Remarks on Tauberian Theorem for Quasiceutral Banach Algebras

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Let A be a semisimple quasiceutral Banach algebra with bounded approximate identity and suppose the center $Z(A)$ of A is completely regular. We denote by $\text{supp}_B(x)$ the hull-kernel closure of $\{P \in \text{Prim } B : x \notin P\}$ whenever B is an algebra with structure space $\text{Prim } B$ and $x \in B$. $\text{supp}_B(x)$ is called the support of x . Also denote by B_{00} the set of all $x \in B$ such that $\text{supp}_B(x)$ is quasicompact, namely it satisfies the Borel-Lebesgue axiom without necessarily being Hausdorff. Let Φ be Dixmier's representation of $Z(M(A))$, the central double centralizer algebra of A (see [4] for definition) and $Z_{00}(A)$ the set of all $z \in Z(A)$ such that $\text{supp}(\Phi_{Lz})$ is quasicompact. Here $Lz(x) = zx$, $x \in A$ and $\text{supp}(\Phi_{Lz})$ is the hull-kernel closure of $\{P \in \text{Prim } A : \Phi_{Lz}(P) \neq 0\}$. Actually $Z_{00}(A) = A_{00} \cap Z(A) \subset (Z(A))_{00}$ (cf. [5, Lemma 3.4 and 3.7]). However $Z_{00}(A) = (Z(A))_{00}$ was left open in [5]. The following result asserts that this is true.

LEMMA. $Z_{00}(A) = (Z(A))_{00}$.

PROOF. We have already shown that $Z_{00}(A) \subset (Z(A))_{00}$ as stated above. To show the converse inclusion, let $z \in (Z(A))_{00}$ and $\{P_\lambda\}$ a net in $\text{supp}(\Phi_{Lz})$. Set $\sigma(P) = P \cap Z(A)$ for each $P \in \text{Prim } A$. In the proof of [5, Lemma 3.7], it can be observed that $\sigma(\text{supp}(\Phi_{Lz})) \subset \text{supp}_{Z(A)}(z')$ for all $z' \in Z(A)$. Then $\{\sigma(P_\lambda)\}$ is a net in $\text{supp}_{Z(A)}(z)$ and hence there exists a convergent subnet $\{\sigma(P_{\lambda'})\}$ of $\{\sigma(P_\lambda)\}$ because $\text{supp}_{Z(A)}(z)$ is compact. Let M be a limit point of $\{\sigma(P_{\lambda'})\}$. By the complete regularity of $Z(A)$, there exists a neighbourhood U_M of M such that $\ker(U_M)$ is modular. We can assume, without loss of generality, that $\sigma(P_{\lambda'}) \in U_M$ for all λ' . We next show that $\text{cl}(\sigma^{-1}(U_M))$ is quasicompact ("cl" denotes the hull-kernel closure.) In fact let P be an arbitrary point in $\sigma^{-1}(U_M)$. By the quasice centrality of A and [3, Lemma 2.7.7], there exists $e_P \in Z(A)$ which is an identity for A modulo P . Since $\sigma(P) \in U_M$, it follows that $e_P(1 - e) \in \ker(U_M) \subset \sigma(P) \subset P$, where e is an identity for $Z(A)$ modulo $\ker(U_M)$. Note also that $e(1 - e_P) \in P$. Therefore $e - e_P$ must be in P and hence

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$$A(1-e) \in A(1-e_p) + A(e_p-e) \subset P + AP = P.$$

We then conclude that $\ker(\sigma^{-1}(U_M))$ is modular and so $\text{cl}(\sigma^{-1}(U_M))$ is quasi-compact. Now since $\{P_\lambda\}$ is a net in $\sigma^{-1}(U_M)$, it has a convergent subnet whose limit point is in $\text{supp}(\Phi_{Lz})$. Thus $\text{supp}(\Phi_{Lz})$ must be quasicompact. In other words, $z \in Z_{00}(A)$ and the proof is complete.

If B is a Banach algebra and B_{00} is norm dense in B then B is said to be Tauberian. Note that if $Z(A)$ is Tauberian then $Z_{00}(A)$ is norm dense in $Z(A)$ from the above lemma. Therefore the first half of [5, Theorem 3.5] is improved as follows:

THEOREM. *If $Z(A)$ is Tauberian then every closed two-sided ideal of A which does not contain $Z(A)$ is contained in some primitive ideal of A .*

Let us now regard A as a Banach $Z(A)$ -module and suppose that A is essential (cf. [2, 15.1] for definition). Then A is obviously quasiceutral. Moreover by the above theorem, if $Z(A)$ is Tauberian then every proper closed two-sided ideal of A is contained in some primitive ideal of A and hence A is also Tauberian because $\text{hull } A_{00} = \emptyset$ from [5, Lemma 3.4].

However we don't know whether the quasice centrality of A implies the essentiality of A or not. We know only that if A is quasiceutral then A is order-free, namely $Z(A)x = \{0\} \Rightarrow x = 0$ for $x \in A$ and that if A is a C^* -algebra then the quasice centrality of A implies the essentiality of A (see [1, Proposition 1]).

References

- [1] R. J. Archbold, Density theorems for a center of a C^* -algebra, J. London Math. Soc. (2) **10** (1975), 187–197.
- [2] R. S. Doran and J. Wichmann, Approximate identities and factorization in Banach modules, Lecture Note in Math. #768, Springer-Verlag Berlin Heidelberg New York (1979).
- [3] C. E. Rickart, General Theory of Banach Algebras, Van Nostrand, N. J. (1960).
- [4] S. Takahasi, Dixmier's representation theorem of central double centralizers on Banach algebras, Trans. Amer. Math. Soc. **253** (1979), 229–236.
- [5] S. Takahasi, Central double centralizers on quasi-central Banach algebras with bounded approximate identity, Can. J. Math. **35** (1983), 373–384.