A Topic on Duality Theorems

Ryûki MATSUDA* and Kôjirô SATÔ**

1. Let G be a topological group and $\mathfrak{R}$ the set of complex finite dimensional representations of G . A map ζ of $\mathfrak{R}$ to $\bigcup_{n=1}^{\infty} GL(n, C)$ (C : the complex numbers) is called a representation of $\mathfrak{R}$ if it preserves

- (a) degree,
  - (b) direct sum,
  - (c) Kronecker product,
  - (d) similarity
- and
- (e) complex conjugate representations.

Then the set $G^*(\mathfrak{R})$ of representations of $\mathfrak{R}$ naturally becomes a topological group and the canonical map Φ of G to $G^*(\mathfrak{R})$ is a continuous homomorphism. G is said to satisfy the Tannaka duality if Φ is a surjective topological isomorphism [10].

2. Let G_1, G_2 be topological groups, $\mathfrak{R}_1, \mathfrak{R}_2$ the sets of representations of G_1 and G_2 respectively, F the set of continuous homomorphisms of G_1 to G_2 , and Ψ the set of maps (called homomorphisms [11]) of $\mathfrak{R}_2$ to $\mathfrak{R}_1$ satisfying (a)~(e). We introduce the compact-open topology on F . Let ϕ be an element of Ψ , K a compact subset of G_1 , D a representation of G_2 , and ε a positive number. We set

$$U(\phi; D; K; \varepsilon) = \{\phi' \in \Psi \mid \|(\phi'(D))(x) - (\phi(D))(x)\| < \varepsilon \text{ for every } x \in K\},$$

where $\|\cdot\|$ is the matrix norm. We introduce a topology on Ψ taking subsets of the form $U(\phi; D; K; \varepsilon)$ as an open subbasis.

3. PROPOSITION. *If G_1, G_2 satisfy the Tannaka duality, then the canonical map θ of F to Ψ is a surjective homeomorphism.*

The proof is the same as one of [11] of the compact case.

4. If $\zeta \in G^*(\mathfrak{R})$ satisfies the following condition, ζ is called a

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Received June 10, 1969.

*Department of Mathematics, Faculty of Science, Ibaraki University, Mito, Ibaraki, Japan.

**Department of Mathematics, Tôhoku Institute of Technology, Sendai, Miyagi, Japan.

special representation of $\mathfrak{R}$ [9]:

(f) If $D \in \mathfrak{R}$ is of the form $\begin{bmatrix} D_1 & * \\ 0 & D_2 \end{bmatrix}$, then $\zeta(D) = \begin{bmatrix} \zeta(D_1) & ** \\ 0 & \zeta(D_2) \end{bmatrix}$.

The set $G(\mathfrak{R})$ of special representations of $\mathfrak{R}$ forms a topological subgroup of $G^*(\mathfrak{R})$. G is said to satisfy the generalized Tannaka duality if Φ is a topological isomorphism of G onto $G(\mathfrak{R})$ [9]. We may call a homomorphism of $\mathfrak{R}_2$ to $\mathfrak{R}_1$ to be special, if it satisfies (f). Then we have the following:

5. PROPOSITION. *If G_1, G_2 satisfy the generalized Tannaka duality, then θ is a homeomorphism of F onto the subspace Ψ_0 of Ψ of special homomorphisms of $\mathfrak{R}_2$ to $\mathfrak{R}_1$.*

6. Next, let $\mathfrak{g}$ be a Lie algebra over a field K , $\mathfrak{R}$ the set of finite dimensional representations of $\mathfrak{g}$ with coefficients in K , and $\mathfrak{gl}(n, K)$ the set of $n \times n$ -matrices over K . A map ζ of $\mathfrak{R}$ to $\bigcup_{n=1}^{\infty} \mathfrak{gl}(n, K)$ is called a representation if it preserves (a), (b), (d) and

(g) Kronecker sum.

If ζ satisfies (f) further, it is said a special representation of $\mathfrak{R}$. Let $\mathfrak{g}^*(\mathfrak{R})$ ($\mathfrak{g}(\mathfrak{R})$) be the set of representations (special representations) of $\mathfrak{R}$. If the canonical map Φ of $\mathfrak{g}$ to $\mathfrak{g}^*(\mathfrak{R})$ ($\mathfrak{g}(\mathfrak{R})$) is bijective, $\mathfrak{g}$ is said to satisfy the Tannaka duality (the generalized Tannaka duality).

7. PROPOSITION. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras, $\mathfrak{R}_1, \mathfrak{R}_2$ the sets of representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively, F the set of homomorphisms of $\mathfrak{g}_1$ to $\mathfrak{g}_2$, Ψ the set of homomorphisms of $\mathfrak{R}_2$ to $\mathfrak{R}_1$ (i. e. satisfying (a), (b), (d), (g)), and Ψ_0 the set of special homomorphisms of $\mathfrak{R}_2$ to $\mathfrak{R}_1$. Then if $\mathfrak{g}_1, \mathfrak{g}_2$ satisfy the Tannaka duality (the generalized Tannaka duality), the map θ of F to Ψ (Ψ_0) is bijective.*

The proof is easy, and [6] is a special case.

8. Next, let G be an algebraic group over a universal domain Ω , and $\mathfrak{R}$ the set of rational representations of G . A map ζ of $\mathfrak{R}$ to $\bigcup_{n=1}^{\infty} GL(n, \Omega)$ is said a representation (a special representation) if it satisfies (a), (b), (c), (d) ((a), (b), (c), (d), (f)). We can define the Tannaka duality and the generalized Tannaka duality for algebraic groups [1], [9].

9. PROPOSITION. *The same duality relations hold for algebraic groups (i. e. the θ are bijections).*

To prove this, let G be an algebraic group satisfying the generalized Tannaka duality. There exists a connected normal algebraic subgroup N of G such that G/N is linear and such that the kernel of any rational homomorphism of G to a linear group contains N [8]. Then for any $n \in N$ and $D \in \mathfrak{R}$, we have

$$(\Phi(n))(D) = (\Phi(1))(D).$$

Accordingly $N=1$ and G is linear.

Assume that G_1, G_2 satisfy the generalized Tannaka duality (the proof for the Tannaka duality case is the same). We see from the above that G_1, G_2 are linear. It is immediate to see that θ is injective. To show surjectiveness, let ψ be any element of Ψ_0 . ψ naturally induces the map ρ of $G_1(\mathfrak{R}_1)$ to $G_2(\mathfrak{R}_2)$. Put $f = \Phi_2^{-1} \circ \rho \circ \Phi_1$, where Φ_1, Φ_2 are duality maps of G_1 onto $G_1(\mathfrak{R}_1)$ and G_2 onto $G_2(\mathfrak{R}_2)$, respectively. The map f of G_1 to G_2 is product preserving. By the definition we have $D \circ f = \psi(D)$ for any $D \in \mathfrak{R}_2$. So $D \circ f$ belongs to $\mathfrak{R}_1$. If we take the biregular isomorphism of G_2 to an algebraic group of matrices for D , we see that f is a rational map of G_1 to G_2 . So $f \in F$ and $\theta(f) = \psi$, proving all of our assertion.

10. REMARKS. The same duality relation may be considered in various cases. We can take for instance a part of $\mathfrak{R}$ instead of $\mathfrak{R}$ (as "ring" in [3] or [2]). Next, we can add or diminish conditions for representation ζ of $\mathfrak{R}$ (e. g. [2]). Next, we may take in 6. various fields K (Especially, in the case of complex numbers, we can define the conjugate representation of a representation). Also we can consider various kinds of ones for our first objects G or g (algebraic, topological, algebraic and topological, ...) (e. g. $\mathcal{Q}$ -modules [5], topological linear spaces, Banach algebras, C^* -algebras, ...). We can consider also the representative rings $\mathfrak{Q}$ (as in [4]) for our dual objects and $\text{Hom}(\mathfrak{Q}, \ )$ for our biduals. We can consider also not necessarily finite dimensional representations ([7] deals for instance with representations on Hilbert spaces of locally compact group and proves the homeomorphism of θ as a corollary of [12]). We conclude our topic on duality with one more remark that other structures (algebraic or

topological) on F or $\Psi(\Psi_0)$ may be considered.

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