The Ring of Arithmetic Functions of Many Variables

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We know various results on arithmetic functions as the Möbius inversion property. And the fact that they form a unique factorization domain [1] is of much interest to the author. In this note, we develop the theory of arithmetic functions of many variables over algebraic number fields.

Let \( n \) be a positive number, \( a_1, \ldots, a_n \) rings of algebraic integers in algebraic number fields of finite degrees. Let \( C \) be complex numbers, \( \mathfrak{I}_1, \ldots, \mathfrak{I}_n \) sets of ideals \( \neq 0 \) of \( a_1, \ldots, a_n \) respectively, \( \mathcal{O}=\mathcal{O}_n \) the set of functions of \( \mathfrak{I}_1 \times \cdots \times \mathfrak{I}_n \) to \( C \). For elements \( \alpha, \beta \) of \( \mathcal{O} \), we define the sum and the product as follows:

\[
(\alpha+\beta)(a) = \alpha(a) + \beta(a),
\]

\[
(\alpha \beta)(a) = \sum_{(d) \mid (a)} \alpha(d) \beta \left( \frac{a}{d} \right),
\]

where e.g. \( (a) = (a_1, \ldots, a_n) \in \mathfrak{I}_1 \times \cdots \times \mathfrak{I}_n \) is of the vector notation.

**Proposition 1.** Under these operations, \( \mathcal{O} \) forms an associative commutative ring with the identity.

**Proposition 2.** \( \mathcal{O} \) is an integral domain.

**Proposition 3.** \( \alpha \in \mathcal{O} \) is invertible if and only if \( \alpha(1) \neq 0 \).

Of course, (1) is \( (a_1, \ldots, a_n) \). Let \( U \) be the group of units of \( \mathcal{O} \). If \( 0 \neq \alpha \in \mathcal{O} \) satisfies \( \alpha((a)(b)) = \alpha(a)\alpha(b) \), whenever \( ((a), (b)) = ((a_1, b_1), \ldots, (a_n, b_n)) = 1 \), we call \( \alpha \) multiplicative, where \( (a) (b) \) is \( (a_1b_1, \ldots, a_nb_n) \).

**Proposition 4.** The set \( M \) of multiplicative functions forms a subgroup of \( U \).

This is the so-called Möbius inversion property. We will give

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examples. For any \(n\)-tuple \((k) = (k_1, \ldots, k_n)\) of real numbers and for \((a) \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_n\), we set

\[\nu_{(k)}(a) = N(a_{1})^{k_1} \cdots N(a_{n})^{k_n}.\]

Since the \(\nu_{(k)}\) is of a multiplicative function, we can define the M"obius function \(\mu = \nu_{(1)}^{-1}\), the Euler-function \(\varphi_{(k)}(a) = \nu_{(k)}^{-1} \nu(a), \sigma_{(k)} = \nu_{(k)} \nu(a)\) and \(\tau = \sigma_{(2)}\). And these become all multiplicative functions. We have following relations among them:

**Proposition 5.**

\[\varphi_{(\sigma_{(\ell)}, (\ell))}(a) = \Pi N(a_{1})^{\varphi_{(\ell)}(a)} \Pi N(d_{1})^{k_{(\ell)} - k_{(\ell)}},\]

\[\Pi N(a_{1})^{k_{(\ell)} - k_{(\ell)}} = \sum_{(\ell) \in (a)} \sigma_{(\ell)} N(d_{1})^{k_{(\ell)} - k_{(\ell)}} \varphi_{(\ell)}(a), \]

\[\Pi N(a_{1})^{k_{(\ell)} \sigma_{(\ell)}(a)} = \sum_{(\ell) \in (a)} \Pi N(d_{1})^{k_{(\ell)} \tau_{(\ell)}(d)} \varphi_{(\ell)}(a), \]

Now we will see about the prime factorization on \(\Omega\). Let \(x_{11}, x_{12}, x_{13}, \ldots; x_{21}, x_{22}, x_{23}, \ldots; \ldots; x_{n1}, x_{n2}, x_{n3}, \ldots\), be independent indeterminates. And let

\[p_{11}, p_{12}, p_{13}, \ldots; p_{21}, p_{22}, p_{23}, \ldots; \ldots; p_{n1}, p_{n2}, p_{n3}, \ldots\]

be distinct prime ideals of \(\mathfrak{o}_1, \mathfrak{o}_2, \ldots, \mathfrak{o}_n\) respectively. For any element \(\alpha\) of \(\Omega\), we set

\[P(\alpha) = \sum a(\Pi p_{11}^{e_{11}}, \ldots; \Pi p_{1\ell}^{e_{1\ell}}, \ldots; \Pi p_{n1}^{e_{n1}}, \Pi x_{11}^{e_{11}} x_{12}^{e_{12}} \ldots x_{n1}^{e_{n1}}),\]

where \(\sum\) ranges over all rational integers \(e_{ij} \geq 0\) almost all of which are zero.

**Proposition 6.** \(P\) gives an isomorphism of the ring \(\Omega\) onto the ring \(F_{\omega}\) of formal power series of complex coefficients with variables \(x_{ij}\).

Since the ring of formal power series of complex coefficients with countably infinite indeterminates is proved to be a unique factorization domain (Cashwell and Everett [1]), we have

**Proposition 7.** \(\Omega\) is a unique factorization domain.

Next we will note that we can form the unitary ring \(\Omega_{\ast}\) on \(\Omega\) by defining the unitary product

\[(\alpha \ast \beta)(a) = \sum a(\Pi p_{11}^{e_{11}}, \ldots; \Pi p_{1\ell}^{e_{1\ell}}, \ldots; \Pi p_{n1}^{e_{n1}}, \Pi x_{11}^{e_{11}} x_{12}^{e_{12}} \ldots x_{n1}^{e_{n1}}),\]

where \((d) \parallel (a)\) means that \(d\) is a unitary divisor of \(a\).
PROPOSITION 8. $\Omega_{\ast}$ is an associative commutative ring with the identity. $\alpha \in \Omega$ is invertible in $\Omega_{\ast}$ if and only if $\alpha(1) \neq 0$. The set of multiplicative functions forms a subgroup of the units group $U$ of $\Omega_{\ast}$ and similar relations with proposition 5 hold. However $\Omega_{\ast}$ is neither an integral domain nor a unique factorization ring.

Finally we will note that the theory of the semi-multiplicativity like [2] is also possible for our high dimensional case. We have for instance the following

PROPOSITION 9. $\alpha \in \Omega$ is semi-multiplicative if and only if $\alpha(a)\alpha(b) = \alpha(a, b)\alpha\langle a, b \rangle$, where $\langle a, b \rangle$ is the l. c. m of $a, b$.

References