On the potential theory for certain degenerated elliptic operators

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Introduction

The potential theory has been developed together with the study of elliptic differential equations. Suitably extended versions are also applicable to non-elliptic equations to some extent, but the value of the potential theory is rather limited in the study of genuinely degenerated operators. In this memoir we therefore develop the potential theory for a certain class of degenerated elliptic operators. Though the main purpose is to illustrate the degeneracy of the operator treated in this memoir in terms of the potential theory, it also seems that one can gain some more insight into the potential theory for elliptic operators by considering it from the point of view of wider classes of operators to be discussed here.

In this memoir we shall work with the following simple degenerated operators defined by

$$(0.1) \quad L_\alpha = -x_n \Delta - \alpha \partial_n \quad \text{for } x = (x', x_n) \in \mathbf{R}_+^n,$$

where $\partial_n = \partial/\partial x_n$, α is a real parameter and $\mathbf{R}_+^n$ is the Euclidian halfspace defined by $\{x = (x', x_n) \mid x' \in \mathbf{R}^{n-1}, x_n > 0\}$.

With the operators L_α , we shall associate nonnegative symmetric functions $K_\alpha(x, y)$ called potential kernel and define the K_α potential of measure μ on the halfspace $\mathbf{R}_+^n$. Then we shall study various properties of the K_α potentials of μ which are deeply concerned with the degeneracy of the operator L_α on the hyperplane $\{x \in \mathbf{R}^n \mid x_n = 0\}$.

Baouendi-Goulaouic [1] studied in the L^2 framework the asymptotic behaviors of the spectrum of elliptic operators degenerated in all directions at the boundary. Goulaouic-Shimakura [4] also studied the same class of operators $\mathcal{A}$ in some Hölder spaces and adopted the operator L_α as the simplest model in $\mathbf{R}_+^n$. Under some assumptions on $\mathcal{A}$ they showed the existence and uniqueness of the classical solution u of the equation " $\mathcal{A}u = f$ " for a given Hölder continuous function f . Later their result was generalized by Bolly, Camus and Metivier [2] to the case of the higher order operators. Graham also treated L_α in his article [5] in the Hölder framework (See also Horiuchi [6]). However there are few papers concerned with the potential theory for the degenerated elliptic operators. Under these conditions we shall take the opportunity to develop a potential

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theory for the operator L_α which could be considered as a substantial extension of the classical theory for the Laplacian in $\mathbf{R}^n$.

This paper is organized in the following way. In §1 we prepare basic notations and definitions. In §2 we prove the mean value theorems for the operators L_α which are analogous to those for the Laplacian in $\mathbf{R}^n$. The general setting of the potential theory will be given in §3 including the definitions of the kernel K_α . We also give our necessary and sufficient condition for the finiteness of the potential K_α there. In §4 we prove the uniqueness of the potential K_α . We also collect mostly without proofs the basic properties of K_α . In §5 we shall prove a theorem on representation of superharmonic function. In §6 we study the potentials with finite energy. Lastly in §7 we introduce the set function called the capacity and study its properties.

§1. Preliminaries

In this section we shall prepare some notations and definitions which are basic in the present paper.

In the first place we define the modified mean value $M_{\alpha,\rho}u(a)$ of $u \in C^0(\overline{\mathbf{R}_+^n})$ at a point $a \in \mathbf{R}_+^n$. By $C^0(\Omega)$ we denote the set of all continuous functions on Ω .

DEFINITION 1.1. Let $a = (a', a_n)$ be an arbitrary point in $\mathbf{R}_+^n$, and let α and ρ be arbitrary positive numbers. For $u \in C^0(\overline{\mathbf{R}_+^n})$ we set

$$(1.1) \quad M_{\alpha,\rho}u(a) = C(\alpha)\rho^{1-n-\alpha} \int_0^1 \{s(1-s)\}^{\alpha/2-1} ds \int_{\partial B_\rho^+} x_n^\alpha u(x' + a', \gamma(x_n, a_n, s)) dS_x$$

$$\text{where} \quad \gamma(x_n, a_n, s) = \sqrt{(x_n - a_n)^2 s + (x_n + a_n)^2 (1-s)},$$

$$\partial B_\rho^+ = \{x = (x', x_n) \in \mathbf{R}^n \mid |x| = \rho, x_n > 0\},$$

$$C(\alpha) = \frac{2^{\alpha-1} \pi^{-n/2} \Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)},$$

and S_x is the $(n-1)$ -dimensional Lebesgue measure.

We also introduce the followings:

DEFINITION 1.2. Let u be of class $C^0(\overline{\mathbf{R}_+^n})$. Then u is said to be α -superharmonic (resp. α -subharmonic) if $M_{\alpha,\rho}u(a) \leq u(a)$ (resp. $M_{\alpha,\rho}u(a) \geq u(a)$) for any $a \in \mathbf{R}_+^n$ and any $\rho > 0$.

DEFINITION 1.3. Let u be of class $C^0(\overline{\mathbf{R}_+^n})$. Then u is said to be α -harmonic if $M_{\alpha,\rho}u(a) = u(a)$ for any $a \in \mathbf{R}_+^n$ and any $\rho > 0$.

In order to make clear in advance the role of $M_{\alpha,\rho}u(x)$ as well as what it means, we shall trace the definition of $M_{\alpha,\rho}u(x)$ to its origin assuming that α is a positive integer in the rest of this subsection.

First we show:

LEMMA 1.1. *Let α be an arbitrary positive integer, and let u be of class $C^0(\mathbf{R}^1)$. Then it holds that*

$$(1.2) \quad \int_{|z-a|=\rho} u(|z|) dS_z = \\ = 2^{\alpha-1} |S^{\alpha-1}| \rho^\alpha \int_0^1 \{s(1-s)\}^{\alpha/2-1} u(\gamma(|a|, \rho, s)) ds,$$

where $\gamma(|a|, \rho, s) = \sqrt{(|a| - \rho)^2 s + (|a| + \rho)^2 (1-s)}$,
 $a = (a_1, a_2, \dots, a_{\alpha+1}) \in \mathbf{R}^{\alpha+1}$, $z = (z_1, z_2, \dots, z_{\alpha+1}) \in \mathbf{R}^{\alpha+1}$.

PROOF. We shall make use of the polar coordinate system defined by

$$(1.3) \quad z = a + \rho\omega, \quad a = r\omega' \quad \text{with } \omega, \omega' \in S^\alpha, \\ |z|^2 = |a|^2 + \rho^2 + 2r\rho \cos \psi, \quad \cos \psi = \omega \cdot \omega', \\ \rho^\alpha dS_\omega^\alpha = (\rho \sin \psi)^{\alpha-1} dS_{\omega''}^{\alpha-1} \rho d\psi \quad \text{with } 0 < \psi < \pi, \omega'' \in S^{\alpha-1}.$$

So we get

$$(1.4) \quad \int_{|z-a|=\rho} u(|z|) dS_z = \\ = |S^{\alpha-1}| \int_0^\pi \rho^\alpha (\sin \psi)^{\alpha-1} u\left(\sqrt{|a|^2 + \rho^2 + 2\rho|a| \cos \psi}\right) d\psi.$$

Carrying out the change of variables defined by

$$(1.5) \quad \sin \psi = 2\sqrt{s(1-s)}, \quad \text{that is,} \quad \cos \psi = 2s - 1,$$

the desired estimate follows.

Q.E.D.

In a similar way we show:

LEMMA 1.2. *Let u be of class $C^0(\mathbf{R}_x^{n-1}, \mathbf{R}_z^{\alpha+1})$, and let (a', b) be an arbitrary point in $\mathbf{R}^{n-1} \times \mathbf{R}^{\alpha+1}$. Then we have*

$$(1.6) \quad \int_{|(x', z) - (a', b)|=\rho} u(x', |z|) dS_{(x', z)} = \\ = 2^{\alpha-1} |S^{\alpha-1}| \int_0^1 \{s(1-s)\}^{\alpha/2-1} ds \int_{\partial B_\rho^+} x_n^\alpha u(x' + a', \gamma(x_n, a_n, s)) dS_x.$$

Here

$$a = (a', a_n) \quad \text{with } a_n = |b|, x = (x', x_n) \quad \text{with } x_n = |z|, \\ (a', b), (x', z) \in \mathbf{R}^{n-1} \times \mathbf{R}^{\alpha+1}, \\ \partial B_\rho^+ = \{x = (x', x_n) \in \mathbf{R}^n \mid |x| = \rho, x_n > 0\}.$$

PROOF. Again we make use of the polar coordinate system defined by
 $(x', z) = (a', b) + \rho\omega$ with $\omega \in S^{n+\alpha-1}$.

Then we have

$$(1.7) \quad dS_{(x', z)} = \rho^{n+\alpha-1} \prod_{p=1}^{n-1} (\sin \theta_p)^{n+\alpha-1-p} \prod_{p=1}^{\alpha-1} (\sin \varphi_p)^{\alpha-p} \wedge \\ \wedge d\theta_1 \wedge d\theta_2 \wedge \cdots \wedge d\theta_{n-1} \wedge d\varphi_1 \wedge d\varphi_2 \wedge \cdots \wedge d\varphi_{\alpha-1} \wedge d\phi.$$

Noting that $|z - b| = \rho \prod_{p=1}^{n-1} \sin \theta_p$, we have

$$(1.8) \quad \int_{|(x', z) - (a', b)| = \rho} u(x', |z|) dS_{(x', z)} = \\ = \rho^{n-1} \int \prod_{p=1}^{n-1} (\sin \theta_p)^{n-1-p} d\theta_1 \wedge d\theta_2 \wedge \cdots \wedge d\theta_{n-1} \int_{\Omega} u(x' + a', |z|) dS_z^{\alpha}.$$

Here Ω is defined by

$$\Omega = \{ z \in \mathbf{R}^{\alpha+1} \mid |z - b| = \rho \prod_{p=1}^{n-1} \sin \theta_p \},$$

and x' is represented by

$$x'_1 = a'_1 + \rho \cos \theta_1, \\ x'_p = a'_p + \rho \prod_{j=1}^{p-1} \sin \theta_j \cos \theta_p \quad \text{for } p = 2, \dots, n-1.$$

From Lemma 1.1 we can compute the integral with respect to z , so that we have

$$(1.9) \quad \int_{\Omega} u(x' + a', |z|) dS_z^{\alpha} = \\ = 2^{\alpha-1} |S^{\alpha-1}| \xi^{\alpha} \int_0^1 \{s(1-s)\}^{\alpha/2-1} u(x' + a', \gamma(\xi, |b|, s)) ds$$

with $\xi = \rho \prod_{j=1}^{n-1} \sin \theta_j$. Since θ_{n-1} satisfies $0 \leq \theta_{n-1} \leq \pi$, we have the desired equality. Q.E.D.

Now it is clear that the definition of $M_{\alpha, \rho} u(a)$ comes from the equality (1.6) provided α is a positive integer.

Here we remark that the integral of the righthand side of (1.1) is not convergent if $\alpha \in (-1, 0]$, nevertheless we see that $M_{\alpha, \rho} u(a)$ can be continued analytically with respect to α in that case. In particular if $\alpha = 0$, we have

$$M_{0, \rho} u(a) = \\ 2^{-1} \pi^{-n/2} \Gamma\left(\frac{n}{2}\right) \rho^{1-n} \int_{\partial B_p^+} \{u(x' + a', x_n + a_n) + u(x' + a', |x_n - a_n|)\} dS_x.$$

The analytic continuation method used here is essentially as follows:

Let $f(\theta)$ be a function of class $C^1([0, 1])$ and τ be a complex parameter with $\operatorname{Re} \tau > 0$. Then we have the equality

$$(1.10) \quad \Gamma(\tau)^{-1} \int_0^1 f(\theta) \theta^{\tau-1} d\theta = \\ = \Gamma(\tau)^{-1} \int_0^1 \{f(\theta) - f(0)\} \theta^{\tau-1} d\theta + \Gamma(\tau+1)^{-1} f(0).$$

The righthand side can be continued analytically to $\operatorname{Re} \tau > -1$. Thus we can extend the definition of the lefthand side to $\operatorname{Re} \tau > -1$ by this equality.

§2. The mean value theorems

In this section we shall describe two important properties of the operator L_α in terms of the mean value $M_{\alpha, \rho} u(a)$ defined in the previous section. It will be the starting point in this paper.

THEOREM 2.1. *Suppose that $\alpha > 0$ and that u is of class $C^2(\overline{\mathbf{R}_+^n})$. Then u is α -superharmonic (resp. α -subharmonic) if and only if $L_\alpha u \geq 0$ (resp. $L_\alpha u \leq 0$) holds.*

THEOREM 2.2. *Suppose that $\alpha > 0$ and that u is of class $C^0(\overline{\mathbf{R}_+^n})$. Moreover we suppose that u is α -harmonic, that is:*

$$(2.1) \quad M_{\alpha, \rho} u(a) = u(a) \quad \text{holds for any } \rho > 0, a \in \mathbf{R}_+^n.$$

Then u is of class $C^\infty(\overline{\mathbf{R}_+^n})$ and satisfies $L_\alpha u = 0$.

From these theorems we immediately have:

COROLLARY 2.1. *Suppose that $\alpha > 0$ and that u is of class $C^2(\overline{\mathbf{R}_+^n})$. Moreover we suppose that u satisfies $L_\alpha u = 0$. Then u is of class $C^\infty(\overline{\mathbf{R}_+^n})$.*

Here we remark that these results remain valid in the case $\alpha > -1$ by the method of analytic continuation (See the previous section.).

PROOF OF THEOREM 2.1. In the first place we recall that

$$\gamma(x_n, a_n, s) = [x_n^2 + a_n^2 + 2a_n x_n(1 - 2s)]^{1/2} \quad \text{and} \quad \gamma(0, a_n, s) = a_n.$$

By Taylor's expansion formula we get

$$(2.2) \quad u(x' + a', \gamma(x_n, a_n, s)) - u(a) = \\ = \sum_{j=1}^{n-1} x_j \partial_j u(a) + (1 - 2s) x_n \partial_n u(a) + \\ + 2^{-1} \sum_{i, j \leq n-1} x_i x_j \partial_i \partial_j u(a) + 2^{-1} (1 - 2s) \sum_{j=1}^{n-1} x_n x_j \partial_n \partial_j u(a) + \\ + 2^{-1} (1 - 2s)^2 x_n^2 \partial_n^2 u(a) + 4s(1 - s) a_n^{-1} x_n^2 \partial_n u(a) + o(|x|^2).$$

We prepare the following lemmas (The proof is omitted.).

LEMMA 2.1. Suppose that $k(j) \geq 0$ for $j = 1, \dots, n$. Then we have

$$(2.3) \quad \int_{S^{n-1}} |x_1^{2k(1)} \cdot x_2^{2k(2)} \dots x_n^{2k(n)}| dS_x = \\ = 2 \prod_{i=1}^n \Gamma\left(\frac{k(i)+1}{2}\right) \Gamma\left(\frac{1}{2}\left(\sum_{j=1}^n k(j) + n\right)\right)$$

LEMMA 2.2. Let $\operatorname{Re} \beta > 0$. Then we have

$$(2.4) \quad \pi^{1/2} \Gamma(\beta) = 2^{\beta-1} \Gamma\left(\frac{\beta}{2}\right) \Gamma\left(\frac{\beta+1}{2}\right).$$

Using these lemmas we have

$$(2.5) \quad M_{\alpha, \rho} u(a) = u(a) - \rho^2 \{2\dot{a}_n(n + \alpha)\}^{-1} L_\alpha u(a) + o(\rho^2).$$

Since ρ and a are arbitrary, we have $L_\alpha u \geq 0$ if u is α -superharmonic.

We proceed to the proof of the sufficiency. Assume that $L_\alpha u \geq 0$. Then it suffices to show that $\partial_\rho M_{\alpha, \rho} u(y) \leq 0$ for any $y \in \mathbf{R}_+^n$. In fact the α -superharmonicity follows from this inequality and the property

$$\lim_{\rho \rightarrow 0} M_{\alpha, \rho} u(y) = u(y).$$

By Green's formula and the homogeneity with respect to ρ we have

$$(2.6) \quad C(\alpha)^{-1} \partial_\rho M_{\alpha, \rho} u(y) = \\ = \rho^{1-\alpha-n} \int_0^1 \{s(1-s)\}^{\alpha/2-1} ds \int_{\partial B_\rho} x_n^\alpha \partial_\nu u(x' + y', \gamma(x_n, y_n, s)) dS_x \\ = \rho^{1-\alpha-n} \int_0^1 \{s(1-s)\}^{\alpha/2-1} ds \times \\ \times \int_{\partial B_\rho^+} x_n^\alpha \left[\left(\Delta_z + \alpha z_n^{-1} \frac{\partial}{\partial z_n} \right) u(x' + z', z_n) \right]_{\substack{z' = y' \\ z_n = \gamma(x_n, y_n, s)}} dS_x \leq 0.$$

Here $\Delta_z = \sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}$ and ν is the outer normal. Q.E.D.

PROOF OF THEOREM 2.2. We prepare more notations. Take a nonnegative smooth function Φ such that $\Phi(\rho) = 1$ in some neighborhood of the origin,

$$(2.7) \quad \int_0^\infty \rho^{n+\alpha-1} \Phi(\rho) d\rho = 1, \quad \Phi(|x|) \in C^\infty(\mathbf{R}^n).$$

Now we define a mollification of u as follows:

$$(2.8) \quad u_{\varepsilon, \alpha}(x) = C(\alpha) \varepsilon^{-n-\alpha} \int_0^1 \{s(1-s)\}^{\alpha/2-1} ds \times \\ \times \int_{\mathbf{R}^{n-1}} \int_0^\infty r^\alpha u(x' - y', r) \Phi\left(\frac{\sqrt{|y'|^2 + \gamma(x_n, r, s)^2}}{\varepsilon}\right) dy' dr,$$

for $u \in C^0(\overline{\mathbf{R}_+^n})$ and $\varepsilon > 0$. Then obviously $u_{\varepsilon, \alpha} \in C^\infty(\overline{\mathbf{R}_+^n})$ for any $\varepsilon > 0$ and $\alpha > 0$. Moreover we show:

LEMMA 2.3. *It holds that*

$$(2.9) \quad u_{\varepsilon, \alpha}(x) = C(\alpha) \varepsilon^{-n-\alpha} \int_0^1 \{s(1-s)\}^{\alpha/2-1} ds \times \\ \times \int_{\mathbf{R}^{n-1}} \int_0^\infty u(x' - y', \gamma(x_n, r, s)) r^\alpha \Phi\left(\frac{\sqrt{|y'|^2 + r^2}}{\varepsilon}\right) dy' dr$$

for any $\varepsilon > 0$ and $\alpha > 0$.

Admitting this for a moment, we show that $u_{\varepsilon, \alpha}(x) = u(x)$ if u is α -harmonic. In fact we immediately have

$$(2.10) \quad u_{\varepsilon, \alpha}(x) = C(\alpha) \int_0^\infty \Phi(\rho/\varepsilon) \varepsilon^{-n-\alpha} d\rho \int_0^1 \{s(1-s)\}^{\alpha/2-1} ds \times \\ \times \int_{\partial B_\rho^+} r^\alpha u(x' - y', \gamma(x_n, r, s)) dS_{(y', r)} = \\ = u(x) \int_0^\infty \rho^{n+\alpha-1} \Phi(\rho) d\rho = u(x),$$

where $S_{(y', r)}$ is the $(n-1)$ -dimensional Lebesgue measure. Therefore we have proved that u is smooth if u is α -harmonic. Q.E.D.

PROOF OF LEMMA 2.3. Let us set

$$(2.11) \quad \varphi \sharp \psi(t) = \\ = 2^{\alpha-1} \int_0^\infty r^\alpha \varphi(r) dr \int_0^1 \{s(1-s)\}^{\alpha/2-1} \psi(\gamma(t, r, s)) ds = \\ = \int_0^\infty r^\alpha \varphi(r) dr \int_0^\pi \psi\left(\sqrt{t^2 + r^2 - 2tr \cos \theta}\right) \sin^{\alpha-1} \theta d\theta,$$

for $\varphi, \psi \in C^0(\overline{\mathbf{R}_+^n})$. Here we used the change of variable defined by $\cos \theta = 2s - 1$.

To prove Lemma 2.3 we have only to show the equality $\varphi \sharp \psi = \psi \sharp \varphi$, which implies the validity of the commutative law. First we put $r = t\tau$ and then subdivide the integral into two terms. So that we obtain

$$(2.12) \quad \varphi \sharp \psi(t) = I_1 + I_2,$$

$$I_1 = t^{\alpha+1} \int_0^1 \varphi(t\tau) \tau^\alpha d\tau \int_0^\pi \psi\left(t\sqrt{1 + \tau^2 - 2\tau \cos \theta}\right) \sin^{\alpha-1} \theta d\theta, \\ I_2 = t^{\alpha+1} \int_1^\infty \varphi(t\tau) \tau^\alpha d\tau \int_0^\pi \psi\left(t\sqrt{1 + \tau^2 - 2\tau \cos \theta}\right) \sin^{\alpha-1} \theta d\theta.$$

Secondly we carry out a change of variables in the inner integral defined by $\sigma^2 = 1 + \tau^2 - 2\tau \cos \theta$, so we get

$$(2.13) \quad \begin{aligned} I_1 &= 2^{1-\alpha} t^{\alpha+1} \int_0^1 \varphi(t\tau) \tau d\tau \int_{1-\tau}^{1+\tau} g(\sigma, \tau)^{\alpha-2} \psi(t\sigma) \sigma d\sigma, \\ I_2 &= 2^{1-\alpha} t^{\alpha+1} \int_1^\infty \varphi(t\tau) \tau d\tau \int_{\tau-1}^{\tau+1} g(\sigma, \tau)^{\alpha-2} \psi(t\sigma) \sigma d\sigma, \end{aligned}$$

where $g(\sigma, \tau) = \sqrt{|2(\sigma^2 + \tau^2) - (\sigma^2 - \tau^2)^2 - 1|}$.

Therefore we get

$$(2.14) \quad \varphi \sharp \psi(t) = 2^{1-\alpha} t^{\alpha+1} \int_{\Lambda^{\sigma, \tau}} \sigma \tau g(\sigma, \tau)^{\alpha-2} \varphi(t\tau) \psi(t\sigma) d\sigma d\tau,$$

where $\Lambda^{\sigma, \tau} = \{(\sigma, \tau) \in \mathbf{R}_+ \times \mathbf{R}_+ \mid |1 - \tau| \leq \sigma \leq 1 + \tau\}$.

After all we have proved the commutative law for the $\sharp$ -product, because $g(\sigma, \tau)$ and $\Lambda^{\sigma, \tau}$ are symmetric with respect to σ and τ .

Here we briefly remark the case $\alpha < -1$ which is excluded in the mean value theorems. Assume that $\alpha < -1$. Let u be a solution of the equation $L_\alpha u = 0$ with Dirichlet boundary condition $u|_{x_n=0} = 0$. Then from the theory of degenerated elliptic equations it follows that:

For any point $x \in \mathbf{R}_+^n$ and any $r > 0$, there is a smooth function $v \in C^\infty(\overline{\mathbf{R}_+^n})$ such that

$$(2.15) \quad u(x) = x_n^{1-\alpha} v(x), \quad \text{for } x \in B_r(x) \cap \mathbf{R}_+^n.$$

Noting that $L_\alpha(x_n^{1-\alpha}) = 0$, we have

$$(2.16) \quad L_\alpha u = x_n^{1-\alpha} L_\alpha v + [L_\alpha, x_n^{1-\alpha}] v = x_n^{1-\alpha} L_{2-\alpha} v.$$

Since $2 - \alpha > 0$, it follows from the theorems that

$$(2.17) \quad u(x) = x_n^{1-\alpha} M_{2-\alpha, \rho} v(x) \quad \text{with } v(x) = x_n^{\alpha-1} u(x) \text{ and } \rho < r.$$

§3. Potential kernels

In this section we shall introduce potential kernels closely related to the operator L_α .

Let us set

$$(3.1) \quad \begin{aligned} E_\alpha(x, y) &= C_\alpha y_n^{\alpha-1} |x - y^*|^{-\alpha} |x - y|^{2-n} F(\alpha, \omega), \quad \alpha > 0, \\ \mathcal{E}_\alpha(x, y) &= C_{2-\alpha} x_n^{1-\alpha} |x - y^*|^{\alpha-2} |x - y|^{2-n} F(2 - \alpha, \omega), \quad \alpha < 2, \\ K_\alpha(x, y) &= C_\alpha |x - y^*|^{-\alpha} |x - y|^{2-n} F(\alpha, \omega), \quad \alpha > 0, \end{aligned}$$

where

$$\begin{aligned} y^* &= (y', -y_n), \\ \omega &= 1 - \frac{|x - y|^2}{|x - y^*|^2} = \frac{4x_n y_n}{|x - y^*|^2}, \\ C_\alpha &= \frac{2^{\alpha-2} \pi^{-n/2} \Gamma\left(\frac{n + \alpha - 2}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha)} \end{aligned}$$

and $F(\alpha, \omega)$ is a hypergeometric function defined by

$$(3.2) \quad F(\alpha, \omega) = \frac{\Gamma(\alpha)}{\Gamma\left(\frac{\alpha+2-n}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \sum_{j=0}^{\infty} \frac{\Gamma\left(j + \frac{\alpha+2-n}{2}\right)\Gamma\left(j + \frac{\alpha}{2}\right)}{\Gamma(j+\alpha)} \frac{\omega^j}{j!}.$$

Moreover $F(\alpha, \omega)$ satisfies the following estimates (The proof is omitted.):

LEMMA 3.1. *There exist positive numbers A and B such that*

$$(3.3) \quad \begin{aligned} A^{-1} &\leq F(\alpha, \omega) \leq A, & n \geq 3, \alpha > 0, \\ B^{-1} &\leq \frac{F(\alpha, \omega)}{\log \left[2 + \frac{|x - y^*|}{|x - y|} \right]} \leq B, & n = 2, \alpha > 0. \end{aligned}$$

Here A and B are independent of x and y .

Then $E_\alpha(x, y)$ and $\mathcal{E}_\alpha(x, y)$ are Green functions for the operator L_α in the following sense (For the proof, see [4] and [6].):

LEMMA 3.2. *Suppose that $f \in C^0(\overline{\mathbf{R}_+^n}) \cap \mathcal{E}'$ and $w \in C^2(\overline{\mathbf{R}_+^n}) \cap \mathcal{E}'$. Let us set*

$$(3.4) \quad \begin{aligned} u(x) &= E_\alpha f(x) = \int_{\mathbf{R}_+^n} E_\alpha(x, y) f(y) dy, \\ v(x) &= \mathcal{E}_\alpha f(x) = \int_{\mathbf{R}_+^n} \mathcal{E}_\alpha(x, y) f(y) dy. \end{aligned}$$

Then it holds that

$$(3.5) \quad \begin{cases} L_\alpha u(x) = f(x), & \operatorname{Re} \alpha > 0, \\ L_\alpha v(x) = f(x) \quad \text{and} \quad v|_{x_n=0} = 0, & \operatorname{Re} \alpha < 1. \end{cases}$$

Moreover it holds that

$$w(x) = E_\alpha(L_\alpha w)(x), \quad \operatorname{Re} \alpha > 0.$$

Here, by $\mathcal{E}'$ we mean the set of distributions having compact support.

The kernel $K_\alpha(x, y)$ is also a fundamental solution in $\mathbf{R}_+^n$ of the symmetric operator $-\nabla \cdot (x_n^\alpha \nabla \cdot)$ with $\alpha > 0$. Among these kernels there exist the following relations: For $\alpha > 0$,

$$(3.7) \quad \begin{aligned} K_\alpha(x, y) &= y_n^{1-\alpha} E_\alpha(x, y) = x_n^{1-\alpha} \mathcal{E}_{2-\alpha}(x, y), \\ E_\alpha(x, y) &= \mathcal{E}_{2-\alpha}(y, x) \quad \text{and} \quad L_\alpha = -x_n^{1-\alpha} \nabla \cdot (x_n^\alpha \nabla \cdot). \end{aligned}$$

In particular if $n \geq 3$ and $\alpha = n - 2$, then we can compute $F(\alpha, \omega)$ to obtain

$$(3.8) \quad K_{n-2}(x, y) = 2^{n-4} \pi^{-n/2} \Gamma\left(\frac{n-2}{2}\right) (|x - y^*| |x - y|)^{2-n}.$$

Now we consider in the Euclidian half space $\mathbf{R}_+^n$ a domain Ω which may coincide with $\mathbf{R}_+^n$. Let $B(\Omega)$ denote the σ -algebra of Borel subset of Ω . The set of all measures in Ω we will denote by $M(\Omega)$ and the subset of all nonnegative measures by $M^+(\Omega)$. We introduce a vague topology in $M(\Omega)$, that is to say, we define a fundamental system of neighborhood of the null element by the following:

$$(3.9) \quad V(\varepsilon, f_1, f_2, \dots, f_p) = \left\{ \mu \in M(\Omega) \mid \left| \int_{\Omega} f_k d\mu(x) \right| < \varepsilon, \quad k = 1, \dots, p \right\}$$

for $\varepsilon > 0$ and $f_k \in C_0^0(\Omega)$, $k = 1, \dots, p$, where we denote by $C_0^0(\Omega)$ the set of continuous functions having compact support in Ω .

We define the $K_\alpha(x, y)$ potential of the measure μ .

DEFINITION 3.1. Let $\mu \in M(\Omega)$. Suppose that $\alpha > 0$. Then we set

$$(3.10) \quad K_\alpha(x) = \int_{\text{supp } \mu} K_\alpha(x, y) d\mu(y) = C_\alpha \int_{\text{supp } \mu} |x - y^*|^\alpha |x - y|^{2-n} F(\alpha, \omega) d\mu(y).$$

In order to insure that $K_\alpha \mu(x)$ exists, we have to impose certain restriction on μ . So we shall mention here the following result which gives a necessary and sufficient condition for finiteness almost everywhere of the potential $K_\alpha \mu(x)$ for $\mu \in M^+(\mathbf{R}_+^n)$.

THEOREM 3.1. Let $\alpha > 0$ and $n \geq 2$. For $\mu \in M^+(\mathbf{R}_+^n)$, a necessary and sufficient condition for the potential $K_\alpha \mu(x)$ to belong to $L_{loc}^1(\mathbf{R}_+^n)$ is the following inequality:

$$(3.11) \quad \begin{aligned} & \int_{|y| \geq 1} |y|^{2-\alpha-n} d\mu(y) < +\infty, \quad 0 < \alpha < 2, \\ & \int_{|y| \geq 1} |y|^{2-\alpha-n} d\mu(y) + \\ & + \int_{|y| \leq 1} \max \left[\log \left(\frac{1}{y_n} \right), y_n^{2-\alpha} \right] d\mu(y) < +\infty, \quad \alpha \geq 2. \end{aligned}$$

PROOF. We suppose that $n \geq 3$. The proof can be easily modified in the case $n = 2$. First we recall

$$(3.12) \quad K_\alpha(x, y) \sim |x - y^*|^{-\alpha} |x - y|^{2-n}.$$

By " $A \sim B$ " we mean that there are two positive constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$ holds for any x and y .

Suppose that μ_1 is the restriction of the measure to the half ball $B_{2r}^+ = \{y \in \mathbf{R}^n \mid |y| < 2r, y_n > 0\}$, and that $\mu_2 = \mu - \mu_1$. Then $K_\alpha \mu = K_\alpha \mu_1 + K_\alpha \mu_2$ and we have

$$(3.13) \quad \int_{|x| < r} K_\alpha \mu_2(x) dx \leq \text{Const.} |B_r^+| \int_{|y| > 2r} |y|^{2-\alpha-n} d\mu(y),$$

where we used the inequality

$$\frac{|y|}{|x-y|} \leq \frac{|y|}{|y|-|x|} \leq 2 \quad \text{for } x \in B_r^+, y \in (B_{2r}^+)^c.$$

This proves the sufficiency. To see the necessity we assume that $K_\alpha \mu(x) < +\infty$ for some x . If $|y| \geq |x| + 1$ holds, then we have

$$(3.14) \quad K_\alpha \mu_2(x) \geq \int_{|y| \geq |x|+1} \left(\frac{|x|+1}{2|x|+1} \right)^{n-2} \cdot \left(\frac{|x|}{|x-y^*|} \right)^\alpha \cdot |y|^{2-\alpha-n} d\mu(y) \geq \\ \geq \text{Const.} \int_{|y| \geq |x|+1} |y|^{2-\alpha-n} d\mu(y).$$

Next we consider $K_\alpha \mu_1$. Let us set for $|y| < r/2$ and $y_n \geq 0$,

$$(3.15) \quad I_\alpha(y) = \int_{|x| < r} |x-y^*|^{-\alpha} |x-y|^{2-n} dx.$$

By virtue of the polar coordinate system we have

$$(3.16) \quad I_\alpha \sim \int_{|y+\rho\omega| \leq r} (\rho+2y_n)^{-\alpha} \rho d\rho dS_\omega \sim y_n^{2-\alpha} \int_0^{r/y_n} (\rho+1)^{-\alpha} \rho d\rho.$$

Here we used the notation $x = y + \rho\omega$, $\omega \in S^{n-1}$ and the obvious inequality $r + y_n \leq |x - y^*| \leq \min\{2x_n + r, 2y_n + r\}$ with $|y| < r/2$. The last integral can be estimated as follows: For $0 < y_n \leq 1/2$,

$$(3.17) \quad \int_0^{1/y_n} (\rho+1)^{-\alpha} \rho d\rho \sim \begin{cases} y_n^{\alpha-2} & 0 < \alpha < 2 \\ \log\left(\frac{1}{y_n}\right) & \alpha = 2 \\ 1 & \alpha > 2. \end{cases}$$

Combining this estimate with (3.16), the assertion is now clear.

Q.E.D.

§4. The basic properties of potential

In this section we shall describe mostly without proofs basic properties concerned with the K_α potential of μ ; semi-continuity, principle of continuity, uniqueness and so on.

Here we remark that: When we consider measures μ we shall always assume that the condition (3.11) is satisfied, without explicit mention on each occasion.

First we note that the potential $K_\alpha\mu(x)$ is lower semi-continuous not only relative to the points but also relative to the measure $\mu \in M^+(\mathbf{R}_+^n)$. Thus we have:

PROPOSITION 4.1 (SEMI-CONTINUITY, PRINCIPLE OF DESCENT). *If $\mu_j \rightarrow \mu$ ($\mu_j, \mu \in M^+(\mathbf{R}_+^n)$) and $x_j \rightarrow x$, then it holds that*

$$(4.1) \quad K_\alpha\mu(x) \leq \lim_{j \rightarrow +\infty} K_\alpha\mu_j(x_j).$$

We also have:

PROPOSITION 4.2 (PRINCIPLE OF CONTINUITY). *Let $\mu \in M^+(\mathbf{R}_+^n)$. Suppose that μ has compact support. If the potential $K_\alpha\mu$ of the measure μ is continuous on the support of μ then it is continuous in $\mathbf{R}_+^n$.*

PROOF. It suffices to prove that $K_\alpha\mu(x)$ is continuous at the points of $\text{supp } \mu$. Take and fix a point $x \in \text{supp } \mu$. Suppose that μ_ϵ is the restriction of μ to the ball $B_\epsilon(x)$ with center x and radius $\epsilon > 0$. Since $K_\alpha\mu(x)$ is finite almost everywhere, for any given $\eta > 0$ we can find $\epsilon > 0$ such that $K_\alpha\mu_\epsilon(y) < \eta$ holds for any $y \in B_\epsilon(x)$. Since $\text{supp } \mu$ is compact, this is true for all points $x \in \text{supp } \mu$. Then the assertion follows from the next lemma (Its proof is omitted.).

LEMMA 4.1 (WEAK MAXIMUM PRINCIPLE). *If $K_\alpha\mu(x) \leq M$ holds on the support of μ , then $K_\alpha\mu(x) \leq CM$ in the whole space $\mathbf{R}_+^n$, where C is a positive constant depending only on n and α .*

Lastly we give the uniqueness theorem for the K_α potential. By a signed measure μ in $\mathbf{R}_+^n$, we mean a real σ -additive set function on $\mathbf{R}_+^n$.

PROPOSITION 4.3 (UNIQUENESS). *If the potential of a signed measure μ equals zero almost everywhere, then $\mu \equiv 0$, and consequently, $K_\alpha\mu(x) \equiv 0$.*

PROOF. Without loss of generality we assume that $\mu \in M^+(\mathbf{R}_+^n)$. It suffices to verify the equation $\mu(\varphi) = 0$ for the set of functions $\varphi \in C_0^0(\overline{\mathbf{R}_+^n})$ dense in $C_0^0(\overline{\mathbf{R}_+^n})$. For such a subset we take the set of functions $f \in C_0^2(\overline{\mathbf{R}_+^n})$, having continuous derivatives of order 2 and compact support. Making use of Lemma 3.2 and the relation (3.7), we shall show

$$(4.2) \quad \begin{aligned} \mu(f) &= - \int d\mu(y) \int K_\alpha(x, y) \nabla \cdot (x_n^\alpha \nabla f(x)) dx (*) \\ &= - \int \nabla \cdot (x_n^\alpha \nabla f(x)) dx \int K_\alpha(x, y) d\mu(y) = 0. \end{aligned}$$

For this, it suffices to see the expression (*) is well-defined. Noting that μ satisfies the condition (3.11), we shall estimate the integral. Suppose that $0 < \alpha < 1$. Then integrating by parts we have

$$(4.3) \quad \int K_\alpha(x, y) \nabla \cdot (x_n^\alpha \nabla f(x)) dx = - \int x_n^\alpha \nabla K_\alpha(x, y) \cdot \nabla f(x) dx.$$

For some positive number C depending only on n and α , we also have

$$(4.4) \quad \begin{aligned} |x_n^\alpha \nabla K_\alpha(x, y)| &\leq C|x - y|^{1-n} \quad \text{for } \alpha > 0, \\ |\nabla f(x)| &\leq C(1 + |x|^{n-1})^{-1}. \end{aligned}$$

From (4.4) we have

$$(4.5) \quad \int x_n^\alpha |\nabla K_\alpha(x, y)| |\nabla f(x)| dx \leq C'|y|^{1-n}.$$

Since $|y|^{1-n} \leq |y|^{2-n-\alpha}$ holds if $|y| \geq 1$ and $0 < \alpha < 1$, it follows from (3.11) that the expression (*) is well-defined.

We proceed to the case $\alpha \geq 1$. Since $x_n^{\alpha-1} L_\alpha f(x)$ is bounded in $\mathbf{R}_+^n$, the assertion follows direct from Theorem 3.1. Q.E.D.

In a similar way we have:

COROLLARY 4.1. *Suppose that in the region Ω the potential of a signed measure $K_\alpha \mu(x)$ equals a α -harmonic function $h(x)$ almost everywhere. Then $\mu \equiv 0$ in Ω .*

PROOF. As in the proof of the previous theorem, it suffices to show that $\mu(f) = 0$ for all functions with two continuous derivatives and with compact support in Ω . Then, we have by Green's formula,

$$(4.6) \quad \begin{aligned} \mu(f) &= \iint K_\alpha(x, y) x_n^{\alpha-1} L_\alpha f(x) dx d\mu(y) \\ &= - \iint K_\alpha(x, y) \nabla \cdot (x_n^\alpha \nabla f(x)) dx d\mu(y) \\ &= - \int h(x) \nabla \cdot (x_n^\alpha \nabla f(x)) dx = - \int \nabla \cdot (x_n^\alpha \nabla h(x)) f(x) dx \\ &= \int x_n^{\alpha-1} L_\alpha h(x) \cdot f(x) dx = 0, \end{aligned}$$

as required. Q.E.D.

§5. Representation of α -superharmonic functions by potentials

In this section we shall prove a theorem on representation of α -superharmonic function.

THEOREM 5.1. *Suppose that f is a α -superharmonic function in $\mathbf{R}_+^n$. Then for any bounded region Ω there exist a measure μ_Ω concentrated on Ω , and a function h_Ω α -harmonic in Ω such that*

$$f(x) = K_\alpha \mu_\Omega(x) + h_\Omega(x) \quad x \in \Omega.$$

Moreover this expression is unique.

PROOF. If f is smooth, then the assertion holds with $\mu_\Omega = -\nabla \cdot (x_n^\alpha \nabla f)$. So we will show in general case that a derivative $\nabla \cdot (x_n^\alpha \nabla f)$ in a distribution sense turns out to be a measure in $\mathbf{R}_+^n$. To this end we consider the following measure ν_ρ^a with $a \in \mathbf{R}_+^n$ and $\rho > 0$:

$$(5.1) \quad \nu_\rho^a(f) = \rho^{-2} [f(a) - M_{\alpha, \rho} f(a)] \quad \text{with } f \in C^0(\mathbf{R}_+^n).$$

Now we assume that f is α -superharmonic. Then from Theorem 1 it follows that $\lim_{\rho \rightarrow 0} \nu_\rho^a(f) = [2(n + \alpha)a_n]^{-1} L_\alpha f(a)$, $\nu_\rho^a(f) > 0$, and $\nu_\rho^a(f)$ is locally integrable with respect to a . So that $x_n^{-1} L_\alpha f(x)$ is a measure as a limit of nonnegative, locally integrable measure in $\mathbf{R}_+^n$, and this proves the assertion because $-\nabla \cdot (x_n^\alpha \nabla f) = x_n^\alpha \{x_n^{-1} L_\alpha f\}$ holds.

Secondly we show that $h_\Omega \equiv f - K_\alpha \mu_\Omega$ is α -harmonic. Indeed, according to the method of mollification in Theorem 2 we can easily verify the smoothness of h_Ω . Lastly we prove the uniqueness. Assume that $K_\alpha \mu_1 + h_1 = K_\alpha \mu_2 + h_2$ holds with h_1, h_2 being α -harmonic. Then the signed measure $\mu = \mu_1 - \mu_2$ has the potential $K_\alpha \mu = h_2 - h_1 = h$. Then by virtue of Corollary 4.1 we immediately have $\mu \equiv 0$. This proves the assertion. Q.E.D

§6. Potential with finite energy

In this section we shall introduce the energy of the signed measure and study the space of all measures with finite energy. We define the energy of measure $\mu \in M^+(\mathbf{R}_+^n)$ by

$$(6.1) \quad I_\alpha(\mu) = \iint_{\mathbf{R}_+^n \times \mathbf{R}_+^n} K_\alpha(x, y) d\mu(x) d\mu(y).$$

We also define the mutual energy of $\mu, \nu \in M^+(\mathbf{R}_+^n)$ as follows.

$$(6.2) \quad I_\alpha(\mu, \nu) = \iint_{\mathbf{R}_+^n \times \mathbf{R}_+^n} K_\alpha(x, y) d\mu(x) d\nu(y).$$

Let μ be a signed measure. By the Jordan decomposition, there exist two non-negative measures μ_+ and μ_- so that we have

$$(6.3) \quad \mu = \mu_+ - \mu_-, \quad |\mu| = \mu_+ + \mu_-,$$

where by $|\mu|$ we denote the total variation of μ . Then the mutual energy of signed measures μ and ν is also defined by (6.2) under the assumption

$$(6.4) \quad \iint_{\mathbf{R}_+^n \times \mathbf{R}_+^n} K_\alpha(x, y) d\mu_\pm(x) d\nu_\mp(y) < +\infty.$$

Now we shall briefly describe the basic properties of the mutual energy. Since $I_\alpha(\mu, \nu)$ is a lower semi-continuous function on $M^+ \times M^+$, we have:

PROPOSITION 6.1. *It holds that if $\mu_j \rightarrow \mu$ vaguely and $\mu_j \in M^+(\mathbf{R}_+^n)$, then $I_\alpha(\mu) \leq \lim_{j \rightarrow \infty} I_\alpha(\mu_j)$.*

We show the positiveness of $I_\alpha(\mu)$:

THEOREM 6.1. *The inequality $I_\alpha(\mu) = \int K_\alpha \mu(x) d\mu(x) \geq 0$ holds for any $\mu \in M(\mathbf{R}_+^n)$. Moreover, $I_\alpha(\mu) = 0$ if and only if $\mu = 0$.*

PROOF. The following is elementary (The proof is omitted.).

LEMMA 6.1. *For any $\mu \in M^+$ and $\nu \in M^+$, we have*

$$(6.5) \quad I_\alpha(\mu, \nu)^2 \leq I_\alpha(\mu) \cdot I_\alpha(\nu).$$

By the Jordan decomposition we have $\mu = \mu_1 - \mu_2$ for some $\mu_1, \mu_2 \in M^+$. Then

$$(6.6) \quad \begin{aligned} I_\alpha(\mu) &= \int K_\alpha(\mu_1 - \mu_2)(x) d(\mu_1 - \mu_2)(x) = \\ &= I_\alpha(\mu_1) + I_\alpha(\mu_2) - 2I_\alpha(\mu_1, \mu_2) \geq 0. \end{aligned}$$

Next we assume that $I_\alpha(\mu) = 0$. Let f be of class $C_0^2(\overline{\mathbf{R}_+^n})$ and put $g = L_\alpha f$. Then

$$(6.7) \quad |\mu(f)| = \left| \int d\mu(y) \int K_\alpha(x, y) x_n^{\alpha-1} g(x) dx \right| \leq I_\alpha(\mu)^{1/2} I_\alpha(x_n^{\alpha-1} g(x))^{1/2}.$$

$$(6.8) \quad \begin{aligned} I_\alpha(x_n^{\alpha-1} g(x)) &= \iint K_\alpha(x, y) x_n^{\alpha-1} g(x) dx y_n^{\alpha-1} g(y) dy \\ &= - \iint E_\alpha(y, x) L_\alpha f(x) dx \nabla \cdot (y_n^\alpha \nabla f(y)) dy \\ &= - \int f(y) \nabla \cdot (y_n^\alpha \nabla f(y)) dy \\ &= \int y_n^\alpha |\nabla f(y)|^2 dy < +\infty. \end{aligned}$$

So that we have $\mu(f) = 0$ as required.

Q.E.D.

By $\mathbf{E}_\alpha$ we denote the space of all measures having finite energy. We define in $\mathbf{E}_\alpha$ the inner product and the norm by

$$(6.10) \quad (\mu, \nu)_\alpha \equiv I_\alpha(\mu, \nu) \quad \text{and} \quad \|\mu\| \equiv \sqrt{(\mu, \mu)_\alpha} \quad \text{respectively.}$$

Then $\mathbf{E}_\alpha$ becomes a Hilbert space, and in addition to the vague convergence we can define two different forms of convergence in $\mathbf{E}_\alpha$. Namely convergence in the norm is called "strong convergence", and we say that a sequence μ_j weakly converges to μ if $(\mu_j, \nu) \rightarrow (\mu, \nu)$ for any $\nu \in \mathbf{E}_\alpha$. We briefly mention the relations among these convergences as a theorem.

THEOREM 6.2.

- (1) $\mu_j \rightarrow \mu$ (vague) and $\|\mu_j\|_\alpha < C$, then $\mu_j \rightarrow \mu$ (weak).
- (2) $\mu_j \rightarrow \mu$ (weak) and $\mu_j \in \mathbf{E}_\alpha \cap M^+$, then $\mu_j \rightarrow \mu$ (vague).
- (3) If a sequence $\{\mu_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{E}_\alpha \cap M^+$ in the strong sense and $\mu_j \rightarrow \mu$ (vague), then $\mu \in \mathbf{E}_\alpha \cap M^+$ and μ_j strongly converges to μ .

PROOF. (1) For the sake of simplicity we assume that $\mu = 0$. For each $m \in \mathbf{N}$, we take a smooth function φ_m such that φ_m has its support in the ball $B_{2m}(0)$ and $\varphi_m \equiv 1$ on $B_m(0)$. Then we have

$$\begin{aligned}
 (6.11) \quad & \int d\nu(x) \int K_\alpha(x, y) d\mu_j(y) \\
 &= \int d\nu(x) \int K_\alpha(x, y) [1 - \varphi_m(y)] d\mu_j(y) + \int d\nu(x) \int K_\alpha(x, y) \varphi_m(y) d\mu_j(y) \\
 &= P_{j,m} + Q_{j,m}.
 \end{aligned}$$

Then by the finiteness of the energy it holds that $\lim_{m \rightarrow \infty} P_{j,m} = 0$ uniformly in j . Also $\lim_{j \rightarrow \infty} Q_{j,m} = 0$ for each m by the definition of the vague convergence.

(2) Take an arbitrary $f \in C_0^2(\mathbf{R}_+^n)$ and put $L_\alpha f = g$. Then

$$\begin{aligned}
 (6.12) \quad & \int f(x) d\mu_j(x) = \iint K_\alpha(x, y) y_n^{\alpha-1} g(y) dy d\mu_j(x) \\
 &= (\mu_j, x_n^{\alpha-1} g(x))_\alpha \rightarrow (\mu, x_n^{\alpha-1} g(x))_\alpha = \int f(x) d\mu(x).
 \end{aligned}$$

(3) Since $\{\mu_j\}$ is a Cauchy sequence, it holds that

$$(6.13) \quad \|\mu_j\|_\alpha < C \quad \text{for some positive number independent of } j.$$

Then by the previous result, $\mu \in \mathbf{E}_\alpha \cap M^+$ and $\mu_j \rightarrow \mu$ weakly. So that

$$(6.14) \quad \|\mu_j - \mu\|_\alpha^2 = \lim_{p \rightarrow \infty} (\mu - \mu_j, \mu_p - \mu_j)_\alpha \leq \lim_{p \rightarrow \infty} \|\mu - \mu_j\|_\alpha \|\mu_p - \mu_j\|_\alpha,$$

Noting that $\|\mu_p - \mu_j\|_\alpha \leq \varepsilon$ for $j > j_0(\varepsilon)$, the assertion has been proved.

Thus we also have proved:

THEOREM 6.3. $\mathbf{E}_\alpha \cap M^+(\mathbf{R}_+^n)$ ($\alpha > 0$) is complete.

PROOF. It suffices to show that for every Cauchy sequence $\{\mu_j\}$, $\mu_j \in \mathbf{E}_\alpha \cap M^+$, a vague limit exists. First, for any $\nu \in \mathbf{E}_\alpha$, $\{(\mu_j, \nu)_\alpha\}$ converges because $\{\mu_j\}$ is a Cauchy sequence. Then again by the previous theorem the asseertion is now clear. Q.E.D.

We remark that $\mathbf{E}_\alpha$ itself is not complete.

§7. Capacity

With a kernel $K_\alpha(x, y)$ we shall associate a certain nonnegative set function defined on the field of Borel sets in the space $\mathbf{R}_+^n$. The function $C_\alpha(E)$, which will be called the α -capacity, is analogous to Lebesgue measure and a variant of the Newton capacity, however it is not additive and not invariant with respect to the motions in direction to x_n -axis.

We use the following notations. For a compact set $K \subset \overline{\mathbf{R}_+^n}$, we denote by M_K^+ the set of all measures whose support lie in K . The set of $\widetilde{M_K^+}$ consisting of measures, normalized by the condition $\int d\mu = 1$, is denoted by $\widetilde{M_K^+}$. It is clear that $\widetilde{M_K^+}$ is convex and vaguely compact.

Let us set

$$(7.1) \quad W_\alpha(K) = \inf\{I_\alpha(\mu); \widetilde{M_K^+}\} \quad \text{for a compact set } K.$$

If $\{\mu_j\}$ ($\mu \in \widetilde{M_K^+}$) is a minimizing sequence, then by the vague compactness of $\widetilde{M_K^+}$ and by the descent principle there is a measure $\lambda \in \widetilde{M_K^+}$ such that we have $I_\alpha(\lambda) = W_\alpha(K)$ holds.

Now we define the capacity:

DEFINITION 7.1. The α -capacity of a compact set K is the number

$$C_\alpha(K) = W_\alpha(K)^{-1}.$$

It is often useful to consider a measure ν which is proportional to λ and fulfills $\gamma = W_\alpha(K)^{-1}\lambda = C_\alpha(K)\lambda$. We shall call this the equilibrium measure.

In view of the classical theory we can easily see that the α -capacity $C_\alpha(K)$ and the equilibrium measure have the following properties:

PROPOSITION 7.1. $C_\alpha(K)$ has the properties listed below:

- (1) $C_\alpha(K)$ is a monotone function of K .
- (2) $C_\alpha(K)$ is continuous on the right.
- (3) $C_\alpha(K)$ is subadditive.

Let γ be the equilibrium measure with respect to K . Then γ has the properties listed below:

- (1) $K_\alpha\gamma(x) = 1$ approximately everywhere on K .
- (2) $K_\alpha\gamma(x) \leq 1$ everywhere.

Here the term "approximately everywhere" means "everywhere" except for a set of inner capacity zero. By the inner capacity of E we mean the supremum of $C_\alpha(F)$ with respect to all compact sets F contained in E . In a similar way we can define the outer capacity of E as the infimum of $C_\alpha(G)$ with respect to all open sets G containing E , and E is called capacitable if and only if the inner

capacity coincides with the outer capacity. We note that these argument can be developed without difficulty according to the classical theory.

Lastly we shall give an example of the ingredients in this section. For the sake of simplicity we assume that $n = 3$ and $\alpha = 1$. Then we have

$$(7.3) \quad K_1(x, y) = E_1(x, y) = (2\pi)^{-1}(|x - y^*| \cdot |x - y|)^{-1} \quad \text{for } x, y \in \mathbf{R}_+^3.$$

We take the half ball $B_R^+ = \{x \in \mathbf{R}^3 \mid |x| < R, x_3 > 0\}$ and put $K \equiv \overline{B_R^+}$. Then $K_1\gamma(x)$ is given by the solution u of the following boundary value problem, that is:

$$(7.4) \quad \sum_{j=1}^3 \partial_j(x_3 \partial_j u) = 0, \quad \text{in } \mathbf{R}_+^3 \setminus B_R^+, \\ u|_{\partial B_R^+} = 1 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} u(x) = 0.$$

To solve this we put $x_1^2 + x_2^2 = \rho^2$, then we get the new equation:

$$(7.5) \quad \frac{\partial_3(x_3 \partial_3 v)}{x_3} + \frac{\partial_\rho(\rho \partial_\rho v)}{\rho} = 0 \quad \text{in } \{(x_3, \rho) \mid x_3^2 + \rho^2 > R^2, \rho > 0\}, \\ v|_{x_3^2 + \rho^2 = R^2} = 1, \quad \lim_{x_3^2 + \rho^2 \rightarrow \infty} v(x_3, \rho) = 0.$$

Then the unique solution of (7.5) is given by

$$(7.6) \quad v(x_3, \rho) = \begin{cases} R^2(x_3^2 + \rho^2)^{-2}, & x_3^2 + \rho^2 > R^2 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the uniqueness of the solution of (7.4), we have

$$(7.7) \quad u(x) = \int K_1(x, y) d\gamma(y) = \begin{cases} R^2|x|^{-2}, & |x| > R \\ 1, & \text{otherwise.} \end{cases}$$

It is not difficult to see that this u is also the unique solution of the next variational problem:

$$\inf \left(\int x_3 |\nabla u|^2 dx; u \geq 1 \text{ on } B_R^+, u \in C_0^1(\overline{B_R^+}) \right)$$

To see the exact form of γ we prepare the following:

LEMMA 7.1. *Suppose that α is a positive integer. Then*

$$(7.9) \quad \int_{\partial B_R^+} E_\alpha(x, y) d\nu_{\alpha, R}(y) = \min\{1, (|x|/R)^{2-n-\alpha}\}.$$

Here $\nu_{\alpha, R}(y)$ is a measure on the half sphere defined by

$$(7.10) \quad \nu_{\alpha, R}(y) = (n + \alpha - 2)R^{-1}y_n S_y,$$

where S_y is the $(n-1)$ -dimensional Lebesgue measure on ∂B_R .

Admitting this for a moment, we see the exact form of γ . Letting $n = 3$ and $\alpha = 1$ we immediately have

$$(7.11) \quad \gamma = \begin{cases} 2R^{-1}y_n S_y, & y \in \partial B_R^+ \\ 0, & \text{otherwise.} \end{cases}$$

PROOF OF LEMMA 7.1. We prepare the following which is well-known:

$$(7.12) \quad \int_{\partial B_R} |x-y|^{2-n} dS_y = 2\pi^{n/2}\Gamma(n/2)^{-1}R \min\{(|x|/R)^{2-n}, 1\}, \quad n \geq 3.$$

Then by virtue of this equality and Lemma 1.2, we can check the desired equality by a simple computation. Q.E.D.

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