On $\Lambda(\varphi, M)$-spaces

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1. Introduction and Preliminary. In 1951, G. G. Lorentz [7] has introduced a class of function spaces called $\Lambda$-spaces. Let $\varphi(t)$ be a positive integrable and almost everywhere equivalent to a non-increasing function defined on $(0, l), l<\infty$. For a measurable function $f$, we denote by $f^*$, the decreasing (truly, non-increasing) rearrangement of $f$ [2; p. 260-299], [6; p. 60]. It is defined as follows. Let $\mu_f(\alpha)$ be the Lebesgue measure of the set $\{t : |f(t)| > \alpha\}$ for any real number $\alpha$. Then $\mu_f(\alpha)$ is right-continuous, i.e., $\lim_{\alpha_n \to \alpha} \mu_f(\alpha_n) = \mu_f(\alpha)$. Now we define the function $f^*(x)$ as the right-inverse of $\mu_f(y)$, i.e.,

\begin{equation}
(1.1) \quad f^*(x) = \inf\{y : \mu_f(y) \leq x\}.
\end{equation}

The space $\Lambda(\varphi, p)$, $p > 1$ is the set of all measurable functions $f$. We shall define the norm $\|f\|$, such that

\begin{equation}
(1.2) \quad \|f\| = \left\{ \int_{0}^{l} \psi(t) f^*(t)^p dt \right\}^{1/p} < \infty.
\end{equation}

Then, $\Lambda(\varphi, p)$ equipped with the norm $\|\cdot\|$ defined by (1.2), is a reflexive Banach space where $1 < p < \infty$ [7].

Hence we can regard as the $p$-th power of function is a convex on the positive real line. Now let $M(u)$, $0 \leq u < \infty$ be a N-function, $\varphi$ be as above, and for a measurable function $f$ we put

\begin{equation}
(1.3) \quad \rho(f) = \int_{0}^{l} \psi(t) M[f^*(t)] dt.
\end{equation}

In this paper, we shall discuss with a class $\Lambda(\varphi, M)$, which extends that of the spaces $\Lambda(\varphi, p)$, where the function $M(u)$ is a N-function in the sense [5; p. 6]. The set $\Lambda(\varphi, M)$ of all $f$ with $\rho(\varphi, f) < \infty$ for some $\alpha > 0$ is a modular space and $\rho$ is a modular on $\Lambda(\varphi, M)$

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In the sense of Nakano [12], i.e.,

\( \Lambda(\varphi, M) = \{ f : \rho(\alpha f) < \infty, \text{ for some } \alpha > 0 \} \).

In §2, we shall show that \( \Lambda(\varphi, M) \) is a modular space and a Banach space with the norm which is induced by the modular (1.3). In §3, we shall treat with the dual space \( \Lambda^*(\varphi, M) \) of \( \Lambda(\varphi, M) \) and show that the spaces \( \Lambda(\varphi, M) \) are reflexive if \( M \) and \( N \), the dual of \( M \), satisfy (\( \Delta_2 \)) and (\( \Delta_2' \))-condition, generalizing the Theorem 4 in [7; p. 417].

For two measurable functions \( f \) and \( g \), if they are equimeasurable to each other, i.e., \( \mu_f(\alpha) = \mu_g(\alpha) \), then we write \( f \sim g \).

For two measurable functions \( f \) and \( g \) defined on \( (0, 1) \), \( f < g \) means that \( \int_0^x f \ast dt \leq \int_0^x g \ast dt \) for all \( x \) with \( 0 < x < 1 \).

Here we present several basic properties about \( f \ast \) and the preorder "\( \ast \)".

(1.5) \( f < g \iff f \ast < g \ast \iff \mathcal{M}(f \ast) < \mathcal{M}(g \ast) \).

In fact, the equivalence of left hand side is obvious from the definition. For the proof of right hand side, see [7; p. 414].

Furthermore if \( \varphi(t) \) is positive decreasing, then

(1.6) \( f \ast < g \ast \iff \mathcal{M}(f \ast) < \mathcal{M}(g \ast) \).

For the proof, see [7; p. 414]. Also we have

(1.7) \( (f + g) \ast < f \ast + g \ast \).

Because,

\[
\int_0^x (f + g) \ast dt = \sup \int_\varepsilon^x | f + g |(t) dt
\]

\[
\mu(e) = x
\]

(\( \mu : \) Lebesgue measure)

\[
\leq \sup \int_\varepsilon^x (| f(t) | + | g(t) |) dt
\]

\[
\leq \sup \int_\varepsilon^x | f(t) | \ dt + \sup \int_\varepsilon^x | g(t) | \ dt
\]

\[
\leq \int_0^x f \ast dt + \int_0^x g \ast dt.
\]

Also we have
(1.8) \( f g \leq f^* g^* \).

For the proof, see [2; p. 278] or [9; p. 102]. Since \( M \) is a convex function, we obtain

(1.9) \( M[(\alpha f + \beta g)^*] \leq \alpha M[f^*] + \beta M[g^*] \), where \( \alpha > 0, \beta > 0, \alpha + \beta = 1 \), by (1.5) and (1.7).

2. **The spaces** \( \Lambda(\varphi, M) \). Let \( M(u) \) be a \( N \)-function [5; p. 6], that is, there exist a function \( p(t) \) which is right-continuous and positive non-decreasing such that \( p(0) = 0 \) and

\[
M(u) = \int_0^u p(t) dt, \quad 0 \leq u < \infty.
\]

**Theorem 1.** \( \Lambda(\varphi, M) \) is a modular space with the modular \( p \).

**Proof.** First \( \Lambda(\varphi, M) \) is a linear space. For any \( f, g \in \Lambda(\varphi, M) \),

\[
\rho(\alpha f + \beta g) \leq \rho\left( \frac{1}{2} \alpha \rho_0(f + g) \right)
\]

\[
\leq \frac{1}{2} \left( \rho(\alpha f) + \rho(\beta g) \right),
\]

for \( \alpha_0 = 2\max(\alpha, \beta) \). We can see that \( \rho(\alpha f), \rho(\alpha g) < \infty \) for \( f \) and \( g \). Therefore \( \Lambda(\varphi, M) \) is linear.

The following properties are easily shown from the definition of \( \rho \).

(\rho. 1) \( 0 \leq \rho(f) \leq \infty \) (for any measurable function \( f \)), and \( \rho(\| f \|) = \rho(f) \) and \( \rho(0) = 0 \);

(\rho. 2) For any \( f \in \Lambda(\varphi, M) \), and a positive real number \( \alpha, \rho(\alpha f) < \infty \);

(\rho. 3) \( \rho(\alpha f) = 0, \alpha > 0 \), then \( f = 0 \) a.e.;

(\rho. 4) If \( \| f \| \leq \| g \| \), then \( \rho(f) \leq \rho(g) \) (monotone);

(\rho. 5) \( \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1 \), then \( \rho(\alpha f + \beta g) \leq \alpha \rho(f) + \beta \rho(g) \), (convex);

(\rho. 6) \( 0 \leq f_n^* f \), then \( \rho(f) = \sup_n \rho(f_n) \) (upper semi-continuous).

To see (\rho. 6) we used the fact that \( 0 \leq f_n \uparrow f \) implies \( f_n^* \downarrow f^* \). Thus we conclude our assertion.

Here, \( M \) is called to satisfy the \( (D) \)-condition, if there exist a constant \( \gamma \) and some \( u_0 \geq 0 \), such that

(2.1) \( M(2u) \leq \gamma M(u) \) for all \( u \geq u_0 \).
Then we define a class \( \Lambda_0(\varphi, M) \) of all measurable functions \( f \) that \( \rho(f) < \infty \), i.e.,

\[
(2.2) \quad \Lambda_0(\varphi, M) = \{ f : \rho(f) < \infty \}.
\]

**Theorem 2.** If \( M \) satisfies \((A_2)\)-condition, then \( \Lambda_0(\varphi, M) = \Lambda(\varphi, M) \).

**Proof.** We have always \( \Lambda_0(\varphi, M) \subseteq \Lambda(\varphi, M) \). Since, for any \( f \in \Lambda(\varphi, M) \), then \( \rho\left(\frac{1}{2} f\right) < \infty \). Thus we have

\[
\rho(f) \leq \int_{\varphi}^{\varphi} M[f^*] dt
\]

\[
\leq \gamma \int_{\varphi}^{\varphi} M\left[\frac{1}{2} f^*\right] dt
\]

\[
= \gamma \rho\left(\frac{1}{2} f\right) < \infty,
\]

i.e.,

\[ f \in \Lambda_0(\varphi, M). \]

Therefore we conclude our assertion.

Further \( \rho(f) \) satisfies some properties as follows. A modular \( \rho \) on \( \Lambda(\varphi, M) \) is said to be **lower semi-additive**,\n
\[
(2.3) \quad \rho(f + g) \leq \rho(f) + \rho(g) \quad \text{for } 0 \leq f, g \in \Lambda(\varphi, M).
\]

In fact, since \( \sup\{f^*, g^*\} < (f + g)^* \) with the preorder \( < \), then we have our assertion.

Furthermore, by \( (\rho, 5) \), \( \rho(\alpha f) \) is a **convex function of \( \alpha \)** for each \( f \), i.e.,

\[
(2.4) \quad \rho\left(\frac{\alpha + \beta}{2} f\right) \leq \frac{1}{2} \left( \rho(\alpha f) + \rho(\beta g) \right).
\]

Now we can define the norm of \( f(\in \Lambda(\varphi, M)) \) which is called **Luxemburg norm**, as follows:

\[
(2.5) \quad \| f \| = \inf \{ \xi : \rho\left(\frac{f}{\xi}\right) \leq 1, \xi > 0 \}.
\]

Then, it is known that \( (2.5) \) satisfies the norm condition. This norm \( (2.5) \) obviously satisfies the following:

\[
(2.6) \quad |f| \leq |g| \iff \| f \| \leq \| g \|;
\]

\[
(2.7) \quad 0 \leq f_n \uparrow f \iff \| f_n \| \uparrow \| f \|.\]
We can see from the definition that \( \| f \| \leq 1 \) is equivalent to \( \rho(f) \leq 1 \).

**Theorem 3.** \( \Lambda(\psi, M) \) is a Banach space with the induced norm (2.5) by the modular \( \rho(f) \).

**Proof.** We shall show the completeness. By (2.7), we obtain the property that is called to be monotone complete (the weak Fatou property) in the sense of Amemiya.

\[
0 \leq f, \quad \sup || f_n || < \infty \implies f \in \Lambda(\psi, M),
\]

and hence \( \Lambda(\psi, M) \) is complete by the theorem in [4].

3. The reflexivity of the space \( \Lambda(\psi, M) \). First we shall construct the dual space of \( \Lambda(\psi, M) \). Let \( N(\psi) \) be the N-function complementary to \( M(\psi) \) in the sense of Young [5; p. 11]. That is to say, the function \( q(s) \) is the right inverse of \( p(t) \) which is defined by the equality:

\[
q(s) = \sup_{p(t) \leq s} t, \quad 0 < s < \infty.
\]

Then we have

\[
N(\psi) = \int_0^\psi q(s) \, ds, \quad 0 < \psi < \infty.
\]

Next we shall give some definitions and propositions which will be needed in the sequel. Now, let \( G(x) = \int_a^x g(t) \, dt \), where \( g(t) \) is integrable and positive on \( (0, 1) \), and \( \psi(x) = \int_a^x \psi(t) \, dt \), \( 0 < x < \infty \). The function \( G(x) \) is said to be \( \psi \)-concave, if

\[
\frac{G(x) - G(a)}{\Phi(x) - \Phi(a)} \geq \frac{G(b) - G(a)}{\Phi(b) - \Phi(a)}, \quad a < x < b.
\]

Lorentz [6; §3.6, Theorem 3.6.3] has shown the following proposition.

**Proposition 1.** The function \( G(x) = \int_a^x g(t) \, dt \) is \( \psi \)-concave if and only if \( g(t) = \psi(t) D(t) \) a.e., where \( D(t) \) is a positive decreasing function.

We define for each measurable function \( g(t) \),

\[
\tau(g) = \inf_{\psi \leq \psi_0} \int_0^\psi N[D] \, dt,
\]

(3.1)
where the infimum is taken for all decreasing positive functions $D(t)$ for which $g^* < \varphi D$.

The conjugate modular $\overline{\rho}$ [12; p. 92] of $\rho$, is defined by

\begin{equation}
\overline{\rho}(g) = \sup_{f \in \Lambda(\varphi, M)} \left\{ \int_{0}^{t} f \, g \, dt \right\}.
\end{equation}

for any measurable function $g$. We consider the dual space of $\Lambda(\varphi, M)$, denoted by $\Lambda(\varphi, M)$, as follows:

\begin{equation}
\Lambda(\varphi, M) = \{ g : \rho(\alpha g) \rightarrow \infty \text{ for some } \alpha > 0, \text{ and } g \text{ is measurable} \}.
\end{equation}

Then $\Lambda(\varphi, M)$ is also a modular space.

**Theorem 4.** For each $g \in \Lambda(\varphi, M)$, we have $\overline{\rho}(g) \leq \tau(g)$.

**Proof.** By Young's inequality and (1.8), we have, for any decreasing positive function $D$ with $g^* < \varphi D$,

\begin{align*}
| \int_{0}^{t} f \, g \, dt | & \leq \int_{0}^{t} f^* \, g^* \, dt \\
& \leq \int_{0}^{t} f^* \, \varphi \, D \, dt \\
& \leq \int_{0}^{t} \varphi M[ f^* ] \, dt + \int_{0}^{t} \varphi N[D] \, dt,
\end{align*}

and hence

\begin{equation}
| \int_{0}^{t} f \, g \, dt | - \rho(f) \leq \int_{0}^{t} \varphi N[D] \, dt.
\end{equation}

Thus we have

\begin{equation}
\sup_{f \in \Lambda(\varphi, M)} \left\{ | \int_{0}^{t} f \, g \, dt | - \rho(f) \right\} \leq \inf_{g < \varphi D} \int_{0}^{t} \varphi N[D] \, dt.
\end{equation}

Hence

\begin{equation}
\overline{\rho}(g) \leq \tau(g).
\end{equation}

Now we shall show that $\overline{\rho}(g) = \tau(g)$ on $\Lambda(\varphi, M)$. To show the result, we need some definitions. For a given $g$, $g^0$ is the smallest (in the sense of the relation $<$, see §1) function among the functions satisfying $g < h = \varphi D$ with a positive decreasing function $D$, and is called the level function of $g$ with respect to $\varphi$. Lorentz has shown the following proposition [4; §3.6, Theorem 3.6.4].
PROPOSITION 2. Let \( g(t) \) be integrable and positive, and let \( D^o \) be defined by \( g^o = \varphi D^o \). For any \( G(x) = \int_a^x g(t) \, dt \), the function \( G^o(x) \) is also of the form \( G^o(x) = \int_a^x g^o(t) \, dt \), \( g^o \geq 0 \). Then \( G^o(x) = G(x) \) holds a.e. (consequently \( g(t) = g^o(t) \) a.e.) except perhaps for the maximal intervals \( (a, b) \) of constancy of \( D \); on each such interval \( (a, b) \), \( \int_a^b g^o \, dt = \int_a^b g \, dt \).

Thus we obtain that for a given integrable function \( g \) the infimum of (3.1) is attained for \( D = D^o \):

\[
\inf_{\varphi \leq \varphi_0} \int_a^t \varphi N[D] \, dt = \int_a^t \varphi N[D^o] \, dt,
\]

and hence

\[
(3.4) \quad \tilde{\varphi}(g) = \int_a^t \varphi N[D^o] \, dt.
\]

Now we shall define the condition of N-function \( N(v) \) which is called \((\tilde{\varphi}_2)\)-condition. There exist real number \( \alpha, \beta \); \( 1 < \alpha < \beta \) and some \( v_0 \), such that

\[
(3.5) \quad N(\alpha v) \leq \beta N(v) \quad \text{for all } v \geq v_0.
\]

If the N-function \( M(u) \) satisfies the \((\tilde{\varphi}_2)\)-condition, then it's complementary N-function \( N(v) \) satisfies \((\tilde{\varphi}_2)\)-condition. Moreover, we obtain that, if N-function satisfies \((\tilde{J}_2)\)-condition, then there exists a constant \( \vartheta(>1) \),

\[
(3.6) \quad vq(v) \leq \vartheta N(v) \quad \text{for all } v \geq v_0.
\]

THEOREM 5. Suppose that the N-function \( M(u) \) satisfies the \((\tilde{J}_2)\)-condition, then for \( g(\in \Lambda(\varphi, M)) \),

\[
\bar{\varphi}(g) = \tilde{\varphi}(g).
\]

PROOF. Since \( g^o = \varphi D^o \) for \( g^* \), in virtue of (3.4), we have

\[
\int_a^t f^* g^* \, dt = \int_a^t f^* \varphi D \, dt.
\]

(Because, for some function \( f^* = q[D^o(t)] \) as \( f(t) \), on each the maximal interval \( (a, b) \) of constancy of \( D^o \), \( D^o \) is constant, then \( q[D^o(t)] \) is constant: otherwise, \( g^* = g^o = \varphi D^o \) a.e., then \( f^* g^* = f^* \varphi D^o \).) In the Young's inequality for such \( f \), the equality sign holds.
Therefore we have
\[ \int_0^t f^* g^* dt = \rho(f) + \int_0^t \varphi N[D^\rho] dt ; \]
\[ \int_0^t f g dt \leq -\rho(f) = \tau(g), \]
\[ i. e., \]
\[ -\rho(f) \geq \tau(g). \]
Now, we have from (3.6) and Young's inequality, for any \( f(=q[D]) \),
\[ \int_0^t \varphi M[f] dt = \int_0^t \varphi D^\rho q[D^\rho] dt - \int_0^t \varphi N[D^\rho] dt \]
\[ \leq (\delta - 1) \int_0^t \varphi N[D^\rho] dt. \]
Therefore there exists a function \( f(=q[D]) \) in \( \Lambda(\varphi, M) \), and we obtain from Theorem 4, \( \rho'(g) = \tau(g) \).
\( \varphi N[D^\rho] \) is integrable whenever \( \int_0^t f g dt - \rho(f) \) is bounded. Then \( g(=\varphi D^\rho) \in \Lambda(\varphi, M) \). For the proof, see [6; p. 73-74].
We denote the Banach dual of \( \Lambda(\varphi, M) \), by \( \Lambda^*(\varphi, M) \), consisting of all linear functionals \( F_g(f) = \int_0^t f g dt \) defined on \( \Lambda(\varphi, M) \). We introduce the norm \( \| g \|_\omega \) of \( g \), which is called Orlicz norm as follows;
\[ \| g \|_\omega = \sup_{f \in \Omega, |f| \leq 1} \int_0^t f g dt \]
Then we have [3; p. 80]
\[ \| g \|_\omega \leq \| g \| \leq 2 \| g \|_\omega \]
Since the norm \( \| F_g \| \) of the linear functional \( F_g \) is defined as
\[ \| F_g \| = \sup_{\| f \|_\omega \leq 1} \int_0^t f g dt \]
we obtain that the norm \( \| g \|_\omega \) on \( \Lambda(\varphi, M) \) is equivalent to the norm \( \| F_g \| \) of the linear functional \( F_g \) on the Banach dual of \( \Lambda^*(\varphi, M) \) (note that \( \| f \|_\omega \leq 1 \Rightarrow \rho(f) \leq 1 \)). Therefore we have \( \Lambda(\varphi, M) \) is isomorphic to \( \Lambda^*(\varphi, M) \). Then we obtain the following theorem.

**Theorem 6.** If both the \( N \)-functions \( M \) and \( N \) satisfy \( (\Delta_2) \)-condition, then \( \Lambda(\varphi, M) \) is reflexive as a Banach space.
PROOF. From the definition of \( \bar{\rho}(g) \)
\[
\sup_{g \in \Lambda(\varphi, M)} \left\{ \left| \int_0^\varphi f g dt \right| - \bar{\rho}(g) \right\} \leq \rho(f).
\]
Here, for some \( f(= \rho[D]) \) [12; Theorem 2.5],
\[
\sup_{g \in \Lambda(\varphi, M)} \left\{ \left| \int_0^\varphi f g dt \right| - \bar{\rho}(g) \right\} = \rho(f).
\]
Hence we have \( \Lambda(\varphi, M) = \Lambda(\varphi, M) \). Since
\[
\Lambda(\varphi, M) = \Lambda^*(\varphi, M),
\]
we have
\[
\Lambda^{**}(\varphi, M) = \Lambda(\varphi, M)^* = \Lambda(\varphi, M) = \Lambda(\varphi, M).
\]
Thus we obtain our assertion.

References