Cut locus in a Riemannian manifold

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Let M be a complete n (≥ 2)-dimensional Riemannian manifold. Let $C(p)$ (resp. $Q(p)$) be the cut locus (resp. the first conjugate locus) in the tangent space $T_p M$ at a point $p \in M$. We put

$$V_i = \{v \in C(p); \#(\exp_p^{-1}(\exp_p v) \cap C(p)) = i + 1\}$$

for a positive integer i .

Sugahara [3], Durumeric [1] and others investigated a Riemannian manifold M which satisfies a condition

$$(A) \quad C(p) \cap Q(p) = \emptyset \quad \text{for a point } p,$$

and they proved many interesting theorems. Some of them are following

THEOREM A ([3]). Assume M satisfies the condition (A) at p . Then V_1 is open and dense in $C(p)$.

THEOREM B ([1]). Assume M is a compact Riemannian manifold which satisfies the condition (A) at p . If $d(p)/i(M) < 1 + f^{-1}(\arccos(-1/4))$ then $C(p) = V_1 \cup V_2 \cup V_3$ and every V_i is a submanifold of M with codimension i .

In this note, we investigate a complete Riemannian manifold M which satisfies a condition

$$(B) \quad i(p) < j(p) \quad \text{for a point } p,$$

where we let $j(p) = \inf \{\|v\|; v \in Q(p)\}$, and $j(p) = \infty$ if $Q(p) = \emptyset$.

1. Condition (B) and cut locus.

We assume hereafter M satisfies the condition (B) at p , and assume sectional curvatures K_σ of M are bounded from below. Let $i(p)$ be the injective radius at p , and let $i(M)$ the injective radius of M . Then we can choose a Riemannian metric g on M as $i(M) = 1$. We denote by $L(\gamma)$ the length of a piecewise smooth curve γ .

Let δ be a lower bound for sectional curvatures K_σ . Let $K_0 = \min\{\delta, 0\}$ and $N(K_0)$ a simply connected complete Riemannian manifold of constant sectional curvature K_0 . We consider a geodesic triangle abc in $N(K_0)$ which consists of

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three minimizing geodesics γ_{ab} , γ_{bc} and γ_{ca} , between a and b , b and c , c and a respectively. When $L(\gamma_{ab})=L(\gamma_{bc})=x+(1/2)$ and $L(\gamma_{ca})=1$, we put $f(x)=\kappa b$. It is obvious from the cosine rule in $N(K_0)$ that (i) $f(x)$ is continuous and strictly decreasing on $[0, \infty)$, (ii) $f(0)=\pi$, and (iii) $f(x)\rightarrow 0$ as $x\rightarrow\infty$.

We put $G=\{v\in C(p); \|v\|<j(p)\}$. Let $v, v_1\in G$ be different two vectors such that $\exp_p v=\exp_p v_1$. Then $\gamma(t)=\exp_p tv$ and $\gamma_1(t)=\exp_p tv_1$, $0\leq t\leq 1$, are minimizing geodesics from p to $\exp_p v$. We define a piecewise curve $\gamma\cdot(-\gamma_1)$ by $\gamma\cdot(-\gamma_1)(t)=\gamma(t)$ if $0\leq t\leq 1/2$, and $\gamma\cdot(-\gamma_1)(t)=\gamma_1(2-2t)$ if $1/2\leq t\leq 1$.

LEMMA 1. If we put $\|v\|/i(M)=1+x(v)$ then $\kappa(\dot{\gamma}(1), \dot{\gamma}_1(1))\geq f(x(v))$.

PROOF. Let $l=1+x(v)$. Let d be the distance function in M . If $d(\gamma(t), \gamma_1(t))<1$ ($=i(M)$) on $[0, 1]$ then there exist unique minimizing geodesics γ_s from $\gamma(t)$ to $\gamma_1(t)$. This shows that $H(s)=(\gamma| [0, s])\cdot\gamma_s\cdot(-\gamma_1| [0, s])$ is a homotopy of $\gamma\cdot(-\gamma_1)$ in a point curve p . Since M satisfies the condition (B) at p , $\exp_p$ has a maximal rank on $\{v\in T_p M; \|v\|<l+\varepsilon\}$ for a sufficiently small $\varepsilon>0$. Therefore, there exist some $s_0\in[0, 1]$ such that $L(H(s_0))\geq 2(l+\varepsilon)$ from Klingenberg's homotopy lemma (cf. [2], 7.1). On the other hand, $L(H(s))\leq 2l$ for any s , which is a contradiction.

If $d(\gamma(t'), \gamma_1(t'))=1$ then $l-t'l\geq l-(1/2)=(1/2)+x(v)$. Therefore, Lemma follows from Toponogov's comparison theorem.

We put $h(v)=\#(\exp_p^{-1}(\exp_p v)\cap C(p))$ for $v\in G$, and $W_i=V_i\cap G$. Then $W_i=h^{-1}(i+1)$.

LEMMA 2 ([3], Lemma 3.6). If h is constant in a neighborhood U in G then $U\subset W_1$.

THEOREM 1. Assume M is a complete Riemannian manifold which satisfies the condition (B) at p , and assume $K_\sigma\geq\delta$. Then

- (1) W_1 is open and dense in G ,
- (2) if we put $\theta(v)=f(x(v))/2$ then

$$h(v)\cdot\int_0^{\theta(v)}\sin^{n-2}\theta d\theta\leq a$$

where $a=\pi$ if n is even, and $a=2$ if n is odd.

PROOF. Since h is upper semi-continuous in G , W_1 is an open subset of G . Let U be a neighborhood of $v\in G$. Then, by the definition of h , h has a minimal value at $w\in U$, and hence h is locally constant around w . Therefore Lemma 2 shows that W_1 is dense in G .

Let $q=\exp_p v$. If $h(v)=k$ then $\exp_p^{-1}(\exp_p v)\cap C(p)=\{v, v_2, \dots, v_k\}$. Let $\gamma_i(t)=\exp_p tv_i$, $0\leq t\leq 1$. Then q is a cut point along every geodesic γ_i . Therefore $\|v_i\|=\|v\|$ and hence $x(v_i)=x(v)$. If we put $v_1=v$ then Lemma 1 shows that

$$\angle(\dot{\gamma}_i(1), \dot{\gamma}_j(1)) \geq f(x(v)) \quad 1 \leq i < j \leq k.$$

Let $u_i = \dot{\gamma}_i(1)/\|\dot{\gamma}_i(1)\|$. Let S be a unit sphere centered at the origin in $T_q M$. Then $B_i = \{u \in S; \angle(u, u_i) < f(x(v))/2\}$ is a neighborhood of u_i in S , and $B_i \cap B_j = \emptyset$ if $i \neq j$. Since $\text{vol}(\bar{B}_i) = \text{vol}(\bar{B}_j)$ for any i, j , we have $k \cdot \text{vol}(\bar{B}_1) \leq \text{vol}(S)$, where $\bar{B}_i$ denotes the closure of B_i . If we put $\theta(v) = f(x(v))/2$ then

$$\text{vol}(\bar{B}_1) = 2\pi^{\frac{n-1}{2}} \cdot \int_0^{\theta(v)} \sin^{n-2} \theta d\theta / \Gamma((n-1)/2),$$

where $\Gamma(s)$ is the gamma function. Since $\text{vol}(S) = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$, we have

$$k \cdot \int_0^{\theta(v)} \sin^{n-2} \theta d\theta \leq \sqrt{\pi} \cdot \Gamma((n-1)/2) / \Gamma(n/2) \leq a.$$

Thus we have proved Theorem.

More precisely, Durumeric proved

LEMMA 3 ([1], Lemma 3). *If $f(x(v)) \leq \arccos(-1/m)$ then $h(v) \leq m+1$.*

2. Case $f(x(v)) > \pi/2$.

LEMMA 4. *Let $w_1, w_2, \dots, w_k$ be non-zero vectors in $T_q M$. If $\angle(w_i, w_j) > \pi/2$ for $i < j$ then $w_1 - w_k, w_2 - w_k, \dots, w_{k-1} - w_k$ are linearly independent.*

PROOF. Lemma is obvious for $k=2$. Let $k > 2$. We assume Lemma is true for any $k-1$ vectors of $w_1, w_2, \dots, w_k$. If Lemma is false for k then

$$w_1 - w_k = \sum_{i=2}^{k-1} a_i (w_i - w_k)$$

and $a_i \neq 0$ for any i . Therefore, we can write

$$w_1 = \sum_{i=2}^k a_i w_i, \quad \sum_{i=2}^k a_i = 1$$

and hence $\sum_{i=2}^k a_i (\langle w_1, w_1 \rangle - \langle w_i, w_1 \rangle) = 0$. This shows that some of a_i are positive and others are negative. We assume $a_2, \dots, a_l$ are positive and $a_{l+1}, \dots, a_k$ are negative. If we put $z = \sum_{i=2}^l a_i w_i$ then $\langle z, w_i \rangle < 0$ for any i . Therefore, we have

$$0 \leq \langle z, z \rangle = \sum_{i=2}^l a_i \langle z, w_i \rangle < 0$$

Thus we have proved Lemma.

THEOREM 2. *Assume M is a complete Riemannian manifold which satisfies the condition (B) at p , and assume $K_\sigma \geq \delta$. If $f(x(v)) > \pi/2$ for any $v \in G$ then $G = W_1 \cup W_2 \cup \dots \cup W_n$ and every W_i is a submanifold of M with codimension i .*

PROOF. We fix a vector $v_1 \in G$. Let $\exp_p^{-1}(\exp_p v_1) \cap C(p) = \{v_1, v_2, \dots, v_k\}$. Let $\gamma_i(t) = \exp_p tv_i$, $0 \leq t \leq 1$. Then Lemma 1 shows that $\angle(\dot{\gamma}_i(1), \dot{\gamma}_j(1)) \geq f(x(v)) > \pi/2$ for $i < j$. Therefore, Lemma 4 shows that $k \leq n+1$. Since $W_i = h^{-1}(i+1)$, we have $G = W_1 \cup W_2 \cup \dots \cup W_n$.

Let $l = \|v_1\|$ and $B = \{u \in T_p M; \|u\| \leq l + \varepsilon\}$ for $\varepsilon > 0$. Let $h_1(u) = \#(\exp_p^{-1}(\exp_p u) \cap B)$ for $u \in B$. Then $1 \leq h_1(u) < \infty$ and $h_1(u)$ is upper semi-continuous on B . If we choose $\varepsilon > 0$ sufficiently small then we have $h_1(v_1) = h(v_1)$. Since $l < j(p)$, there exists a neighborhood U_i of v_i such that (i) $h_1(u) \leq k$ in U_i , and (ii) $\exp_p|_{U_i}$ is diffeomorphic and $\exp_p U_i = \exp_p U_1$. For $u_i \in U_i$, there exists from (ii) unique $u_i \in U_i$, and $\exp_p u_i = \exp_p u_1$. This shows that $h_1(u)$ is constant k in U_1 . On the other hand, we can find a minimizing geodesic $\exp_p tu_0$, $0 \leq t \leq 1$, from p to $\exp_p u_1$. Since $h_1(u_1) = k$, we have $u_0 = u_i$ for some i . Therefore $u_1 \in C(p)$ if $\|u_i\| = \|u_1\|$ for any i . If we define a mapping $\varphi: U_1 \rightarrow \mathbb{R}^{k-1}$ by $\varphi(u_1) = (\|u_1\|^2 - \|u_k\|^2, \dots, \|u_{k-1}\|^2 - \|u_k\|^2)$ then Lemma 4 shows that φ has rank $k-1$ at v_1 . Thus we have proved Theorem.

When M is compact, we put $d(p) = \sup \{\|v\|; v \in C(p)\}$.

COROLLARY. Assume M is a compact Riemannian manifold which satisfies the condition (A) at p . If $d(p)/i(M) < 1 + f^{-1}(\pi/2)$ then $C(p) = V_1 \cup V_2 \cup \dots \cup V_n$ and every V_i is a submanifold of M with codimension i .

From Lemma 3, we can prove following

THEOREM 3. Assume M is a complete Riemannian manifold which satisfies the condition (B) at p , and assume $K_\sigma \geq \delta$. If $f(x(v)) \geq \arccos(-1/m)$ then $G = W_1 \cup W_2 \cup \dots \cup W_m$.

COROLLARY. Assume M is a compact Riemannian manifold which satisfies the condition (A) at p . If $d(p)/i(M) \leq 1 + f^{-1}(\arccos(-1/m))$ then $C(p) = V_1 \cup V_2 \cup \dots \cup V_m$.

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