$ for some $t_0 \in T$. Since $t_0 \in (t/t_\lambda)D_N \cap D = (t/t_\lambda)D \cap D$ for each λ , we have $n_0 t_0 D \subset \alpha$.

Conversely let $x \in \mathfrak{a}$. We write x as ms , where $m \in N$ and $s \in T$. Since $x = a_\lambda d_\lambda$ for some $d_\lambda \in D$, we have $mn_\lambda st_\lambda = ntd_\lambda$. Since $(st_\lambda) \cap (n) = (st_\lambda n)$, we have $mn_\lambda = nd'_\lambda$ for some $d'_\lambda \in D$. Hence $m \in (n_0)$. It follows that

$$sD = sD_N \cap D \subset \bigcap_{\lambda} (nt)/(n_\lambda t_\lambda) D_N \cap D = t_0 D_N \cap D = t_0 D. \quad \text{Hence } x = ms \in (n_0 t_0).$$

We have shown that $\mathfrak{a} = (n_0 t_0)$. Therefore D is a pseudo-principal ring.

Lemma (12.12) is a pseudo-principal version of [6, Theorem 14.3].

LEMMA (12.13). *Let D be a pseudo-principal domain with quotient field K . If $K[X; S]$ is a pseudo-principal ring, then $D[X; S]$ is a pseudo-principal ring.*

PROOF. We set $N = \{a; 0 \neq a \in D\}$ in Lemma (12.12). And let T be the subset of elements $t \in D[X; S]$ that is prime to N . We have $T = \{f \in D[X; S]; \text{GCD } c(f) = 1\} \cup \{0\}$ by Lemma (12.4), (5). Let $\{a_\lambda; \lambda \in \Lambda\}$ be a subset of K . Since D is a pseudo-principal ring, we have $\bigcap_{\lambda} a_\lambda D \cap D = aD$ for some $a \in D$. It follows that $\bigcap_{\lambda} a_\lambda D[X; S] \cap D[X; S] = aD[X; S]$. Therefore the condition (1) of Lemma (12.12) is satisfied for $D[X; S]$ and N . The condition (2) of Lemma (12.12) clearly holds. Since $D[X; S]_N = K[X; S]$, we see that $D[X; S]$ is a pseudo-principal ring.

LEMMA (12.14). *Let S be a pseudo-principal semigroup with quotient group G , and let K be a field. If $K[X; G]$ is a pseudo principal ring, then $K[X; S]$ is a pseudo-principal ring.*

PROOF. We set $N = \{X^s; s \in S\}$ in Lemma (12.12). Then the subset T of elements $t \in K[X; S]$ that is prime to N coincides with $\{f \in K[X; S]; \text{GCD } e(f) = 0\} \cup \{0\}$ by Lemma (12.7), (5). Let $\{\alpha_\lambda; \lambda \in \Lambda\}$ be a subset of G . Since S is a pseudo-principal semigroup, we have $\bigcap_{\lambda} (\alpha_\lambda) \cap S = (s)$ for some $s \in S$. It follows that $\bigcap_{\lambda} X^{\alpha_\lambda} K[X; S] \cap K[X; S] = X^s K[X; S]$. Since $K[X; S]_N = K[X; G]$, $K[X; S]$ is a pseudo-principal ring by Lemma (12.12).

THEOREM (12.15). *$D[X; S]$ is a pseudo-principal ring if and only if D is a pseudo-principal ring, S is a pseudo-principal semigroup and each nonzero element of $q(S)$ is of type $(0, 0, 0, \dots)$.*

PROOF. Suppose that $D[X; S]$ is a pseudo-principal ring. Then D is a pseudo-principal ring by Lemma (12.1). S is a pseudo-principal semigroup by Lemma (12.3). Each nonzero element of G is of type $(0, 0, 0, \dots)$ by Lemma (12.11), where $G = q(S)$. Conversely suppose that D is a pseudo-principal domain, S is a pseudo-principal semigroup and each nonzero element of G is of type $(0, 0, 0, \dots)$. Then $K[X; G]$ is a pseudo-principal ring by Lemma (12.10), where $K = q(D)$. $K[X; S]$ is a pseudo-principal ring by Lemma (12.14). Therefore $D[X; S]$ is a pseudo-principal ring by Lemma (12.13).

13. Remarks. If D is a Krull domain and if the divisor class group $C(D)$ is a torsion group, then D is called an almost factorial ring. We know the following results.

LEMMA (13.1) ([8, Corollary 5.4]). $D[X; G]$ is an almost factorial ring if and only if D is an almost factorial ring and G is of type $(0, 0, 0, \dots)$.

LEMMA (13.2) ([5]). $D[X; S]$ is a Krull domain if and only if D is a Krull domain and $S \cong H \oplus S_1$, where H is a group of type $(0, 0, 0, \dots)$ and S_1 is a sub-semigroup of a free group $F = \Sigma \oplus \mathbb{Z}u_i$ such that $S_1 = \{\Sigma n_i u_i \in q(S_1); \mathbb{Z} \ni n_i \geq 0\}$. Moreover $C(D[X; S]) \cong C(D) \oplus C(S_1)$.

REMARK (13.3). $D[X; S]$ is an almost factorial ring if and only if $D[X; S]$ is a Krull domain and both $C(D)$ and $C(S_1)$ are torsion groups.

Next, [6, p. 22] has the following statement. Namely, let S be a commutative associative set with the identity. If S admits a total order compatible with the operation on S , then S is torsion-free and cancellative.

EXAMPLE (13.4). Let S be a semigroup. S admits a total order compatible with the operation. Let ∞ be a symbol and set $T = S \cup \{\infty\}$. If we set $t + \infty = \infty$ and $t \leq \infty$ for each $t \in T$, then T is a commutative associative set with the identity and T admits a total order compatible with the operation on T . But $n\infty = m\infty$ for $0 < n, m \in \mathbb{Z}$ and $s + \infty = \infty + \infty$ for each $s \in S$. Therefore S is neither torsion-free nor cancellative.

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