Notes on Anderson-Anderson-Johnson questions and Bouvier questions

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§0. Introduction.

In this paper A denotes an integral domain. A domain A is called a locally factorial ring if A_M is a factorial ring for each maximal ideal M of A . If each ideal of A is a product of primary ideals of A , then A is called a Q-ring.

Let $X^{(1)}(A)$ be the set of height one prime ideals of A , and $\text{Spec}(A)$ the set of prime ideals of A .

[6, p.17] asks the following question :

(B-0) Let A be a locally factorial ring with $\text{Pic}(A) = 0$. Is A a factorial ring?

Let A be a Krull ring. [3, §2] asks the following questions :

Are the following statements true?

(AAJ-1) Let P be a height one prime ideal of A . If $P \cap Q = PQ$ for each $Q \in X^{(1)}(A)$ with $Q \neq P$, then P is an invertible ideal.

(AAJ-2) Let P be a non-zero prime ideal of A . If $P \cap Q = PQ$ for each $Q \in \text{Spec}(A)$ with $ht\ Q \leq 2$ that is incomparable with P , then P is a maximal ideal or an invertible ideal of A .

(AAJ-3) A is a locally factorial ring if and only if $P \cap Q = PQ$ for each pair of distinct height one prime ideals P, Q of A .

(AAJ-4) A is a factorial ring if and only if $\text{Pic}(A) = 0$ and $P \cap Q = PQ$ for each pair of distinct height one prime ideals P, Q of A .

(AAJ-5) A is a locally factorial ring if and only if PQ is a divisorial ideal for each distinct height one prime ideals P, Q of A .

(AAJ-6) The following conditions are equivalent:

(1) A is a locally factorial ring with $\dim A \leq 2$.

(2) $P \cap Q = PQ$ for each incomparable prime ideals P, Q of A .

(3) A is a Q-ring.

Let A be a Krull ring. Relating with the above questions, [6, (3.9)] asks the following questions;

(B-1) Are the following conditions equivalent?

(1) A is a locally factorial ring.

(2) $P \cap Q = PQ$ for each distinct height one prime ideals P, Q of A .

(3) PQ is a divisorial ideal for each height one prime ideals P, Q of A .

(B-2) If $\text{Pic}(A) = 0$ and $P \cap Q = PQ$ for each distinct height one prime ideals P, Q of A , then is A a factorial ring?

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(B-3) Let A be a locally factorial domain. Then are the following conditions equivalent?

- (1) $\dim A \leq 2$.
- (2) $P \cap Q = PQ$ for each incomparable prime ideals P, Q of A .
- (3) A is a Q-ring.

§1. On (B-0).

(1.1). Let $\mathbf{Z}$ be the ring of integers, $\{p_i; i = 1, 2, \dots\}$ the set of positive prime numbers, and X, Y indeterminates over $\mathbf{Z}$. Set $A =$

$\bigcup_{i=1}^{\infty} \mathbf{Z}[X/p_1, Y/p_1, X/p_2, Y/p_2, \dots, X/p_n, Y/p_n]$ (cf. [5, Example]). Then we have the following,

- (a) A is not a Krull domain. Thus A is not a factorial ring.
- (b) A is a locally factorial ring with $\text{Pic}(A) = 0$.

PROOF. (a): [5, Example, (a)] shows that A is not a Krull domain.

(b): Let $\mathbf{Q}$ be the field of rational numbers, and M be a maximal ideal of A . Then A_M is a localization of $\mathbf{Q}[X, Y]$ or A_M is a localization of the polynomial ring $\mathbf{Z}_{(p)}[X/p, Y/p]$ for a prime number p . Hence A_M is a factorial ring. Thus A is a locally factorial ring. Set $A_n = \mathbf{Z}[X/(p_1 \cdots p_n), Y/(p_1 \cdots p_n)]$ for each natural number n . We have $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, and $A = \bigcup_{i=1}^{\infty} A_n$. Thus $\text{Pic}(A) = \varinjlim \{\text{Pic}(A_n)\}$ by [8, Theorem 1.3]. Since each A_n is a factorial ring, we have $\text{Pic}(A) = 0$.

Thus Question (B-0) was solved negatively.

§2. Lemmas.

(2.1)([2, COROLLARY 2.2]). Let A be a Krull ring, I a nonzero ideal of A . Then I is an invertible ideal of A if and only if I is a locally principal ideal of A .

(2.2)(A PART OF [4]). (1) If A is a Q-ring, then any quotient ring is a Q-ring.

(2) Let (A, M) be a quasi-local ring. Then A is a Q-ring if and only if A is either $\dim A \leq 1$ or A is a 2-dimensinal factorial ring.

(3) A is a Q-ring if and only if A is Laskerian and every nonmaximal prime ideal is finitely generated and locally principal.

(2.3)(A PART OF [1, THEOREM 1]). Let A be a Krull ring. Then the followings are equivalent:

- (1) A is a locally factorial ring.
- (2) Each height one prime ideal of A is an invertible ideal.
- (3) The product of two divisorial ideals of A is a divisorial ideal.

§3. The implications of the questions.

(3.1).

(1) Let $P, Q \in \text{Spec}(A)$. Then $P \cap Q = PQ$ if and only if

$PA_M \cap QA_M = PA_M \cdot QA_M$ for each $M \in \text{Max}(A)$.

(2) Let $P \in \text{Spec}(A)$ and let $P \subset M \in \text{Max}(A)$. Then $P \supsetneq PM$ if and only if $PA_M \supsetneq PA_M \cdot MA_M$.

(3.2). Let A be a Krull ring and let $P, Q \in X^{(1)}(A)$ with $P \neq Q$. Then

(1) PQ is a divisorial ideal of A if and only if $P \cap Q = PQ$.

(2) If PQ is a divisorial ideal of A , then $PA_M \cdot QA_M$ is a divisorial ideal of A_M for each $M \in \text{Max}(A)$.

PROOF. (1): Assume that PQ is a divisorial ideal. We have

$\text{div}(P \cap Q) = \sup(\text{div}P, \text{div}Q) = \text{div}P + \text{div}Q = \text{div}PQ$.

Thus $P \cap Q = (PQ)^v = PQ$.

(2) follows by (1).

(3.3).

(1) In (AAJ-3), the necessity holds.

(2) In (AAJ-4), the necessity holds.

(3) In (AAJ-5), the necessity holds.

(4) In (AAJ-6); (3) implies (1), and (1) implies (2).

(5) In (B-1); (1) implies (3), and (3) implies (2).

(6) In (B-3); (3) implies (1), and (1) implies (2).

PROOF. (1) and (3) follow by (2. 3).

(4) and (6) follow by (2. 2).

(5): (1) implies (3) by (2. 3). (3) implies (2) by (3. 2).

Let (A) be a ‘yes or no’ question for domains. Assume that if (A) is yes for each quasi-local domain, then (A) is yes for each domain. Then we will say that (A) is of local property.

(3.4).

(1) (AAJ-1) is of local property.

(2) (AAJ-2) is of local property.

(3) (AAJ-3) is of local property.

(4) (AAJ-4) is of local property.

(5) (AAJ-5) is of local property.

(6) (B-1) is of local property.

(7) (B-2) is of local property.

PROOF. (1) and (2) follow by (2. 1).

(4): Let A be a Krull ring. We will show the sufficiency in (AAJ-4). By assumption, A is a locally factorial ring. By (2. 3), each height one prime ideal of A is an invertible ideal. Since $\text{Pic}(A) = 0$, A is a factorial ring.

(5) follows by (3. 2).

(3.5). If (AAJ-2) is yes, then (AAJ-1) is yes.

PROOF. Let (A, M) be a quasi-local Krull domain and let $P \in X^{(1)}(A)$. Assume that $P \cap Q = PQ$ for each $Q \in X^{(1)}(A)$ with $Q \neq P$. We will show that P is an invertible ideal.

If $\dim A \leq 1$, then P is an invertible ideal.

Let $\dim A \geq 2$ and let $Q \in \text{Spec}(A)$ with $ht\ Q \leq 2$ that is incomparable with P . If $ht\ Q \leq 1$, then $P \cap Q = PQ$. Suppose that $ht\ Q = 2$. Take $a \in P \cap Q$ with $a \neq 0$. Since A_Q is a Krull domain and $a \in QA_Q$, there exists $Q' \in X^{(1)}(A_Q)$ such that $a \in Q'$. Q' is of the form Q_0A_Q where $Q \supsetneq Q_0 \in X^{(1)}(A)$, $P \neq Q_0$ and $a \in P \cap Q_0 = PQ_0 \subset PQ$. Thus $P \cap Q = PQ$. Since (AAJ-2) is yes, P is a maximal ideal or an invertible ideal. Since $\dim A \geq 2$, P is an invertible ideal of A .

Let (A') and (A'') be two 'yes or no' questions. In the followings ' $(A') \implies (A'')$ ' means that if (A') is yes then (A'') is yes.

(3.6).

$$\begin{aligned} (1) \quad & (AAJ-2) \implies (AAJ-1) \implies (AAJ-3) \iff (AAJ-4) \\ & \iff (AAJ-5) \iff (B-1) \iff (B-2). \\ (2) \quad & (AAJ-6) \implies (B-3). \end{aligned}$$

(3.7). If (AAJ-2) is yes, then we have the followings:

- (1) In (AAJ-6); $(1) \iff (2)$.
- (2) In (B-3); $(1) \iff (2)$.
- (3) $(AAJ-6) \iff (B-3)$.

(3.8). Let A be a Krull ring and let $P \in X^{(1)}(A)$. Then P is an invertible ideal of A if and only if $P \cap Q = PQ$ for each $P \neq Q \in X^{(1)}(A)$ and $P \not\subseteq PM$ for each $P \subset M \in \text{Max}(A)$.

PROOF. The sufficiency: We may assume that (A, M) is a quasi-local ring. Let $P \in X^{(1)}(A)$. Since A is a Krull ring, A is an atomic ring. Take $a \in P - PM$. Then a is an irreducible element of A . Because, if a is a reducible element with $a = p_1 p_2 \cdots p_n$, where $n \geq 2$ and each p_i is irreducible element, then $p_i \in P$ for some i . Thus $a \in PM$; a contradiction.

Let $Q \in X^{(1)}(A)$ that is distinct with P . If $a \in Q$, then $a \in P \cap Q = PQ \subseteq PM$; a contradiction. Thus P is the only height one prime ideal containing a .

Let v be the valuation with the valuation ring A_P . Then we have $v(a) = v(b)$ for each $b \in P - PM$. Because, if, say, $v(a) < v(b)$ then there exists x of A such that $x = b/a$. Then $b = ax \in PM$; a contradiction. Thus $P = aA \cup PM$. It follows that $P = aA$.

§4. The integral closure of a Noetherian domain.

A Noetherian ring A is called a local ring if A has only one maximal ideal. Let B be a Noetherian domain with quotient field L , K a finite algebraic extension field of L and A the integral closure of B in K . By Mori-Nagata's Integral Closure Theorem, A is a Krull ring.

(4.1)(GOTO). Let A be as above. Then we have $\bigcap_{n=1}^{\infty} P^n A_P = (0)$ for each $P \in \text{Spec}(A)$.

PROOF. Set $P \cap B = P'$. Then $A_{P'}$ is the integral closure of $B_{P'}$ in K . Thus we may assume that (B, M) is a local ring and $P \cap B = M$.

Set $d = \dim B$. Thus we may use the induction on d .

If $d = 0, 1$, then the statement is clear. Suppose that $d \geq 2$ and the statement holds for all positive integer lower than d . Set $n = [K : L]$. Then $K = \sum_{i=1}^n Lx_i$ for some $x_1, x_2, \dots, x_n \in A$. Set $M' = P \cap B[x_1, \dots, x_n]$. We may replace B with $B[x_1, x_2, \dots, x_n]_{M'}$ and K with L .

Then A is the integral closure of B in K .

We may assume that $ht P \geq 2$. Take $Q \in X^{(1)}(A)$ with $Q \subset P$. Set $Q \cap B = Q'$. Then $Q' \neq (0)$, $B/Q' \subset A/Q$ and the quotient field of A/Q is a finite algebraic extension of that of B/Q' by Mori-Nagata's Theorem. Let R be the integral closure of B/Q' in the quotient field of A/Q . We have $\dim(B/Q') < d$. Take $N \in \text{Spec}(R)$ with $N \cap (A/Q) = P/Q$. Since $\dim R < d$, we have

$$(0) = \bigcap_{n=1}^{\infty} N^n R_N \supset \bigcap_{n=1}^{\infty} P^n (A_P/QA_P).$$

Thus

$$\bigcap_{n=1}^{\infty} P^n A_P \subset \bigcap_{\substack{Q \in X^{(1)}(A) \\ Q \subset P}} QA_P$$

Since $\dim A_P \geq 2$, it follows that $\bigcap_{\substack{Q \in X^{(1)}(A) \\ Q \subset P}} QA_P = (0)$.

Thus $\bigcap_{n=1}^{\infty} P^n A_P = (0)$.

(4.2). Let B be a Noetherian domain with quotient field L . Let K be a finite algebraic extension field of L , and A the integral closure of B in K . then (AAJ-1) is yes for A .

PROOF. Let $P \in X^{(1)}(A)$. Assume that $P \cap Q = PQ$ for each $Q \in X^{(1)}(A)$ with $P \neq Q$. Let $P \subset M \in \text{Max}(A)$. If $P = PM$, then

$$PA_M = PMA_M = PM^2A_M = PM^3A_M = \dots = (0)$$

by (4. 1); a contradiction. Therefore $P \not\supseteq PM$. (3. 8) implies that P is an invertible ideal.

(4.3). Let A be as (4.2). Then for A we have

$$(AAJ - 6) \iff (B - 3).$$

PROOF. The sufficiency: we assume (2) of (AAJ-6). We will prove (3) of (AAJ-6). Each height one prime ideal of A is an invertible ideal by (4.2). Therefore A is a locally factorial ring. Since (B-3) is yes, A is a Q-ring.

(4.4). Let A be as (4. 2). Then for A we have

$$(AAJ - 2) \implies (B - 3).$$

PROOF. We assume moreover that A is locally factorial. Assume (2) of (B-3). We will show (3) of (B-3). Since (AAJ-2) is yes for A , we have $\dim A \leq 2$. By Nagata's Theorem, A is a Noetherian ring. By [3, Corollary (2. 8)], A is a Q-ring.

§5. An example

Relating with (4.1), there naturally arises the following question:

Let (A, M) be a quasi-local Krull domain. Then is $\bigcap_{n=1}^{\infty} M^n = (0)$?

If the answer to this question is yes, then (AAJ-1) is yes.

(5.1). There exists a quasi-local factorial ring (A, M) with $M \neq (0)$ such that $\bigcap_{n=1}^{\infty} M^n = M$.

In fact, let A be a factorial ring, X, U_1, X_1, Y_1 be indeterminates. Set $V_1 = (X - U_1 X_1)/Y_1$, $D' = A[U_1, X_1, V_1, Y_1]$ and $D = A[X]$. Then we have the following,

(1) $ht(P' \cap D) \leq 1$ for each $P' \in X^{(1)}(D')$.

(2) For each $P \in X^{(1)}(D)$ there exists a unique $P' \in X^{(1)}(D')$ such that $P' \cap D = P$. Further $P'D'_{P'} = PD'_{P'}$.

Let k be a field and let $\{X(i_1 i_2 \cdots i_n); n, i_j \in \mathbb{N}, 1 \leq i_j \leq 4\}$ be a system of indeterminates. Set

$$Y(i_1 \cdots i_n) = X(i_1 \cdots i_n) - X(i_1 \cdots i_{n-1})X(i_1 \cdots i_{n-2}) \\ - X(i_1 \cdots i_{n-3})X(i_1 \cdots i_{n-4}).$$

Let $k[X(i_1 \cdots i_n); i_1, \dots, i_n]$ be a polynomial ring over k of the infinite number of indeterminates. Let $(Y(i_1 \cdots i_n); i_1, \dots, i_n)$ be the ideal of

$k[X(i_1, \dots, i_n); i_1, \dots, i_n]$ generated by the elements

$\{Y(i_1 \cdots i_n); i_1, \dots, i_n\}$. Then

$(Y(i_1 \cdots i_n); i_1, \dots, i_n)$ is a prime ideal of $k[X(i_1 \cdots i_n); i_1, \dots, i_n]$. Set

$D = k[X(i_1 \cdots i_n); i_1, \dots, i_n]/(Y(i_1 \cdots i_n); i_1, \dots, i_n)$. Let $x(i_1 \cdots i_n)$ be the image of $X(i_1 \cdots i_n)$ under the natural homomorphism of

$k[X(i_1 \cdots i_n); i_1, \dots, i_n]$ to D . Then we have $D = \bigcup_{i=1}^{\infty} k[x(i_1 \cdots i_n); n \leq i, 1 \leq i_j \leq 4]$, which we denote by $k[x(i_1 \cdots i_n); i_1, \dots, i_n]$. We have

$$x(i_1 \cdots i_n) = x(i_1 \cdots i_{n-1})x(i_1 \cdots i_{n-2}) + x(i_1 \cdots i_{n-3})x(i_1 \cdots i_{n-4}).$$

Set

$$\begin{aligned} D_0 &= k[x(1), x(2), x(3), x(4)], \\ D_1 &= D_0[x(11), x(12), x(13), x(14)], \\ D_2 &= D_1[x(21), x(22), x(23), x(24)], \\ D_3 &= D_2[x(31), x(32), x(33), x(34)], \\ D_4 &= D_3[x(41), x(42), x(43), x(44)], \\ D_5 &= D_4[x(111), x(112), x(113), x(114)], \\ D_6 &= D_5[x(121), x(122), x(123), x(124)], \\ &\dots \end{aligned}$$

Then we have $D = \bigcup_{i=1}^{\infty} D_i$.

Each D_i is a polynomial ring over k . Hence each D_i is a factorial ring. For each $n \in \mathbb{N}$, D_n and D_{n+1} are of the following forms:

$$D_{n+1} = A[x(\alpha_1), x(\alpha_2), x(\alpha_3), x(\alpha_4)],$$

$$D_n = A[x(\alpha)], \text{ where } A \text{ is some factorial ring and}$$

$x(\alpha) = x(\alpha_1)x(\alpha_2) + x(\alpha_3)x(\alpha_4)$. Thus $ht(P' \cap D_n) \leq 1$ for each $P' \in X^{(1)}(D_{n+1})$ by (1).

Let $m \geq n$ and let P_m be a height one prime ideal of D_m such that $P_m \cap D_n = P_n$ for some $P_n \in X^{(1)}(D_n)$. Then we have $P_n(D_m)_{P_m} = P_m(D_m)_{P_m}$ by (2). Thus $P_n \not\subset (P_m)^{(2)}$ (the 2-nd symbolic power of P_m). Thus D is a Krull domain by [7, Theorem 3]. By [7, Remark 2] we have $X^{(1)}(D) = \bigcup_{n=1}^{\infty} X^{(1)}(D_n)$. Hence D is a factorial domain. Set $N = (x(i_1 \cdots i_n); i_1, \dots, i_n)$. N is a maximal ideal of D . We have $N = N^2 = N^3 = \dots$. Set $A = D_N$. Then A is a quasi-local factorial domain with maximal ideal $M = ND_N$. Moreover we have $\bigcap_{n=1}^{\infty} M^n = M$.

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