On a Generalization of Euler's ϕ -function

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Professor Tanaka extended Euler's ϕ -function and obtained the formula to give the value of the function [2]. In this note, we extend these further to any algebraic number field.

Let $\mathfrak{O}$ be the ring of algebraic integers of a (finite) algebraic number field. Let $S_{a,m}$ be the set of elements $x+m$ of $\mathfrak{O}/m$ such that

$$(x, m) = (x + \alpha, m) = 1$$

for fixed $\alpha \in \mathfrak{O}$ and an (integral) ideal m of $\mathfrak{O}$.

DEFINITION. We define the generalized Euler's ϕ -function $\phi_a(m)$ to be the number of elements of $S_{a,m}$.

LEMMA 1. *Let m, m' be ideals of $\mathfrak{O}$ such that*

$$(m, m') = 1,$$

then we have

$$\phi_a(mm') = \phi_a(m)\phi_a(m').$$

PROOF. We have

$$\mathfrak{O}/mm' \cong \mathfrak{O}/m \oplus \mathfrak{O}/m'$$

by the map

$$x + mm' \longleftrightarrow (x + m, x + m').$$

And the result follows from the fact that

$$(x, mm') = (x + \alpha, mm') = 1$$

if and only if

$$(x, m) = (x + \alpha, m) = (x, m') = (x + \alpha, m') = 1.$$

Now let $T_{a,m}$ be the set of elements $x+m$ of $\mathfrak{O}/m$ such that

$$x^2 + \alpha x \equiv 0 \pmod{m}.$$

We set $N_{a,m}$ to be the number of elements of $T_{a,m}$.

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LEMMA 2. For a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ and a natural number a , we have

$$\phi_a(\mathfrak{p}^a) = N(\mathfrak{p})^a - N(\mathfrak{p})^{a-1}N_{a,\mathfrak{p}},$$

where $N(\)$ denotes the norm.

PROOF. Let

$$T_{a,\mathfrak{p}} = \{x_1 + \mathfrak{p}, \dots, x_k + \mathfrak{p}\}, \quad k = N_{a,\mathfrak{p}}$$

and

$$\mathfrak{p}/\mathfrak{p}^a = \{y_1 + \mathfrak{p}^a, \dots, y_c + \mathfrak{p}^a\}, \quad c = N(\mathfrak{p})^{a-1}.$$

For any x of $\mathfrak{O}$,

$$x + \mathfrak{p}^a \in S_{a,\mathfrak{p}}^a$$

if and only if

$$x^2 + \alpha x \equiv 0(\mathfrak{p}).$$

Hence the complement of $S_{a,\mathfrak{p}}^a$ in $\mathfrak{O}/\mathfrak{p}^a$ is just

$$x_1 + y_1 + \mathfrak{p}^a, \dots, x_i + y_j + \mathfrak{p}^a, \dots, x_k + y_c + \mathfrak{p}^a\}.$$

Therefore it holds:

$$\phi_a(\mathfrak{p}^a) = N(\mathfrak{p}^a) - kc = N(\mathfrak{p})^a - N(\mathfrak{p})^{a-1}N_{a,\mathfrak{p}}.$$

PROPOSITION.

$$\phi_a(\mathfrak{m}) = N(\mathfrak{m}) \prod_{\mathfrak{p} \mid \mathfrak{m}} \left(1 - \frac{N_{a,\mathfrak{p}}}{N(\mathfrak{p})}\right).$$

PROOF. Let

$$\mathfrak{m} = \prod \mathfrak{p}^a.$$

We have, by lemmas 1 and 2,

$$\begin{aligned} \phi_a(\mathfrak{m}) &= \phi_a(\prod \mathfrak{p}^a) = \prod \phi_a(\mathfrak{p}^a) \\ &= \prod (N(\mathfrak{p})^a - N(\mathfrak{p})^{a-1}N_{a,\mathfrak{p}}) \\ &= \prod N(\mathfrak{p})^a \prod \left(1 - \frac{N_{a,\mathfrak{p}}}{N(\mathfrak{p})}\right) \\ &= N(\mathfrak{m}) \prod_{\mathfrak{p} \mid \mathfrak{m}} \left(1 - \frac{N_{a,\mathfrak{p}}}{N(\mathfrak{p})}\right). \end{aligned}$$

COROLLARY. If $\alpha \equiv 0(\mathfrak{m})$,

$$\phi_a(\mathfrak{m}) = N(\mathfrak{m}) \prod_{\mathfrak{p} \mid \mathfrak{m}} \left(1 - \frac{1}{N(\mathfrak{p})}\right).$$

PROOF. Because $N_{a,b} = 1$ in this case.

REMARK. We can define also a generalized Euler's ϕ -function $\phi_{\mathfrak{A}}(\mathfrak{m})$ for an ideal $\mathfrak{m}$ of $\mathfrak{O}$ and for a residue class $\mathfrak{A} \bmod \mathfrak{m}$. And it is obvious that we get the similar results.

References

- [1] T. Takagi, Theory of algebraic numbers, Iwanami (1948).
- [2] K. Tanaka, A generalization of Euler's ϕ -function, Res. Rep. Tokyo Electrical Engrg. College, 15 (1967), 24-29.